On a Class of Reflected Backward Stochastic Volterra Integral Equations and Related Time-Inconsistent Optimal Stopping Problems

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FSVIE

Forward stochastic Volterra integral equations (FSVIE):

Dynamics (drift b, volatility σ, Brownian motion W)

$$X(t) = \phi(t) + \int_0^t b(t, s, X(s))ds + \int_0^t \sigma(t, s, X(s))dW(s)$$

They represent:

- Systems with memory
- Fractional Brownian motion

$$W^H(t) = \int_0^t K(t, s)dW(s)$$

Fredholm, Bellman, Wiener, Volterra...

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Motivation

Consider the equation

$$dY(t) = 0 \quad Y(T) = \xi$$

According to ξ, we distinguish two cases:

- if $\xi \in \mathbb{R}$ **(BODE)**

$$Y(t) = \xi$$

- if $\xi \in L^2(\Omega, \mathcal{F}_T)$ **(BSDE)**

$$Y(t) = \xi \qquad \text{Problem!!}$$

$\mathcal{F}_t$-adapted $\qquad$ $\mathcal{F}_T$-measurable

BSDE

The process Y is required to be $\mathcal{F}_t$-adapted . Set

$$Y(t) = \mathbb{E}[\xi|\mathcal{F}_t]$$

MRT, $\exists! Z \in L^2$

$$Y(t) = \mathbb{E}[\xi|\mathcal{F}_t] = \mathbb{E}[\xi] + \int_0^t Z(s)dW(s)$$

Simple calculations apply

$$Y(t) = \xi - \int_t^T Z(s)dW(s).$$

BSDE

BSDE (Y, Z)

BSDE (driver f, terminal condition ξ, Brownian motion W)

$$-dY(t) = f(t, Y(t), Z(t))dt - Z(t)dW(t), \quad Y(T) = \xi$$

equivalent to

$$Y(t) = \xi + \int_t^T f(s, Y(s), Z(s))ds - \int_t^T Z(s)dW(s)$$

BSVIE [1]

Consider the following equation (still $\xi(t)$ $\mathcal{F}_T$-measurable)

$$Y(t) = \xi(t) \qquad\qquad \text{BSDE } Y(t) = \xi$$

Define $M_t(u) = \mathbb{E}[\xi(t)|\mathcal{F}_u]$, MRT, $\exists! Z(t,\cdot) \in L^2$

$$M_t(u) = \mathbb{E}[\xi(t)] + \int_0^u Z(t,s)dW(s)$$

Putting $u = t$ and $Y(t) = M_t(t)$

$$Y(t) = \mathbb{E}[\xi(t)] + \int_0^t Z(t,s)dW(s)$$

After some computation, we get

$$Y(t) = \xi(t) + \int_t^T f(t,s,Y(s),Z(t,s))dt - \int_t^T Z(t,s)dW(s)$$

[1] Yong

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RBSDE

Reflected BSDE (Y, Z, L)

RBSDE (driver f, terminal condition ξ, Brownian motion W, obstacle L,local time K)

- $$-dY(t) = f(t, Y(t), Z(t))dt - Z(t)dW(t), \quad Y(T) = \xi$$
- $Y(t) \geq L(t)$
- $K(t)$ is an increasing, continuous process, $K(0) = 0$ and

$$\int_0^T (Y(s) - L(s))\, dK(s) = 0, \quad \mathbb{P}\text{-a.s.}$$

Question

What is the corresponding reflected BSVIE?

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We define the following spaces for the solution.

- $\mathcal{S}^2$ is the set of $\mathbb{R}$-valued $\mathbb{F}$-adapted processes $(Y(t))_{t\in[0,T]}$:

$$\|Y\|^2_{\mathcal{S}^2} := \mathbb{E}[\sup_{t\in[0,T]} |Y(t)|^2] < \infty .$$

- $\mathcal{H}^2$ is the space of progressively measurable processes $(v(t))_{t\in[0,T]}$:

$$\|v\|^2_{\mathcal{H}^2} := \mathbb{E}\left[\int_0^T |v(t)|^2 dt\right] < \infty.$$

- $\mathbb{L}^2$ is the set of $\mathbb{R}$-valued processes $(Z(t,s))_{(t,s)\in[0,T]\times[0,T]}$: for a.a. $t \in [0, T]$ $Z(t,\cdot) \in \mathcal{H}^2$ and satisfy

$$\|Z\|^2_{\mathbb{L}^2} := \mathbb{E}\left[\int_0^T \int_t^T |Z(t,s)|^2 ds dt\right] < \infty .$$

- $\mathcal{K}^2$ is the space of processes K which satisfy
  - for each $t \in [0, T]$, $u \mapsto K(t,u)$ is an $\mathbb{F}$-adapted and increasing process with $K(t,0) = 0$;
  - $(t,u) \mapsto K(t,u)$ is continuous and $K(\cdot, T) \in \mathcal{H}^2$.

Let (Y, Z, K) be the solution of

Reflected BSVIE (driver f, obstacle $K(t, ds)$)

$$Y(t) = \xi(t) + \int_t^T f(t, s, Y(s), Z(t, s))ds + \int_t^T K(t, ds) - \int_t^T Z(t, s)dW(s) \tag{1}$$

(a) $Y \in \mathcal{H}^2$, $t \mapsto Y(t)$ is continuous and $Z \in \mathbb{L}^2$;

(b) $Y(t) \geq L(t) \quad \mathbb{P}\text{-a.s.}, \quad 0 \leq t \leq T$;

(c) $K(t, ds)$ is the Lebesgue-Stieltjes measure induced by the function $s \mapsto K(t, s)$, it enjoys the following properties:

(c1) $K \in \mathcal{K}^2$;

(c2) The Skorohod flatness condition holds: for each $0 \leq \alpha < \beta \leq T$,

$$K(t, \alpha) = K(t, \beta) \quad \text{whenever} \quad Y(u) > L(u) \text{ for each } u \in [\alpha, \beta] \quad \mathbb{P}\text{-a.s.}$$

Remark

The Skorohod flatness condition (c2) implies

$$\int_t^T K(t, ds) = 0 \ \textit{ whenever } Y(t) > L(t) \textit{ for each } t \in [0, T] \quad \mathbb{P}\text{-a.s.}$$

Mimicking Lin (2002), Yong (2006) and Wang & Zhang (2007) (for BSVIEs), we construct Y so that, for every $t \in [0, T]$, $Y(t) = \widetilde{Y}(t, t)$ parametrized by t:

Accompanying reflected BSVIE $\widetilde{Y}(t, \cdot)$

$$\widetilde{Y}(t, u) = \xi(t) + \int_u^T f(t, s, Y(s), Z(t, s))ds + \int_u^T K(t, ds) - \int_u^T Z(t, s)dW(s), \quad u \in [t, T]$$

We make the following assumptions on (f, ξ, L)

(A1) $\xi(t)$ is a $\mathcal{B}([0,T]) \otimes \mathcal{F}_T$-measurable map $\xi : \Omega \times [0,T] \longrightarrow \mathbb{R}$:

$$\sup_{0 \le t \le T} \mathbb{E}[|\xi(t)|^2] < \infty;$$

(A2) The obstacle $(L(u), 0 \le u \le T)$ is a real-valued and $\mathbb{F}$-adapted continuous process:

$$L(T) \le \xi(t), \quad t \in [0,T] \quad \text{and} \quad \mathbb{E}\Big[\sup_{0 \le u \le T} (L(u))^2\Big] < \infty. \qquad (2)$$

(A3) The driver f is a map from $\Omega \times [0, T] \times [0, T] \times \mathbb{R} \times \mathbb{R}$ onto $\mathbb{R}$, for any fixed $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}$, $f(t, \cdot, y, z)$ is progressively measurable.

(i) $$\sup_{0 \le t \le T} \mathbb{E}\left[\left(\int_t^T |f(t, s, 0, 0)| ds\right)^2\right] < \infty,$$

(ii) $\exists c_f \ge 0$, for all $(t, s) \in [0, T]^2$ and $y, y', z, z' \in \mathbb{R}$,

$$|f(t, s, y, z) - f(t, s, y', z')| \le c_f \left(|y - y'| + |z - z'|\right).$$

(iii) For some $\alpha \in (0, 1/2]$ and $c_1 > 0$, for all $(y, z) \in \mathbb{R} \times \mathbb{R}$ and all $0 \le t, t' \le s \le T$,

$$|f(t', s, y, z) - f(t, s, y, z)| \le c_1 |t' - t|^\alpha,$$

and for some $\beta > 1/\alpha$ and $c_2 > 0$,

$$\mathbb{E}[|\xi(t) - \xi(t')|^\beta] \le c_2 |t' - t|^{\alpha\beta}$$

and

$$\mathbb{E}\left[\left(\int_0^T |f(0, s, 0, 0)|^2 ds\right)^{\beta/2}\right] < \infty.$$

Existence and Uniqueness

Remark

Assumption (iii) yields the continuity of Y and the bi-continuity of $K(\cdot,\cdot)$ which in turn guarantee the Skorohod flatness condition (c2).

Theorem

Under the above assumptions, the reflected BSVIE (1) *associated with* (f,ξ,L) *admits a unique solution* (Y,Z,K).
Moreover, for every $t\in[0,T]$,

$$Y(t)=\operatorname*{ess\,sup}_{\tau\geq t}\mathbb{E}\Big[\int_t^{\tau} f(t,s,Y(s),Z(t,s))ds+L(\tau)\mathbb{1}_{\{\tau<T\}}+\xi(t)\mathbb{1}_{\{\tau=T\}}\Big|\mathcal{F}_t\Big] \tag{3}$$

Comparison Theorem

Let (Y, Z, K) be a solution of the **reflected BSVIE** associated with (f, ξ, L) and (Y', Z', K') be a solution of the **reflected BSVIE** associated with (f', ξ', L').

Theorem

Assume that (f, ξ, L) and (f', ξ', L') satisfy the assumptions (A1), (A2) and (A3) and that either the map $y \mapsto f(t, s, y, z)$ or $y \mapsto f'(t, s, y, z)$ is nondecreasing. Assume further that

(H1) $\xi(t) \leq \xi'(t)$, $\mathbb{P}$-a.s. , $0 \leq t \leq T$,

(H2) $f(t, s, y, z) \leq f'(t, s, y, z)$, *for all* $(t, y, z) \in [0, s] \times \mathbb{R} \times \mathbb{R}$, *a.s., a.e.* $s \in [0, T]$,

(H3) $L(t) \leq L'(t)$, $0 \leq t \leq T$, $\mathbb{P}$-a.s.

Then

$$Y(t) \leq Y'(t), \quad 0 \leq t \leq T, \quad \mathbb{P}\text{-a.s.}$$

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Consider

$$f(t,s,X(s)) := \varphi(s-t)\psi(X(s)), \quad \xi(t) := \varphi(T-t)h(X(T)),$$

where φ is a (deterministic) discounting function (deflator).
Maximize $\mathcal{J}(t,\tau) := \mathbb{E}\left[\int_t^\tau f(t,s,X(s))ds + L(\tau)\mathbb{1}_{\{\tau<T\}} + \xi(t)\mathbb{1}_{\tau=T\}}\right]$

Optimal stopping problem

Find an $\mathbb{F}$-stopping time τ_t^*, indexed by t, such that

$$\tau_t^* = \arg\max_{\tau\geq t} \mathcal{J}(t,\tau).$$

- $\mathcal{J}$=investment in a commodity
- $X(t)$=price of a commodity
- f=utility rate per unit time
- L=utility function at the stopping time τ, ξ=utility at the final time T

The associated value-function is

$$Y(t) := \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}\left[\int_t^\tau f(t,s,X(s))ds + L(\tau)\mathbb{1}_{\{\tau<T\}} + \xi(t)\mathbb{1}_{\{\tau=T\}} \middle| \mathcal{F}_t\right]. \tag{4}$$

Examples of hyperbolic discounting functions used in utility maximization include

- Loewenstein and Drazen (1992), Laibson (1997), Loewenstein and O'Donoghue (2002):

  $$\phi(s-t) := (1+\alpha(s-t))^{-\frac{\delta}{\alpha}}, \ \alpha, \delta > 0, \ \ s > t.$$

  Note that $\phi(s-t) \to e^{-\delta(s-t)}$ as $\alpha \to 0$.

- Strulik (2017):

  $$\phi(t,s) := \left(\frac{1+\alpha t}{1+\alpha s}\right)^\beta \ \alpha \geq 0, \ \beta \geq 1, \ s \geq t.$$

• It is easily seen that, apart from $e^{-\delta t}$, for any other choice of the discounting function φ, $(Y(t) + \int_0^t f(t,s,X(s))ds)_{t\leq T}$ will not be a supermartingale. i.e. the optimal stopping problem is time-inconsistent. We approach the problem as follows.

• Note that

$$\sup_{\tau\geq t} \mathcal{J}(t,\tau) \leq \mathbb{E}[Y(t)]. \tag{5}$$

Now, if we can find an $\mathcal{F}$-stopping time τ_t^* such that

$$Y(t) = \mathbb{E}\left[\int_t^{\tau_t^*} f(t,s,X(s))ds + L(\tau_t^*)\mathbb{1}_{\{\tau_t^*<T\}} + \xi(t)\mathbb{1}_{\{\tau_t^*=T\}}\middle|\mathcal{F}_t\right]$$

then τ_t^* is optimal for $\mathcal{J}(t,\cdot)$ since

$$\mathcal{J}(t,\tau_t^*) = \mathbb{E}[Y(t)] \leq \sup_{\tau\geq t} \mathcal{J}(t,\tau) \leq \mathbb{E}[Y(t)]. \tag{6}$$

- It is tempting to suggest that Y solves a Reflected BSVIE of the form

$$Y(t) = \xi(t) + \int_t^T f(t,s,X(s))ds + \int_t^T K(t,ds) - \int_t^T Z(t,s)dW(s)$$

for some processes $(Z(t,s), K(t,s))$, where $K(t,ds)$ is the Lebesgue-Stieltjes measure induced by the 'increasing function' $s \mapsto K(t,s)$.

Proposition

Suppose the assumptions (A1), (A2) and (A3) are satisfied. For each $t \in [0, T]$, *denote by* τ_t^* *the stopping time*

$$\tau_t^* = \inf\{t \leq u \leq T;\ \widetilde{Y}(t,u) = L(u)\}$$

with the convention that $\tau_t^* = T$ *if* $\widetilde{Y}(t,u) > L(u),\ t \leq u \leq T$.
Then τ_t^* *is optimal in the sense that*

$$Y(t) = \mathbb{E}\left[\int_t^{\tau_t^*} f(t,s,X(s))ds + L(\tau_t^*)\mathbb{1}_{\{\tau_t^* < T\}} + \xi(t)\mathbb{1}_{\{\tau_t^* = T\}} \middle| \mathcal{F}_t\right]. \tag{7}$$

Moreover, τ_t^* *is an optimal strategy for* $\mathcal{J}(t,\cdot)$ *i.e.*

$$\tau_t^* = \arg\max_{\tau \geq t} \mathcal{J}(t,\tau).$$

Remark

The choice of the optimal stopping time τ_t^* as the first hitting time of the accompanying Snell envelope $\widetilde{Y}(t,\cdot)$ of the obstacle L instead of the value function Y (as it is the case for standard reflected BSVIE) is simply due to fact that

$$Y(t) \neq Y(u) + \int_u^T f(t,s,X(s))ds + \int_u^T K(t,ds) - \int_u^T Z(t,s)dW(s), \quad u \geq t.$$

Thank you for your attention