Additional information disclosed progressively in time
Enlargement of filtrations and martingale representation

Anna Aksamit
The University of Sydney

9th Colloquium on BSDEs and Mean Field Systems
Annecy, 30th of June 2022
• Given a measurable space \((\Omega, \mathcal{F})\) a financial model consists of

\[
S \quad \mathbb{F} := (\mathcal{F}_t)_{t \geq 0} \quad \mathbb{P}
\]

• Information, mathematically expressed by a filtration

\[
\mathbb{F} = (\mathcal{F}_t)_{t \geq 0} \quad \mathcal{F}_s \subset \mathcal{F}_t \quad s \leq t
\]

• Asymmetry of information: additional information via filtration enlargement \(\mathbb{F} \subset \mathbb{G}\), i.e., \(\mathcal{F}_t \subset \mathcal{G}_t\) for all \(t \geq 0\)

• Default time modelling
• Given a measurable space \((\Omega, \mathcal{F})\) a financial model consists of prices, information, probability:

\[ S \mathbb{F} := (\mathcal{F}_t)_{t \geq 0} \mathbb{P} \]

• Information, mathematically expressed by a filtration:

\[ \mathbb{F} = (\mathcal{F}_t)_{t \geq 0} \quad \mathcal{F}_s \subset \mathcal{F}_t \quad s \leq t \]

• Asymmetry of information: additional information via filtration enlargement \( \mathbb{F} \subset \mathbb{G} \), i.e., \( \mathcal{F}_t \subset \mathcal{G}_t \) for all \( t \geq 0 \)

• Default time modelling

• Questions on arbitrages, utility maximisation, value of information, market completeness, hedging strategies,...
• Given a measurable space \((\Omega, \mathcal{F})\) a financial model consists of
  
  \[
  S \mathbb{F} := (\mathcal{F}_t)_{t \geq 0} \quad \mathbb{P}
  \]
  
  • Information, mathematically expressed by a filtration
    \[
    \mathbb{F} = (\mathcal{F}_t)_{t \geq 0} \quad \mathcal{F}_s \subset \mathcal{F}_t \quad s \leq t
    \]

  • Asymmetry of information: additional information via filtration enlargement \(\mathbb{F} \subset \mathbb{G}\), i.e., \(\mathcal{F}_t \subset \mathcal{G}_t\) for all \(t \geq 0\)
  
  • Default time modelling

  • Questions on arbitrages, utility maximisation, value of information, market completeness, hedging strategies,...


Aksamit, A. & Fontana, C. Marked thin random times, Working paper
Martingale hypotheses

In *Questions* on arbitrages, utility maximisation, value of information, market completeness, hedging strategies, etc it is crucial to know how martingales behave under filtration enlargement.
Martingale hypotheses

In Questions on arbitrages, utility maximisation, value of information, market completeness, hedging strategies, etc it is crucial to know how martingales behave under filtration enlargement.

Set a probability space \((\Omega, \mathcal{G}, \mathbb{P})\). Consider two filtrations \(\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}\) and \(\mathcal{G} = (\mathcal{G}_t)_{t \geq 0}\) such that for every \(t \geq 0\), \(\mathcal{F}_t \subset \mathcal{G}_t\), i.e., \(\mathcal{F} \subset \mathcal{G}\).

Then we talk about two hypotheses (Brémaud, Yor, Jeulin):

- \((\mathcal{H})\) hypothesis: each \(\mathcal{F}\)-martingale remains a \(\mathcal{G}\)-martingale;
- \((\mathcal{H}')\) hypothesis: each \(\mathcal{F}\)-martingale remains a \(\mathcal{G}\)-semimartingale.
Martingale hypotheses

In Questions on arbitrages, utility maximisation, value of information, market completeness, hedging strategies, etc it is crucial to know how martingales behave under filtration enlargement.

Set a probability space $(\Omega, \mathcal{G}, \mathbb{P})$. Consider two filtrations $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ and $\mathcal{G} = (\mathcal{G}_t)_{t \geq 0}$ such that for every $t \geq 0$, $\mathcal{F}_t \subset \mathcal{G}_t$, i.e., $\mathcal{F} \subset \mathcal{G}$.

Then we talk about two hypotheses (Brémaud, Yor, Jeulin):

$(\mathcal{H})$ hypothesis: each $\mathcal{F}$-martingale remains a $\mathcal{G}$-martingale;

$(\mathcal{H}')$ hypothesis: each $\mathcal{F}$-martingale remains a $\mathcal{G}$-semimartingale.

Aim: in case when $(\mathcal{H}')$ hypothesis holds, we aim to derive $\mathcal{G}$-semimartingale decomposition of $\mathcal{F}$-martingales.
Lévy transformation example

• Let $B$ be a Brownian motion and $\mathbb{F}^B = (\mathcal{F}_t^B)_{t \geq 0}$ its natural filtration.

• Tanaka formula gives

$$|B_t| = W_t + L_t^0$$

with

$$W_t = \int_0^t \text{sgn}(B_s) dB_s$$

• Then, $\mathbb{F}^W = \mathbb{F}^{|B|} \subsetneq \mathbb{F}^B$ as $L_t^0 = \sup_{s \leq t} (-W_s)$ and the hypothesis $(\mathcal{H})$ is satisfied.
For a reference filtration $\mathbb{F}$ and a random variable $\xi$, the initially enlarged filtration $\mathcal{G}$ is the smallest filtration containing $\mathbb{F}$ such that $\xi$ is $\mathcal{G}_0$-measurable.

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\xi)$$
Initial enlargement under Jacod’s hypothesis

For a reference filtration $\mathcal{F}$ and a random variable $\xi$, the initially enlarged filtration $\mathcal{G}$ is the smallest filtration containing $\mathcal{F}$ such that $\xi$ is $\mathcal{G}_0$-measurable.

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\xi)$$

Under which conditions on $\xi$ and $\mathcal{F}$ the hypothesis ($\mathcal{H}'$) is satisfied?
Initial enlargement under Jacod’s hypothesis

For a reference filtration $\mathbb{F}$ and a random variable $\xi$, the initially enlarged filtration $\mathbb{G}$ is the smallest filtration containing $\mathbb{F}$ such that $\xi$ is $\mathbb{G}_0$-measurable.

$$\mathbb{G}_t = \mathbb{F}_t \vee \sigma(\xi)$$

Under which conditions on $\xi$ and $\mathbb{F}$ the hypothesis ($\mathcal{H}'$) is satisfied? A random variable $\xi$ with law $\eta$ satisfies Jacod’s hypothesis if

$$\mathbb{P}(\xi \in du | \mathbb{F}_t)(\omega) \ll \eta(du) \quad \mathbb{P}\text{-a.s. for every } t \geq 0$$
Initial enlargement under Jacod’s hypothesis

The initially enlarged filtration $\mathcal{G}$ is given by $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\xi)$. If $\xi$ satisfies Jacod’s condition, there exists a family of non-negative $\mathcal{F}$-martingales $(p(u), u \in \mathbb{R})$ such that

$$P(\xi \in du | \mathcal{F}_t) = p_t(u)\eta(du).$$

Moreover any $\mathcal{F}$-semimartingale is a $\mathcal{G}$-semimartingale, and for an $\mathcal{F}$-local martingale $X$ we have

$$X_t = \hat{X}_t + \int_0^t \frac{1}{p_s(u)} d\langle X, p(u) \rangle^F_{s \mid u = \xi},$$

where $\hat{X}$ is a $\mathcal{G}$-local martingale.
Initial enlargement under Jacod’s hypothesis

The initially enlarged filtration $\mathcal{G}$ is given by $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\xi)$.

If $\xi$ satisfies Jacod’s condition, there exists a family of non-negative $\mathcal{F}$-martingales $(p(u), u \in \mathbb{R})$ such that

$$
P(\xi \in du | \mathcal{F}_t) = p_t(u)\eta(du).
$$

Moreover any $\mathcal{F}$-semimartingale is a $\mathcal{G}$-semimartingale, and for an $\mathcal{F}$-local martingale $X$ we have

$$
X_t = \hat{X}_t + \int_0^t \frac{1}{p_{s-}(u)}d\langle X, p(u) \rangle^F_s|_{u=\xi},
$$

where $\hat{X}$ is a $\mathcal{G}$-local martingale.

Examples: $B_1 + \varepsilon$ where $\varepsilon$ is a noise, discrete random variable
Progressive enlargement of filtration

Enlarged filtration $\mathcal{G}$ is obtained from a reference filtration $\mathcal{F}$ and a random time $\tau$ by defining

$$\mathcal{F}_t^\tau = \mathcal{F}_t \vee \sigma(\{\tau \leq s\} : s \leq t).$$
Progressive enlargement of filtration

Enlarged filtration $\mathcal{G}$ is obtained from a reference filtration $\mathcal{F}$ and a random time $\tau$ by defining

$$\mathcal{F}_t^\tau = \mathcal{F}_t \vee \sigma(\{\tau \leq s\} : s \leq t).$$

$\mathcal{F}$-information on the random time $\tau$:

- Azéma supermartingales $Z_t$ defined as $Z_t = \mathbb{P}(\tau \geq t | \mathcal{F}_t)$
- $\mathcal{F}$-dual optional projection of the process $A = 1_{[\tau, \infty]}$, $A^\circ$, i.e.

for each optional process $H$, $A^\circ$ satisfies

$$\mathbb{E}[H_\tau 1_{\{\tau < \infty\}}] = \mathbb{E}\left[\int_{[0, \infty[} H_s dA^\circ_s\right]$$

Denote by $m$ an $\mathcal{F}$-martingale defined as $m_t = \mathbb{E}[A^\circ_\infty | \mathcal{F}_t]$. Then

- $Z_+ = m - A^\circ$ and $Z = m - A^\circ_-$
- $\Delta m = Z - Z_-$ and $\Delta A^\circ = Z - Z_+$
What may happen after a random time?

- $(\mathcal{H}')$ hypothesis is always satisfied for $\mathbb{F}$-martingales stopped at $\tau$
What may happen after a random time?

- \((\mathcal{H}')\) hypothesis is always satisfied for \(\mathbb{F}\)-martingales stopped at \(\tau\) and in some cases also after \(\tau\), e.g. for honest times.
- Random time \(\tau\) is an \(\mathbb{F}\)-honest time if for each \(t\) there exists \(\mathcal{F}_t\)-measurable random variable \(\tau_t\) such that \(\tau = \tau_t\) on \(\{\tau \leq t\}\)
- Equivalently, honest times are last passage times (or end of optional sets), i.e., \(\sup\{t : X_t = a\}\) for an optional process \(X\)
- Each \(\mathbb{F}\)-local martingale \(X\) is a \(\mathbb{F}^\tau\)-semimartingale with

\[
X_t = \hat{X}_t + \int_0^{t\wedge\tau} \frac{1}{Z_s} d\langle X, m \rangle_s - \int_\tau^t \frac{1}{1 - Z_s} d\langle X, m \rangle_s
\]

where \(\hat{X}\) is an \(\mathbb{F}^\tau\)-local martingale.
What may happen after a random time?

• In general it is not true that \((H')\) hypothesis is satisfied
• Let \(\tau := \sup\{t \leq 1 : B_1 - 2B_t = 0\}\) and

\[
\begin{align*}
\mathbb{F}^{\sigma(B_1)} : \quad & \mathcal{F}^\sigma_t(B_1) = \mathcal{F}_s \vee \sigma(B_1) \\
\mathbb{F}^{\sigma(B_1), \tau} : \quad & \mathcal{F}^\tau_t = \mathcal{F}_t \vee \sigma(\tau \wedge t) \\
\end{align*}
\]
What may happen after a random time?

• In general it is not true that \((\mathcal{H}')\) hypothesis is satisfied

• Let \(\tau := \sup\{ t \leq 1 : B_1 - 2B_t = 0 \}\) and

\[
\mathbb{F}^{\sigma(B_1)} : \quad \mathcal{F}^{\sigma(B_1)}_t = \mathcal{F}_s \vee \sigma(B_1) \quad \mathbb{F}^{\tau} : \quad \mathcal{F}_{\tau}^t = \mathcal{F}_t \vee \sigma(\tau \wedge t)
\]

\[
\mathbb{F}^{\sigma(B_1),\tau} = \mathbb{F}^{\sigma(B_1)} \vee \mathbb{F}^{\tau}
\]

• \(\tau\) is not an \(\mathbb{F}\)-honest time, \(\tau\) is not an \(\mathbb{F}^{\sigma(B_1)}\)-stopping time. But it is an \(\mathbb{F}^{\sigma(B_1)}\)-honest time.
What may happen after a random time?

- In general it is not true that \((\mathcal{H}')\) hypothesis is satisfied
- Let \(\tau := \sup\{ t \leq 1 : B_1 - 2B_t = 0 \}\) and

\[
\mathbb{F}^{\sigma(B_1)} : \quad \mathcal{F}_t^{\sigma(B_1)} = \mathcal{F}_s \lor \sigma(B_1) \quad \mathbb{F}^\tau : \quad \mathcal{F}_t^\tau = \mathcal{F}_t \lor \sigma(\tau \land t)
\]

\[
\mathbb{F}^{\sigma(B_1),\tau} = \mathbb{F}^{\sigma(B_1)} \lor \mathbb{F}^\tau
\]

- \(\tau\) is not an \(\mathbb{F}\)-honest time, \(\tau\) is not an \(\mathbb{F}^{\sigma(B_1)}\)-stopping time. But it is an \(\mathbb{F}^{\sigma(B_1)}\)-honest time.
- After \(\tau\), \(\mathbb{F}^{\sigma(B_1)} \subset \mathbb{F}^\tau\) and \(\mathbb{F}^{\sigma(B_1),\tau} = \mathbb{F}^\tau\)

\[
B_t = \hat{B}_t + \int_0^{t \land \tau} \frac{1}{Z_s} d\langle m, B \rangle_s + \int_{\tau}^{t \land 1} \frac{B_1 - B_s}{1 - s} ds - \int_{\tau}^{t \land 1} \frac{1}{1 - Y_s} d\langle m^Y, B \rangle_s
\]
Avoidance of $\mathcal{F}$-stopping times

Often in the literature, the following assumption on a random time is used:

**(A) condition:** $\tau$ avoids $\mathcal{F}$ stopping times, i.e. $P(\tau = T < \infty) = 0$, for any $\mathcal{F}$-stopping time $T$. 

Thick and thin random times

Theorem

For any random time $\tau$ there exists a pair of random times $(\tau_1, \tau_2)$ such that

(a) $\tau_1$ is a thick random time, i.e., $\mathbb{P}(\tau = T < \infty) = 0$ for any $\mathbb{F}$-stopping time $T$;
Thick and thin random times

Theorem

For any random time $\tau$ there exists a pair of random times $(\tau_1, \tau_2)$ such that

(a) $\tau_1$ is a thick random time, i.e., $\mathbb{P}(\tau = T < \infty) = 0$ for any $\mathcal{F}$-stopping time $T$;

(b) $\tau_2$ is a thin random time, i.e., there exists a sequence of $\mathcal{F}$-stopping times $(T_n)_{n=1}^{\infty}$ with disjoint graphs such that $\mathbb{P}\left(\bigcup_n \{\tau = T_n\}\right) = 1$; the sequence $(T_n)_{n}$ is then called an exhausting sequence of a thin time;
Thick and thin random times

**Theorem**

For any random time \( \tau \) there exists a pair of random times \((\tau_1, \tau_2)\) such that

(a) \( \tau_1 \) is a **thick random time**, i.e., \( \mathbb{P}(\tau = T < \infty) = 0 \) for any \( \mathcal{F} \)-stopping time \( T \);

(b) \( \tau_2 \) is a **thin random time**, i.e., there exists a sequence of \( \mathcal{F} \)-stopping times \((T_n)_{n=1}^{\infty}\) with disjoint graphs such that \( \mathbb{P}(\bigcup_n \{\tau = T_n\}) = 1 \); the sequence \((T_n)_{n}\) is then called an **exhausting sequence of a thin time**;

(c) and \( \tau = \tau_1 \wedge \tau_2 \quad \tau_1 \vee \tau_2 = \infty \).

Moreover such a pair is unique.
Decomposition of a random time

Let

\[ \tau_1 = \tau\{\Delta A^\varphi = 0\} \quad \text{and} \quad \tau_2 := \tau\{\Delta A^\varphi > 0\} \]

We see that the time \( \tau_1 \) is a thick time as

\[
\mathbb{P}(\tau_1 = T < \infty) = \mathbb{E}\left[ \int_0^\infty \mathbb{1}_{\{u = T\}} \mathbb{1}_{\{\Delta A^\varphi_u = 0\}} dA^\varphi_u \right] = 0.
\]

and the time \( \tau_2 \) is a thin time as

\[
[\tau_2] = [\tau] \cap \{\Delta A^\varphi > 0\} = [\tau] \cap \bigcup_n [T_n] \subset \bigcup_n [T_n]
\]

- The random time \( \tau \) is a thin time if and only if its dual optional projection \( A^\varphi \) is a pure jump process.
- The random time \( \tau \) is a thick time if and only if its dual optional projection \( A^\varphi \) is a continuous process.
Non quasi-left continuous filtrations

- Since $\tau_2$ is an $\mathbb{F}^\tau$-stopping time, it can be decomposed into $\mathbb{F}^\tau$-accessible and $\mathbb{F}^\tau$-totally inaccessible parts. Thus, we can consider the decomposition of $\tau$ into three parts as:

\[
\tau_2^i = \tau\{\Delta A^0_\tau > 0, \Delta A^p_\tau = 0\} \quad \tau_2^a = \tau\{\Delta A^o_\tau > 0, \Delta A^p_\tau > 0\} \quad \tau_1 = \tau\{\Delta A^o_\tau = 0\}.
\]

Since $\tau_1$ is $\mathbb{F}^\tau$-totally inaccessible, it follows that $\tau_1 \wedge \tau_2^i$ is the $\mathbb{F}^\tau$-totally inaccessible part and $\tau_2^a$ is the $\mathbb{F}^\tau$-accessible part of the $\mathbb{F}^\tau$-stopping time $\tau$.

- Assume that the $\mathbb{F}^\tau$-accessible stopping time $\tau_2^a$ is not equal to an $\mathbb{F}$-stopping time on $\{\tau > 0\}$. Then, the filtration $\mathbb{F}^\tau$ is not quasi-left continuous. This provides a systemic way to construct examples of non quasi-left continuous filtrations.
Brownian filtration example: local time approximation

• Let $B$ be a Brownian motion. For $\varepsilon > 0$, define a double sequence of stopping times by: $V_0 = 0$ and

\[
U_n = \inf\{t \geq V_{n-1} : B_t = \varepsilon\}, \quad V_n = \inf\{t \geq U_n : B_t = 0\},
\]

and the process $D_t = \max\{n : V_n \leq t\}$ which is the number of downcrossings of $B$ from level $\varepsilon$ to level 0 before time $t$. 
Brownian filtration example: local time approximation

• Let $B$ be a Brownian motion. For $\varepsilon > 0$, define a double sequence of stopping times by: $V_0 = 0$ and

$$U_n = \inf\{t \geq V_{n-1} : B_t = \varepsilon\}, \quad V_n = \inf\{t \geq U_n : B_t = 0\},$$

and the process $D_t = \max\{n : V_n \leq t\}$ which is the number of downcrossings of $B$ from level $\varepsilon$ to level 0 before time $t$.

• Define the random time

$$\tau^\varepsilon = \sup\{V_n : V_n \leq T_1\}$$

with $T_1 = \inf\{t : B_t = 1\}$. 
Brownian filtration example: local time approximation

- Let $B$ be a Brownian motion. For $\varepsilon > 0$, define a double sequence of stopping times by: $V_0 = 0$ and
  \[
  U_n = \inf\{t \geq V_{n-1}: B_t = \varepsilon\}, \quad V_n = \inf\{t \geq U_n: B_t = 0\},
  \]
  and the process $D_t = \max\{n: V_n \leq t\}$ which is the number of downcrossings of $B$ from level $\varepsilon$ to level 0 before time $t$.

- Define the random time
  \[
  \tau^\varepsilon = \sup\{V_n: V_n \leq T_1\}
  \]
  with $T_1 = \inf\{t: B_t = 1\}$.

- $A_t^\circ = \varepsilon D_{t \wedge T_1} + \varepsilon$ and $\{\Delta A^\circ > 0\} = [0, T_1] \cap \bigcup_{n=0}^{\infty}[V_n]$. 

Thin times

- In Poisson filtration, or more generally in jumping filtration, every honest time is thin:
  - A filtration $\mathbb{F}$ is called a jumping filtration if there exists a localizing sequence $(S_n)_n$ with $S_0 = 0$ and such that for all $n$ and $t > 0$ the $\sigma$-fields $\mathcal{F}_t$ and $\mathcal{F}_{S_n}$ coincide up to null sets on $\{S_n \leq t < S_{n+1}\}$. 


Thin times

• In Poisson filtration, or more generally in jumping filtration, every honest time time is thin:
  • A filtration $\mathbb{F}$ is called a **jumping filtration** if there exists a localizing sequence $(S_n)_n$ with $S_0 = 0$ and such that for all $n$ and $t > 0$ the $\sigma$-fields $\mathcal{F}_t$ and $\mathcal{F}_{S_n}$ coincide up to null sets on $\{S_n \leq t < S_{n+1}\}$.

• Thin honest times were not studied in the previous arbitrage papers (eg Jeanblanc, Fontana and Song (2014)), and they are qualitatively different.
Thin times

• In Poisson filtration, or more generally in jumping filtration, every honest time is thin:
  • A filtration $\mathbb{F}$ is called a **jumping filtration** if there exists a localizing sequence $(S_n)_n$ with $S_0 = 0$ and such that for all $n$ and $t > 0$ the $\sigma$-fields $\mathcal{F}_t$ and $\mathcal{F}_{S_n}$ coincide up to null sets on $\{S_n \leq t < S_{n+1}\}$.

• Thin honest times were not studied in the previous arbitrage papers (eg Jeanblanc, Fontana and Song (2014)), and they are qualitatively different.
Thin times

- The family of bounded $\mathbb{F}$-martingales $(z^n)_{n \geq 0}$ given by

$$z^n_t = \mathbb{P}(C_n|\mathcal{F}_t) = \mathbb{P}(\tau = T_n < \infty | \mathcal{F}_t)$$

is a martingale family of the thin time $\tau$

- $z^n_t > 0$ and $z^n_\tau > 0$ for all $t \geq 0$ a.s. on $C_n$ for each $n \geq 0$.

- The progressive enlargement $\mathcal{G}$ of filtration $\mathbb{F}$ with $\tau$ satisfies

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(C_n \cap \{T_n \leq s\}, s \leq t, n \geq 1).$$

- For any $n \geq 1$ and any $\mathcal{G}$-measurable integrable random variable $X$, we have

$$\mathbb{E} [X|\mathcal{F}_t^\tau] \mathbb{1}_{\{t \geq T_n\} \cap C_n} = \mathbb{1}_{\{t \geq T_n\} \cap C_n} \frac{\mathbb{E} [X \mathbb{1}_{C_n}|\mathcal{F}_t]}{z^n_t}.$$
Theorem

Let $\tau$ be a thin time. Then the hypothesis $(\mathcal{H}')$ is satisfied for $(F, G)$. 
(\mathcal{H}') hypothesis

**Theorem**

Let $\tau$ be a thin time. Then the hypothesis $(\mathcal{H}')$ is satisfied for $(F, G)$.

**Theorem**

Let $\tau$ be a random time and $(\tau_1, \tau_2)$ its thin-thick decomposition. Then, $F^\tau = F^{\tau_1, \tau_2}$. Furthermore, the hypothesis $(\mathcal{H}')$ is satisfied for $(F, F^\tau)$ if and only if the hypothesis $(\mathcal{H}')$ is satisfied for $(F, F^{\tau_2})$. 
### Proposition

Let $\tau$ be a thin time with exhausting sequence $(T_n)_{n \geq 0}$, partition $(C_n)_{n \geq 0}$ and martingale family $(z^n)_{n \geq 0}$. Then, $\mathcal{F}$ is immersed in $\mathcal{F}^\tau$ if and only if one of the following conditions hold:

(a) $z^n_\infty = z^n_{T_n}$ for each $n \geq 1$,

(b) $z^n_t = z^n_{T_n \wedge t}$ for each $t \geq 0$ for each $n \geq 1$,

(c) for each $n \geq 1$, $C_n$ is independent of $\mathcal{F}_\infty$ conditionally w.r.t. $\mathcal{F}_{T_n}$. 
Immersion

Proposition

Let $\tau$ be a thin time with exhausting sequence $(T_n)_{n \geq 0}$, partition $(C_n)_{n \geq 0}$ and martingale family $(z^n)_{n \geq 0}$. Then, $F$ is immersed in $F^\tau$ if and only if one of the following conditions hold:

(a) $z^n_\infty = z^n_{T_n}$ for each $n \geq 1$,
(b) $z^n_t = z^n_{T_n \wedge t}$ for each $t \geq 0$ for each $n \geq 1$,
(c) for each $n \geq 1$, $C_n$ is independent of $F_\infty$ conditionally w.r.t. $F_{T_n}$.

Proposition

$F$ is immersed in $F^\tau$ if and only if $F$ is immersed in $F^{\tau_1}$ and in $F^{\tau_2}$. In that case, $F^{\tau_1}$ and $F^{\tau_2}$ are immersed in $F^\tau$.
Progressive enlargement with \((\tau, \xi)\)

\(G\) is obtained from a reference filtration \(F\), a random time \(\tau\) and a mark \(\xi\) by defining

\[
G_t = F_t \vee \sigma(\xi 1_{\{\tau \leq s\}} : s \leq t).
\]

So \(G\) is the smallest filtration containing \(F\) and making \(\tau\) a stopping time and \(\xi\) a \(G_\tau\)-measurable r.v.
Progressive enlargement with \((\tau, \xi)\)

\(\mathcal{G}\) is obtained from a reference filtration \(\mathcal{F}\), a random time \(\tau\) and a mark \(\xi\) by defining

\[
\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\xi 1_{\{\tau \leq s\}} : s \leq t).
\]

So \(\mathcal{G}\) is the smallest filtration containing \(\mathcal{F}\) and making \(\tau\) a stopping time and \(\xi\) a \(\mathcal{G}_\tau\)-measurable r.v.

How can we use here the previous Jacod’s result?
Progressive enlargement with \((\tau, \xi)\)

\(\mathcal{G}\) is obtained from a reference filtration \(\mathcal{F}\), a random time \(\tau\) and a mark \(\xi\) by defining

\[
\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\xi \mathbb{1}_{\{\tau \leq s\}} : s \leq t).
\]

So \(\mathcal{G}\) is the smallest filtration containing \(\mathcal{F}\) and making \(\tau\) a stopping time and \(\xi\) a \(\mathcal{G}_\tau\)-measurable r.v.

How can we use here the previous Jacod’s result?

\(\tau\) satisfies Jacod’s cond., \(\xi = 1\), initial times Le Cam, Jeanblanc09

\((\tau, \xi)\) satisfy Jacod’s condition jointly, Kchia, Larsson, Protter 13

\(\tau\) is a stopping time, \(\xi\) satisfies Jacod’s cond., Jiao, Kharroubi 18
Marked thin random time

Let \((\tau, \xi)\) be such that

- \(\tau\) is thin random time with exhausting sequence \((T_n)_n\)
- \(\xi\) is a random variable satisfying generalised Jacod’s condition
Marked thin random time

Let \((\tau, \xi)\) be such that

- \(\tau\) is thin random time with exhausting sequence \((T_n)_n\)
- \(\xi\) is a random variable satisfying generalised Jacod’s condition

**Generalised Jacod’s condition:** For each \(n\), let

\[
\xi^n = \begin{cases} 
\xi & \text{on } C_n \\
\Delta & \text{on } C_n^c
\end{cases}
\]

and assume that

\[
P(\xi^n \in du|\mathcal{F}_{T_n \vee t}) = p^n_t(u)P(\xi^n \in du|\mathcal{F}_{T_n}) \quad t \geq 0 \quad \text{on} \quad C_n.
\]
Marked thin random time

Under generalised Jacod’s condition, ($\mathcal{H}'$) hypothesis is satisfied and an $\mathbb{F}$-martingale $X$ decomposes as

$$X_t = \hat{X}_t + \int_0^{t \wedge \tau} \frac{1}{Z_s} d\langle X, m \rangle_s + \sum_n \mathbb{1}_{C_n} \int_{T_n}^t \frac{1}{p^n_s(u)} d\langle X, p^n(u) \rangle_{s \mid u = \xi}$$
Marked thin random time

Under generalised Jacod’s condition, \((\mathcal{H}')\) hypothesis is satisfied and an \(\mathbb{F}\)-martingale \(X\) decomposes as

\[
X_t = \hat{X}_t + \int_0^{t \wedge \tau} \frac{1}{Z_s} d\langle X, m \rangle_s + \sum_n \mathbb{1}_{C_n} \int_{T_n}^t \frac{1}{p^n_s(u)} d\langle X, p^n(u) \rangle_{s \mid u = \xi}
\]

Divide a problem into two steps \(\mathbb{F} \subset \mathbb{F}^\tau \subset \mathbb{G}\) and note that:

- \((\mathcal{H}')\) hypothesis holds for \(\mathbb{F} \subset \mathbb{F}^\tau\) since \(\tau\) is thin
- assume that \(\xi\) satisfies \(\mathbb{P}(\xi \in du \mid \mathcal{F}^\tau_{\tau \vee t}) = q_t(u) \mathbb{P}(\xi \in du \mid \mathcal{F}^\tau_{\tau})\)
  which ensures \((\mathcal{H}')\) hypothesis for \(\mathbb{F}^\tau \subset \mathbb{G}\)
Marked thin random time

Under generalised Jacod’s condition, \((\mathcal{H}')\) hypothesis is satisfied and an \(\mathbb{F}\)-martingale \(X\) decomposes as

\[
X_t = \hat{X}_t + \int_0^{t\wedge\tau} \frac{1}{Z_s} d\langle X, m \rangle_s + \sum_n \mathbb{1}_{C_n} \int_{T_n}^t \frac{1}{p^n_s(u)} d\langle X, p^n(u) \rangle_{s|u=\xi}
\]

Divide a problem into two steps \(\mathbb{F} \subset \mathbb{F}^\tau \subset \mathbb{G}\) and note that:

- \((\mathcal{H}')\) hypothesis holds for \(\mathbb{F} \subset \mathbb{F}^\tau\) since \(\tau\) is thin
- assume that \(\xi\) satisfies \(\mathbb{P}(\xi \in du|\mathcal{F}^\tau_{\tau\vee t}) = q_t(u)\mathbb{P}(\xi \in du|\mathcal{F}^\tau_{\tau})\)
  which ensures \((\mathcal{H}')\) hypothesis for \(\mathbb{F}^\tau \subset \mathbb{G}\)

Above condition is exactly generalised Jacod’s condition and

\[
p^n_t(u) = \frac{Z^n_{T_n\vee t}}{Z^n_{T_n}} q_t(u) \quad \text{on } C_n.
\]

Example: \(\xi = B_1 + \mathbb{1}_{\{B_2 > 0\}}\) and \(\tau = 1\)
PRP for marked thin time

Representation can be studied separately

- on $[0, \tau]$, i.e., for $\mathbb{1}_{[0,\tau]} \cdot M$ where $M$ is an $\mathbb{F}$-martingale. We then use results from:
  
  

- and on $[\tau, \infty]$, i.e. for $\mathbb{1}_{[\tau,\infty]} \cdot M$ where $M$ is an $\mathbb{F}$-martingale. We then use results from:
  
On \( \tau, \infty \] \], we divide a problem into two steps

\[
F \subset F^C = F \lor \sigma\{C_n : n\} \subset G
\]

and note that:

- \( F \subset F^C \) is initial enlargement with discrete random variable, hence Fontana results apply
- \( G \) can be viewed as initial enlargement at a \( F^\tau \)-stopping time \( \tau \), hence once again Fontana’s result apply
- Finally we wrap up two parts using key lemma for thin times (similarly as for semmartingale decomposition)


Aksamit, A. & Fontana, C. Marked thin random times, Working paper


