Additional information disclosed progressively in time

Enlargement of filtrations and martingale representation

Anna Aksamit

The University of Sydney

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- Given a measurable space $(\Omega, \mathcal{F})$ a financial model consists of

$$\begin{array}{ccc} \text{prices} & \text{information} & \text{probability} \\ S & \mathbb{F} := (\mathcal{F}_t)_{t \geq 0} & \mathbb{P} \end{array}$$

- Information, mathematically expressed by a filtration

$$\mathbb{F} = (\mathcal{F}_t)_{t \geq 0} \quad \mathcal{F}_s \subset \mathcal{F}_t \quad s \leq t$$

- Asymmetry of information: additional information via filtration enlargement $\mathbb{F} \subset \mathbb{G}$, i.e., $\mathcal{F}_t \subset \mathcal{G}_t$ for all $t \geq 0$
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Aksamit, A., Choulli, T., & Jeanblanc, M. Thin times and random times' decomposition. (2021) Electronic Journal of Probability 26, 1-22.

Aksamit, A. & Fontana, C. Marked thin random times, Working paper

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Set a probability space $(\Omega, \mathcal{G}, \mathbb{P})$. Consider two filtrations $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ and $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ such that for every $t \geq 0$, $\mathcal{F}_t \subset \mathcal{G}_t$, i.e., $\mathbb{F} \subset \mathbb{G}$.

Then we talk about two hypotheses (Brémaud, Yor, Jeulin):

($\mathcal{H}$) hypothesis: each $\mathbb{F}$-martingale remains a $\mathbb{G}$-martingale;

($\mathcal{H}'$) hypothesis: each $\mathbb{F}$-martingale remains a $\mathbb{G}$-semimartingale.

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$(\mathcal{H}')$ hypothesis: each $\mathbb{F}$-martingale remains a $\mathbb{G}$-semimartingale.

Aim: in case when $(\mathcal{H}')$ hypothesis holds, we aim to derive $\mathbb{G}$-semimartingale decomposition of $\mathbb{F}$-martingales.

Lévy transformation example

- Let B be a Brownian motion and $\mathbb{F}^B = (\mathcal{F}_t^B)_{t\geq 0}$ its natural filtration.
- Tanaka formula gives

$$|B_t| = W_t + L_t^0 \quad \text{with} \quad W_t = \int_0^t \operatorname{sgn}(B_s) dB_s$$

- Then, $\mathbb{F}^W = \mathbb{F}^{|B|} \subsetneq \mathbb{F}^B$ as $L_t^0 = \sup_{s\leq t}(-W_s)$ and the hypothesis $(\mathcal{H})$ is satisfied.

Initial enlargement under Jacod's hypothesis

For a reference filtration $\mathbb{F}$ and a random variable ξ, the initially enlarged filtration $\mathbb{G}$ is the smallest filtration containing $\mathbb{F}$ such that ξ is $\mathcal{G}_0$-measurable.

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\xi)$$

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Under which conditions on ξ and $\mathbb{F}$ the hypothesis $(\mathcal{H}')$ is satisfied?
A random variable ξ with law η satisfies Jacod's hypothesis if

$$\mathbb{P}(\xi \in du | \mathcal{F}_t)(\omega) \ll \eta(du) \quad \mathbb{P}\text{-a.s.} \quad \text{for every } t \geq 0$$

Initial enlargement under Jacod's hypothesis

The initially enlarged filtration $\mathbb{G}$ is given by $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\xi)$.

If ξ satisfies Jacod's condition, there exists a family of non-negative $\mathbb{F}$-martingales $(p(u), u \in \mathbb{R})$ such that

$$\mathbb{P}(\xi \in du | \mathcal{F}_t) = p_t(u)\eta(du).$$

Moreover any $\mathbb{F}$-semimartingale is a $\mathbb{G}$-semimartingale, and for an $\mathbb{F}$-local martingale X we have

$$X_t = \widehat{X}_t + \int_0^t \frac{1}{p_{s-}(u)} d\langle X, p(u)\rangle^{\mathbb{F}}_{s\,|u=\xi},$$

where $\widehat{X}$ is a $\mathbb{G}$-local martingale.

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Examples: $B_1 + \varepsilon$ where ϵ is a noise, discrete random variable

Progressive enlargement of filtration

Enlarged filtration $\mathbb{G}$ is obtained from a reference filtration $\mathbb{F}$ and a random time τ by defining

$$\mathcal{F}_t^\tau = \mathcal{F}_t \vee \sigma(\{\tau \leq s\} : s \leq t).$$

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$\mathbb{F}$-information on the random time τ:

- Azéma supermartingales Z_t defined as $Z_t = \mathbb{P}(\tau \geq t | \mathcal{F}_t)$
- $\mathbb{F}$-dual optional projection of the process $A = \mathbb{1}_{[\![\tau,\infty[\![}$, A^o, i.e.

for each optional process H, A^o satisfies

$$\mathbb{E}[H_\tau \mathbb{1}_{\{\tau<\infty\}}] = \mathbb{E}[\int_{[0,\infty[} H_s dA_s^o]$$

Denote by m an $\mathbb{F}$-martingale defined as $m_t = \mathbb{E}[A_\infty^o | \mathcal{F}_t]$. Then

- $Z_+ = m - A^o$ and $Z = m - A_-^o$
- $\Delta m = Z - Z_-$ and $\Delta A^o = Z - Z_+$

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- $(\mathcal{H}')$ hypothesis is always satisfied for $\mathbb{F}$-martingales stopped at τ and in some cases also after τ, e.g. for honest times.
- Random time τ is an $\mathbb{F}$-honest time if for each t there exists $\mathcal{F}_t$-measurable random variable τ_t such that $\tau = \tau_t$ on $\{\tau \leq t\}$
- Equivalently, honest times are last passage times (or end of optional sets), i.e., $\sup\{t : X_t = a\}$ for an optional process X
- Each $\mathbb{F}$-local martingale X is a $\mathbb{F}^\tau$-semimartingale with

$$X_t = \widehat{X}_t + \int_0^{t\wedge\tau} \frac{1}{Z_{s-}} d\langle X, m\rangle_s - \int_\tau^t \frac{1}{1 - Z_{s-}} d\langle X, m\rangle_s$$

where $\widehat{X}$ is an $\mathbb{F}^\tau$-local martingale.

What may happen after a random time?

- In general it is not true that $(\mathcal{H}')$ hypothesis is satisfied
- Let $\tau := \sup\{t \leq 1 : B_1 - 2B_t = 0\}$ and

$$\mathbb{F}^{\sigma(B_1)} : \quad \mathcal{F}_t^{\sigma(B_1)} = \mathcal{F}_s \vee \sigma(B_1) \qquad \mathbb{F}^{\tau} : \quad \mathcal{F}_t^{\tau} = \mathcal{F}_t \vee \sigma(\tau \wedge t)$$

$$\mathbb{F}^{\sigma(B_1),\tau} = \mathbb{F}^{\sigma(B_1)} \vee \mathbb{F}^{\tau}$$

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- τ is not an $\mathbb{F}$-honest time, τ is not an $\mathbb{F}^{\sigma(B_1)}$-stopping time. But it is an $\mathbb{F}^{\sigma(B_1)}$-honest time.
- After τ, $\mathbb{F}^{\sigma(B_1)} \subset \mathbb{F}^{\tau}$ and $\mathbb{F}^{\sigma(B_1),\tau} = \mathbb{F}^{\tau}$

$$B_t = \widehat{B}_t + \int_0^{t\wedge\tau} \frac{1}{Z_s} d\langle m, B\rangle_s + \int_{\tau}^{t\wedge 1} \frac{B_1 - B_s}{1-s} ds - \int_{\tau}^{t\wedge 1} \frac{1}{1-Y_s} d\langle m^Y, B\rangle_s$$

Avoidance of $\mathbb{F}$-stopping times

Often in the literature, the following assumption on a random time is used:

(A) condition: τ avoids $\mathbb{F}$ stopping times, i.e. $\mathbb{P}(\tau = T < \infty) = 0$, for any $\mathbb{F}$-stopping time T.

Thick and thin random times

Theorem

For any random time τ there exists a pair of random times (τ_1, τ_2) such that

(a) τ_1 is a thick random time, i.e., $\mathbb{P}(\tau = T < \infty) = 0$ for any $\mathbb{F}$-stopping time T;

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(a) τ_1 is a thick random time, i.e., $\mathbb{P}(\tau = T < \infty) = 0$ for any $\mathbb{F}$-stopping time T;
(b) τ_2 is a thin random time, i.e., there exists a sequence of $\mathbb{F}$-stopping times $(T_n)_{n=1}^{\infty}$ with disjoint graphs such that $\mathbb{P}\left(\bigcup_n \{\tau = T_n\}\right) = 1$; the sequence $(T_n)_n$ is then called an exhausting sequence of a thin time;

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(c) and $\tau = \tau_1 \wedge \tau_2 \qquad \tau_1 \vee \tau_2 = \infty$.
Moreover such a pair is unique.

Decomposition of a random time

Let

$$\tau_1 = \tau_{\{\Delta A^o_\tau = 0\}} \quad \text{and} \quad \tau_2 := \tau_{\{\Delta A^o_\tau > 0\}}$$

We see that the time τ_1 is a thick time as

$$\mathbb{P}(\tau_1 = T < \infty) = \mathbb{E}\left[\int_0^\infty \mathbb{1}_{\{u=T\}} \mathbb{1}_{\{\Delta A^o_u = 0\}} dA^o_u\right] = 0.$$

and the time τ_2 is a thin time as

$$[\![\tau_2]\!] = [\![\tau]\!] \cap \{\Delta A^o > 0\} = [\![\tau]\!] \cap \bigcup_n [\![T_n]\!] \subset \bigcup_n [\![T_n]\!]$$

- The random time τ is a thin time if and only if its dual optional projection A^o is a pure jump process.
- The random time τ is a thick time if and only if its dual optional projection A^o is a continuous process.

Non quasi-left continuous filtrations

- Since τ_2 is an $\mathbb{F}^\tau$-stopping time, it can be decomposed into $\mathbb{F}^\tau$-accessible and $\mathbb{F}^\tau$-totally inaccessible parts. Thus, we can consider the decomposition of τ into three parts as:

  $$\tau_2^i = \tau_{\{\Delta A^o_\tau > 0,\ \Delta A^p_\tau = 0\}}\ \tau_2^a = \tau_{\{\Delta A^o_\tau > 0,\ \Delta A^p_\tau > 0\}}\ \tau_1 = \tau_{\{\Delta A^o_\tau = 0\}}.$$

  Since τ_1 is $\mathbb{F}^\tau$-totally inaccessible, it follows that $\tau_1 \wedge \tau_2^i$ is the $\mathbb{F}^\tau$-totally inaccessible part and τ_2^a is the $\mathbb{F}^\tau$-accessible part of the $\mathbb{F}^\tau$-stopping time τ
- Assume that the $\mathbb{F}^\tau$-accessible stopping time τ_2^a is not equal to an $\mathbb{F}$-stopping time on $\{\tau > 0\}$. Then, the filtration $\mathbb{F}^\tau$ is not quasi-left continuous. This provides a systemic way to construct examples of non quasi-left continuous filtrations.

Brownian filtration example: local time approximation

- Let B be a Brownian motion. For $\varepsilon > 0$, define a double sequence of stopping times by: $V_0 = 0$ and

$$U_n = \inf\{t \geq V_{n-1} : B_t = \varepsilon\}, \quad V_n = \inf\{t \geq U_n : B_t = 0\},$$

  and the process $D_t = \max\{n : V_n \leq t\}$ which is the number of downcrossings of B from level ε to level 0 before time t.

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- Define the random time

$$\tau^{\varepsilon} = \sup\{V_n : V_n \leq T_1\}$$

  with $T_1 = \inf\{t : B_t = 1\}$.

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  with $T_1 = \inf\{t : B_t = 1\}$.
- $A^o_t = \varepsilon D_{t \wedge T_1} + \varepsilon$ and $\{\Delta A^o > 0\} = [\![0, T_1]\!] \cap \bigcup_{n=0}^{\infty} [\![V_n]\!]$.

Thin times

- In Poisson filtration, or more generally in jumping filtration, every honest time time is thin:
  - A filtration $\mathbb{F}$ is called a jumping filtration if there exists a localizing sequence $(S_n)_n$ with $S_0 = 0$ and such that for all n and $t > 0$ the σ-fields $\mathcal{F}_t$ and $\mathcal{F}_{S_n}$ coincide up to null sets on $\{S_n \leq t < S_{n+1}\}$.

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Thin times

- The family of bounded $\mathbb{F}$-martingales $(z^n)_{n\geq 0}$ given by

$$z_t^n = \mathbb{P}(C_n|\mathcal{F}_t) = \mathbb{P}(\tau = T_n < \infty|\mathcal{F}_t)$$

  is a martingale family of the thin time τ
- $z_t^n > 0$ and $z_{t-}^n > 0$ for all $t \geq 0$ a.s. on C_n for each $n \geq 0$.
- The progressive enlargement $\mathbb{G}$ of filtration $\mathbb{F}$ with τ satisfies

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(C_n \cap \{T_n \leq s\},\ s \leq t,\ n \geq 1).$$

- For any $n \geq 1$ and any $\mathcal{G}$-measurable integrable random variable X, we have

$$\mathbb{E}\left[X|\mathcal{F}_t^\tau\right] \mathbb{1}_{\{t\geq T_n\}\cap C_n} = \mathbb{1}_{\{t\geq T_n\}\cap C_n} \frac{\mathbb{E}\left[X \mathbb{1}_{C_n}|\mathcal{F}_t\right]}{z_t^n}$$

$(\mathcal{H}')$ hypothesis

Theorem

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Theorem

Let τ be a random time and (τ_1, τ_2) its thin-thick decomposition. Then, $\mathbb{F}^{\tau} = \mathbb{F}^{\tau_1, \tau_2}$. Furthermore, the hypothesis $(\mathcal{H}')$ is satisfied for $(\mathbb{F}, \mathbb{F}^{\tau})$ if and only if the hypothesis $(\mathcal{H}')$ is satisfied for $(\mathbb{F}, \mathbb{F}^{\tau_2})$.

Immersion

Proposition

Let τ be a thin time with exhausting sequence $(T_n)_{n\geq 0}$, partition $(C_n)_{n\geq 0}$ and martingale family $(z^n)_{n\geq 0}$. Then, $\mathbb{F}$ is immersed in $\mathbb{F}^\tau$ if and only if one of the following conditions hold:

(a) $z^n_\infty = z^n_{T_n}$ *for each* $n \geq 1$,

(b) $z^n_t = z^n_{T_n \wedge t}$ *for each* $t \geq 0$ *for each* $n \geq 1$,

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Proposition

$\mathbb{F}$ is immersed in $\mathbb{F}^\tau$ if and only if $\mathbb{F}$ is immersed in $\mathbb{F}^{\tau_1}$ and in $\mathbb{F}^{\tau_2}$. In that case, $\mathbb{F}^{\tau_1}$ and $\mathbb{F}^{\tau_2}$ are immersed in $\mathbb{F}^\tau$.

Progressive enlargement with (τ, ξ)

$\mathbb{G}$ is obtained from a reference filtration $\mathbb{F}$, a random time τ and a mark ξ by defining

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\xi \mathbb{1}_{\{\tau \leq s\}} : s \leq t).$$

So $\mathbb{G}$ is the smallest filtration containing $\mathbb{F}$ and making τ a stopping time and ξ a $\mathcal{G}_\tau$-measurable r.v.

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How can we use here the previous Jacod's result?

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$\cdot\tau$ satisfies Jacod's cond., $\xi = 1$, initial times Le Cam, Jeanblanc09
$\cdot(\tau, \xi)$ satisfy Jacod's condition jointly, Kchia, Larsson, Protter 13
$\cdot\tau$ is a stopping time, ξ satisfies Jacod's cond., Jiao, Kharroubi 18

Marked thin random time

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Generalised Jacod's condition: For each n, let

$$\xi^n = \begin{cases} \xi & \text{on } C_n \\ \Delta & \text{on } C_n^c \end{cases}$$

and assume that

$$\mathbb{P}(\xi^n \in du | \mathcal{F}_{T_n \vee t}) = p_t^n(u) \mathbb{P}(\xi^n \in du | \mathcal{F}_{T_n}) \quad t \geq 0 \quad \text{on} \quad C_n.$$

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Under generalised Jacod's condition, $(\mathcal{H}')$ hypothesis is satisfied and an $\mathbb{F}$-martingale X decomposes as

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Divide a problem into two steps $\mathbb{F} \subset \mathbb{F}^\tau \subset \mathbb{G}$ and note that:

- $(\mathcal{H}')$ hypothesis holds for $\mathbb{F} \subset \mathbb{F}^\tau$ since τ is thin
- assume that ξ satisfies $\mathbb{P}(\xi \in du|\mathcal{F}^\tau_{\tau\vee t}) = q_t(u)\mathbb{P}(\xi \in du|\mathcal{F}^\tau_\tau)$ which ensures $(\mathcal{H}')$ hypothesis for $\mathbb{F}^\tau \subset \mathbb{G}$

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$$X_t = \widehat{X}_t + \int_0^{t\wedge\tau} \frac{1}{Z_s} d\langle X, m\rangle_s + \sum_n \mathbb{1}_{C_n} \int_{T_n}^t \frac{1}{p_s^n(u)} d\langle X, p^n(u)\rangle_{s|_{u=\xi}}$$

Divide a problem into two steps $\mathbb{F} \subset \mathbb{F}^\tau \subset \mathbb{G}$ and note that:

- $(\mathcal{H}')$ hypothesis holds for $\mathbb{F} \subset \mathbb{F}^\tau$ since τ is thin
- assume that ξ satisfies $\mathbb{P}(\xi \in du|\mathcal{F}^\tau_{\tau\vee t}) = q_t(u)\mathbb{P}(\xi \in du|\mathcal{F}^\tau_\tau)$ which ensures $(\mathcal{H}')$ hypothesis for $\mathbb{F}^\tau \subset \mathbb{G}$

Above condition is exactly generalised Jacod's condition and

$$p_t^n(u) = \frac{z^n_{T_n\vee t}}{z^n_{T_n}} q_t(u) \quad \text{on } C_n.$$

Example: $\xi = B_1 + \mathbb{1}_{\{B_2>0\}}$ and $\tau = 1$

PRP for marked thin time

Representation can be studied separately

- on $[\![0, \tau]\!]$, i.e., for $\mathbb{1}_{[\![0,\tau]\!]} \cdot M$ where M is an $\mathbb{F}$-martingale. We then use results from:

Choulli, T., Daveloose, C., & Vanmaele, M. (2020). A martingale representation theorem and valuation of defaultable securities. Mathematical Finance, 30(4), 1527-1564

Jeanblanc, M., & Song, S. (2015). Martingale representation property in progressively enlarged filtrations. Stochastic Processes and their Applications, 125(11), 4242-4271.

- and on $]\!]\tau, \infty[\![$, i.e. for $\mathbb{1}_{]\!]\tau,\infty[\![} \cdot M$ where M is an $\mathbb{F}$-martingale. We then use results from:

Fontana, C. (2018). The strong predictable representation property in initially enlarged filtrations under the density hypothesis. Stochastic Processes and their Applications, 128(3), 1007-1033.

PRP after marked thin time

On $]\!]\tau, \infty[\![$, we divide a problem into two steps

$$\mathbb{F} \subset \mathbb{F}^{\mathcal{C}} = \mathbb{F} \vee \sigma\{C_n : n\} \subset \mathbb{G}$$

and note that:

- $\mathbb{F} \subset \mathbb{F}^{\mathcal{C}}$ is initial enlargement with discrete random variable, hence Fontana results apply
- $\mathbb{G}$ can be viewed as initial enlargement at a $\mathbb{F}^{\tau}$-stopping time τ, hence once again Fontana's result apply
- Finally we wrap up two parts using key lemma for thin times (similarly as for semmartingale decomposition)

Aksamit, A., & Jeanblanc, M. Enlargement of filtration with finance in view. Springer. (2017).

Aksamit, A., Choulli, T., & Jeanblanc, M. Thin times and random times' decomposition. (2021) Electronic Journal of Probability 26, 1-22.

Aksamit, A. & Fontana, C. Marked thin random times, Working paper

Amendinger, J., Imkeller, P., & Schweizer, M. (1998). Additional logarithmic utility of an insider. Stochastic processes and their applications, 75(2), 263-286.

Fontana, C., Jeanblanc, M., & Song, S. (2014). On arbitrages arising with honest times. Finance and Stochastics, 18(3), 515-543.

Jacod, J. (1985). Grossissement initial, hypothese (H') et theoreme de Girsanov. In Grossissements de filtrations: exemples et applications (pp. 15-35). Springer, Berlin, Heidelberg.

Jeanblanc, M., & Le Cam, Y. (2009). Progressive enlargement of filtrations with initial times. Stochastic Processes and their Applications, 119(8), 2523-2543.

Jeulin, T., & Yor, M. (1979). Inégalité de Hardy, semimartingales, et faux-amis. In Séminaire de Probabilités XIII (pp. 332-359). Springer, Berlin, Heidelberg.

Jiao, Y., & Kharroubi, I. (2018). Information uncertainty related to marked random times and optimal investment. Probability, Uncertainty and Quantitative Risk, 3(1), 1-24.

Kardaras, C., & Ruf, J. (2020). Filtration shrinkage, the structure of deflators, and failure of market completeness. Finance and Stochastics, 24(4), 871-901.

Kchia, Y., Larsson, M., & Protter, P. (2013). Linking progressive and initial filtration expansions. In Malliavin Calculus and Stochastic Analysis (pp. 469-487). Springer, Boston, MA.

Meyer, P.-A. *Sur un théorème de J. Jacod.* In *Séminaire de Probabilités XII*, pages 57–60.

Yor, M. *Entropie d'une partition, et grossissement initial d'une filtration*. In *Grossissements de filtrations: exemples et applications*, pages 45–58. Springer, 1985.