Solving fully coupled FBSDEs by minimizing a directly calculable error functional

FRIEDRICH-SCHILLER-UNIVERSITÄT JENA

Stefan Ankirchner and Alexander Fromm

9th Colloquium on *BSDEs and Mean Field Systems*, Annecy

Fully coupled FBSDEs

$$
\begin{aligned}
X_t &= X_0 + \int_0^t \mu(s, X_s, Y_s, Z_s)ds + \int_0^t \sigma(s, X_s, Y_s, Z_s)dW_s, \\
Y_t &= \xi(X_T) - \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s.
\end{aligned}
$$

Vision: How to check solvability numerically?

Before: Some known facts on existence and uniqueness

Existence on a small interval

Known fact:

- Under a standard Lipschitz condition (SLC) on μ, σ, f and ξ
- and if $L_\xi < L_{\sigma,z}^{-1}$,

then there exists an interval $[t, T]$ on which a solution of the FBSDE exists.

Idea of the proof: Picard interations

Useful: decoupling fields

A function $u : \Omega \times [t, T] \times \mathbb{R} \to \mathbb{R}$ with $u(T, \cdot) = \xi$ is a *decoupling field* if for all $[t_1, t_2] \subset [t, T]$ and $x \in \mathbb{R}$ there exist progr. mb. processes (X, Y, Z) such that

$$X_s = x + \int_{t_1}^{s} \mu(r, X_r, Y_r, Z_r)dr + \int_{t_1}^{s} \sigma(r, X_r, Y_r, Z_r)dW_r,$$
$$Y_s = Y_{t_2} - \int_{t}^{t_2} f(r, X_r, Y_r, Z_r)dr - \int_{s}^{t_2} Z_r dW_r.$$

and

$$u(s, X_s) = Y_s, \qquad s \in [t_1, t_2].$$

- u is the same, regardless of $[t_1, t_2]$
- u is an additional structure

Theoretical background

Theorem

Under (SLC) and if $L_\xi < L_{\sigma,z}^{-1}$, then there exists $t < T$ such that on $[t, T]$

- *the FBSDE has a unique solution*
- *th. ex. a decoupling field u*
- *u is Lipschitz continuous*
- $\sup_{s\in[t,T]} L_{u(s,\cdot)} < L_{\sigma,z}^{-1}$

Characterizing the maximal existence interval with the decoupling field

By pasting existence intervals together one can show

$$I_{\max} = [0, T] \qquad \text{or} \qquad I_{\max} = (t_{\min}, T]$$

Theorem

If $I_{\max} \neq [0, T]$, then

$$\lim_{t \downarrow t_{\min}} L_{u(t,\cdot)} = L_{\sigma,z}^{-1} \tag{1}$$

(1) is very useful: If no explosion, then $I_{\max} = [0, T]$.

Method of Decoupling fields (as described by Alexander Fromm)

Prove *global solvability via the following steps:*

1. *Consider arbitrary* $[t, T]$ *on which* u *exists. Let* $x \in \mathbb{R}^n$ *be the initial condition.*
2. *Differentiate the FBSDE wrt* x *to obtain the dynamics of* $\frac{\mathrm{d}}{\mathrm{d}x}X, \frac{\mathrm{d}}{\mathrm{d}x}Y, \frac{\mathrm{d}}{\mathrm{d}x}Z$.
3. *Use the Itô formula to obtain dynamics of* $\Psi_s := u_x(s, X_s) = \frac{\mathrm{d}}{\mathrm{d}x}Y_s \left(\frac{\mathrm{d}}{\mathrm{d}x}X_s\right)^{-1}$.
4. *Show that* $\|\Psi\|_\infty$ *is bounded away from* $L_{\sigma,z}^{-1}$ *independently of* $t < T$.

Then the equation must be solvable on $[0, T]$.

A measure for how well two Ito processes solve an FBSDE

Ito processes

Let X and Y be two Ito processes on $I = [a, b] \subset [0, T]$.

Notation:

$$X_s = X_a + \int_a^s DX_r dr + \int_a^s \mathfrak{D} X_r dW_r,$$
$$Y_s = Y_a + \int_a^s DY_r dr + \int_a^s \mathfrak{D} Y_r dW_r,$$

$s \in [a, b]$.

An error functional for a pair of Ito processes

For 2 Ito processes X and Y we define

$$G_{x,I,\zeta}(X,Y) :=$$
$$\mathbb{E}\Bigg[|X_a - x|^2 + |I| \cdot \int_a^b |DX_r - \mu(r,\Theta_r)|^2 \,\mathrm{d}r + \int_a^b |\mathfrak{D}X_r - \sigma(r,\Theta_r)|^2 \,\mathrm{d}r$$
$$+ |I| \cdot \int_a^b |DY_r - f(r,\Theta_r)|^2 \,\mathrm{d}r + |Y_b - \zeta(X_b)|^2\Bigg],$$

where

- $|I|$ = the length of the interval I
- x is the initial condition for the forward equation
- ζ defines the terminal condition of the backward equation
- $\Theta_r = (X_r, Y_r, \mathfrak{D}Y_r)$

Properties of the error functional

- If $G_{x,I,\zeta}(X, Y) = 0$, then $(X, Y, \mathfrak{D}Y)$ solve the FBSDE on I with initial condition $X_a = x$ and terminal condition $Y_b = \zeta(X_b)$.
- Evaluation of G does not require the knowledge of (X^*, Y^*), the minimizer.

Properties of the error functional cont'd

Informal theorem There exists a nice norm $\| \cdot \|_1$ on the set of Ito processes and constants $c < C$ such that for all X, Y

$$c\|(X, Y) - (X^*, Y^*)\|_1 \leq \sqrt{G(X, Y)} \leq C\|(X, Y) - (X^*, Y^*)\|_1$$

Minimizing G is in a sense equivalent to solving the FBSDE

Stochastic polynomials

Definition of stochastic polynomials

$L^2(I) := \{X : \Omega \times I \to \mathbb{R}$ progr. mb. with $E \int_a^b X_s^2 ds < \infty\}$.

$\mathcal{P}(I)$ is defined as the smallest subset of $L^2(I)$ satisfying

1. $\mathcal{P}(I)$ is an $\mathbb{R}$ - vector space, which contains all constants;
2. if $X \in \mathcal{P}(I)$, then $s \mapsto \int_a^s X_r r$ is also in $\mathcal{P}(I)$;
3. if $X \in \mathcal{P}(I)$, then $s \mapsto \int_a^s X_r dW_r$ is also in $\mathcal{P}(I)$.

Example

$$\alpha_1 + \alpha_2(s - a) + \alpha_3(W_s - W_a) + \alpha_4 \int_a^s (W_r - W_a)\,\mathrm{d}W_r, \quad (2)$$

Let $\mathcal{P}^{3/2}$ be the set of Ito processes of the form (2).

Outlook. $\mathcal{P}^{3/2}$ is the minimal set of polynomials for approximating (general) FBSDEs

Why is $\frac{3}{2}$ minimal?

To explain this, let for $\alpha, \beta \in \mathbb{R}^4$

$$X^\alpha := \alpha_1 + \alpha_2(s-a) + \alpha_3(W_s - W_a) + \alpha_4 \int_a^s (W_r - W_a)\, dxW_r,$$
$$Y^\beta := \beta_1 + \beta_2(s-a) + \beta_3(W_s - W_a) + \beta_4 \int_a^s (W_r - W_a)\, dxW_r,$$

Aim: finding a good $\mathcal{P}^{3/2} \times \mathcal{P}^{3/2}$ approximation of the FBSDE components X, Y. To this end minimize

$$G_{x,I,\zeta}(\alpha, \beta) := G_{x,I,\zeta}(X^\alpha, Y^\beta)$$

over all $\alpha, \beta \in \mathbb{R}^4$.

Why is $\frac{3}{2}$ minimal?

Theorem

For the minimizers α^*, β^* we have

$$\sqrt{G_{x,I,\zeta}}(\alpha^*, \beta^*) \in \mathcal{O}(|I|^{\frac{3}{2}})$$

Total error

$$\sqrt{G_{x,I,\zeta}}(\alpha^*, \beta^*) \in \mathcal{O}(|I|^{\frac{3}{2}})$$

- Split $[0, T]$ into N subintervals of length $\frac{T}{N}$.
- At time $t_j = \frac{j}{N}T$ choose $\zeta = \tilde{u}(t_{j+1}, \cdot)$, the approximation of the dec. field at t_{j+1}
- Error on each subinterval: $\in \mathcal{O}((\frac{1}{N})^{\frac{3}{2}})$
- If the error propagation can be controlled, then the total error is of the order

$$N\,(\frac{1}{N})^{\frac{3}{2}} = \sqrt{\frac{1}{N}}$$

Idea for an algorithm

1. Select a time discretization $0 = t_0 < t_1 < \cdots < t_N = T$.
2. Select a set of supporting points $\{x_1, x_2, \ldots, x_S\} \subset \mathbb{R}$.
3. Set $j = N - 1$.
4. For every supporting point x_k find the parameters $\alpha_1^*, \ldots, \alpha_4^*$ and $\beta_1^*, \ldots, \beta_4^*$ that minimize $(\alpha, \beta) \mapsto G(X^\alpha, Y^\beta)$. Set $w(t_j, x_k) = \beta_1^*$.
5. Choose a smooth function $\tilde{u}(t_j, \cdot)$ such that $\tilde{u}(t_j, x_k)$ is close to $w(t_j, x_k)$ for all $k \in \{1, \ldots, S\}$.
6. If $j \neq 0$, then set $j = j - 1$ and go to 4.

Literature

Stefan Ankirchner, Alexander Fromm: *Solving fully coupled FBSDEs by minimizing a directly calculable error functional Preprint on hal, 2018.*

Alexander Fromm: Theory and applications of decoupling fields for FBSDEs. PhD thesis, 2015.

Thank you!