On optimal stochastic control problem of McKean-Vlasov type with some applications via the derivative with respect to the law of probability

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We establish a general necessary optimality conditions for stochastic continuous-singular control of McKean-Vlasov type equations. The coefficients of the state equation depend on the state of the solution process as well as of its probability law and the control variable. The coefficients of the system are nonlinear and depend explicitly on the absolutely continuous component of the control. The control domain under consideration is not assumed to be convex.
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The proof of our general maximum principle is based on the first and second-order derivatives with respect to measure in Wasserstein space of probability measures, and by using variational method with some estimations.
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The proof of our general maximum principle is based on the first and second-order derivatives with respect to measure in Wasserstein space of probability measures, and by using variational method with some estimations.
Our work distinguishes itself from the above ones in the following aspects.

1. We study the more general controlled nonlinear McKean-Vlasov type system, where the coefficients of the equation depend on the state of the solution process $X^{u,\eta}$ as well as of its probability measures $P_{X^{u,\eta}(t)}$.

2. Second, we apply the first and second-order derivatives with respect to probability measures to establish our Peng’s type necessary optimality conditions.

3. Third, we study the general continuous-singular control problem, where the control domain is not assumed to be convex.
Novelty in this work

4. Forth, the second-order derivative with respect to probability measures in *Wasserstein space* is applied to establish our result without convexity conditions.

5. Our McKean-Vlasov control problem occur naturally in the probabilistic analysis of financial optimization problems. Moreover, the above mathematical McKean-Vlasov approaches play an important role in different fields of economics, finance, physics, chemistry and game theory.
An admissible continuous-singular control is a pair \((u(\cdot), \eta(\cdot))\) of measurable \(U_1 \times U_2\)-valued, \(\mathcal{F}_t\)-adapted processes, such that

1. \(\eta(\cdot)\) is of bounded variation process, nondecreasing, continuous on the left with right limits and \(\eta(0) = 0\).
2. \(E\left[\sup_{t \in [0,T]} |u(t)|^2 + |\eta(T)|^2\right] < \infty\).

where

- \(U_1\) is a nonempty subset of \(\mathbb{R}^n\),
- \(U_2 = ([0, +\infty))^m\).
Definition

Let $\mathcal{U}_1$ : the class of measurable, adapted processes $u(\cdot) : [0, T] \times \Omega \rightarrow \mathbb{U}_1$. Let $\mathcal{U}_2$ : the class of measurable, adapted processes $\eta(\cdot) : [0, T] \times \Omega \rightarrow \mathbb{U}_2$ such that $\eta(\cdot)$ is of bounded variation, nondecreasing continuous on the left with right limits and $\eta(0) = 0$.

We denote by $\mathcal{U}_1 \times \mathcal{U}_2 ([0, T])$, the set of all admissible continuous-singular controls.
Formulation of the continuous-singular control problem

$(\Omega, \{\mathcal{F}_t\}, P)$ be a fixed filtered probability space satisfying the usual conditions, $B(t):$ Brownian motion.
Formulation of the continuous-singular control problem

1. $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a fixed filtered probability space satisfying the usual conditions, $B(t)$ : Brownian motion.

2. The criteria to be minimized over the class of admissible controls has the form

$$J(u(\cdot), \eta(\cdot)) = E \left[ \int_0^T f(t, X^{u,\eta}_t, \mathbb{P}_{X^{u,\eta}_t}, u(t)) dt + h(X^{u,\eta}_T, \mathbb{P}_{X^{u,\eta}_T}) + \int_{[0,T]} M(t) d\eta(t) \right],$$

(1)
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$$+ h(X^{u,\eta}(T), P_{X^{u,\eta}(T)}) + \int_{[0,T]} M(t) d\eta(t) \right] ,$$

3. Such that $X^{u,\eta}(t)$ solution of the following McKean-Vlasov SDEs:

$$dX^{u,\eta}(t) = f \left( t, X^{u,\eta}(t), P_{X^{u,\eta}(t)}, u(t) \right) dt $$

$$+ \sigma \left( t, X^{u,\eta}(t), P_{X^{u,\eta}(t)}, u(t) \right) dB(t) $$

$$+ G(t) d\eta(t),$$

$$X^{u,\eta}(0) = x_0,$$
Formulation of the continuous-singular control problem

1. \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, P)\) be a fixed filtered probability space satisfying the usual conditions, \(B(t)\) : Brownian motion.

2. The criteria to be minimized over the class of admissible controls has the form

\[
J(u(\cdot), \eta(\cdot)) = E \left[ \int_0^T f(t, X_{u,\eta}(t), P_{X_{u,\eta}(t)}, u(t)) dt 
+ h(X_{u,\eta}(T), P_{X_{u,\eta}(T)}) + \int_{[0,T]} M(t) d\eta(t) \right],
\]

(1)

3. Such that \(X_{u,\eta}(t)\) solution of the following McKean-Vlasov SDEs:

\[
\begin{cases}
    dX_{u,\eta}(t) = f \left( t, X_{u,\eta}(t), P_{X_{u,\eta}(t)}, u(t) \right) dt \\
    + \sigma \left( t, X_{u,\eta}(t), P_{X_{u,\eta}(t)}, u(t) \right) dB(t) \\
    + G(t) d\eta(t), \\
    X_{u,\eta}(0) = x_0,
\end{cases}
\]

(2)

4. The control domain \(U_1 \times U_2\) is not assumed to be convex.
We now recall briefly an important notion in McKean-Vlasov control problems: the differentiability with respect to probability measures, in *Wasserstein space* which was introduced by Lions [3].
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The main idea is to identify a distribution $\mu \in Q_2(\mathbb{R}^n)$ with a random variable $X \in L^2(F, \mathbb{R}^n)$ so that $\mu = P_X$. 

We assume that probability space $(\Omega, \mathcal{F}, P)$ is rich-enough in the sense that for every $\mu \in Q_2(\mathbb{R}^n)$, there is a random variable $X \in L^2(F, \mathbb{R}^n)$ such that $\mu = P_X$. (3)
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We suppose that there is a sub-$\sigma$–field $\mathcal{F}_0 \subset \mathcal{F}$ such that $\mathcal{F}_0$ is rich-enough i.e,

$$Q_2(\mathbb{R}^n) \triangleq \{ P_X : X \in L^2(\mathcal{F}_0, \mathbb{R}^n) \}.$$ (3)
for any function $f : Q_2(\mathbb{R}^n) \to \mathbb{R}$ we define a function
\[ \tilde{f} : L^2(\mathcal{F}, \mathbb{R}^n) \to \mathbb{R} \text{ such that} \]
\[ \tilde{f}(X) = f(P_X), \quad X \in L^2(\mathcal{F}, \mathbb{R}^n). \quad (4) \]
Clearly, the function $\tilde{f}$, called the *lift* of $f$, depends only on the law of
$X \in L^2(\mathcal{F}, \mathbb{R}^n)$ and is independent of the choice of the representative $X$.
(see [3])
A function $g : Q_2(\mathbb{R}^n) \rightarrow \mathbb{R}$ is said to be differentiable at a distribution $\mu_0 \in Q_2(\mathbb{R}^n)$ if there exists $X_0 \in L^2(\mathcal{F}, \mathbb{R}^n)$, with $\mu_0 = P_{X_0}$ such that its lift $\tilde{g}$ is Fréchet-differentiable at $X_0$.

More precisely, there exists a continuous linear functional $D\tilde{g}(X_0) : L^2(\mathcal{F}, \mathbb{R}^n) \rightarrow \mathbb{R}$ such that

$$\tilde{g}(X_0 + \zeta) - \tilde{g}(X_0) = \langle D\tilde{g}(X_0) \cdot \zeta \rangle + o(\|\zeta\|_2) = D\zeta g(\mu_0) + o(\|\zeta\|_2),$$

where $\langle . \cdot . \rangle$ is the dual product on $L^2(\mathcal{F}, \mathbb{R}^n)$. 

(5)
Differentiability with respect to measure

We called $\mathcal{D}_\zeta g(\mu_0)$ the Fréchet-derivative of $g$ at $\mu_0$ in the direction $\zeta$. In this case we have

$$\mathcal{D}_\zeta g(\mu_0) = \langle \mathcal{D}\tilde{g}(X_0) \cdot \zeta \rangle = \left. \frac{d}{dt} \tilde{g}(X_0 + t\zeta) \right|_{t=0}, \text{ with } \mu_0 = P_{X_0}.$$  

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By applying Riesz representation theorem, there is a unique random variable $\Theta_0 \in L^2(\mathcal{F}, \mathbb{R}^n)$ such that

$$\langle D\tilde{g}(X_0) \cdot \zeta \rangle = (\Theta_0 \cdot \zeta)_2 = E \left[ (\Theta_0 \cdot \zeta)_2 \right] \text{ where } \zeta \in L^2(\mathcal{F}, \mathbb{R}^n).$$
Differentiability with respect to measure

Definition

We say that the function $g \in C_{b}^{1,1}(Q_{2}(\mathbb{R}^{n}))$ if for all $X \in L^{2}(\mathcal{F}, \mathbb{R}^{n})$ there exists a $P_{X}$—modification of $\partial_{\mu}g(P_{X}, \cdot)$ (denoted by $\partial_{\mu}g$) such that

$$\partial_{\mu}g : Q_{2}(\mathbb{R}^{n}) \times \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$$

is bounded and Lipschitz continuous. That is for some $C > 0$, it holds that
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1. $\partial_\mu g : Q_2(\mathbb{R}^n) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is bounded and Lipschitz continuous. That is for some $C > 0$, it holds that

$$\left| \partial_\mu g(\mu, x) \right| \leq C, \ \forall \mu \in Q_2(\mathbb{R}^n), \ \forall x \in \mathbb{R}^n.$$
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1. \( \partial_\mu g : Q_2(\mathbb{R}^n) \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) is bounded and Lipschitz continuous. That is for some \( C > 0 \), it holds that

\[
|\partial_\mu g(\mu, x)| \leq C, \quad \forall \mu \in Q_2(\mathbb{R}^n), \quad \forall x \in \mathbb{R}^n.
\]

2. The derivatives \( \partial_\mu g \) satisfied the following

\[
|\partial_\mu g(\mu, x) - \partial_\mu g(\mu', x')| \leq C \left[ \mathbb{I}(\mu, \mu') + |x - x'| \right],
\]

\[
\forall \mu, \mu' \in Q_2(\mathbb{R}^n), \quad \forall x, x' \in \mathbb{R}^n.
\]
Second-order derivatives with respect to measure

Definition
We say that the function $g \in C^{2,1}_b(Q_2(\mathbb{R}^n))$ if $g \in C^{1,1}_b(Q_2(\mathbb{R}^n))$ such that $\partial_\mu g(\cdot, x) : Q_2(\mathbb{R}^n) \to \mathbb{R}^n$

- $\partial_\mu g(\cdot, y) \in C^{1,1}_b(Q_2(\mathbb{R}^n))$, $\forall y \in \mathbb{R}^n$ and $i \in \{1, 2, \ldots, n\}$.
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2. $\partial_\mu g(\mu, \cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is differentiable, for evry $\mu \in Q_2(\mathbb{R}^n)$.
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1. \( \partial_\mu g(\cdot, y) \in C^{1,1}_b(Q_2(\mathbb{R}^n)), \forall y \in \mathbb{R}^n \) and \( i \in \{1, 2, \ldots, n\} \).
2. \( \partial_\mu g(\mu, \cdot) : \mathbb{R}^n \to \mathbb{R}^n \) is differentiable, for evry \( \mu \in Q_2(\mathbb{R}^n) \).
3. The mapps \( \partial_x \partial_\mu g(\cdot, \cdot) : Q_2(\mathbb{R}^n) \times \mathbb{R}^n \to \mathbb{R}^n \otimes \mathbb{R}^n \) and \( \partial_\mu^2 g(P_{\mathcal{X}_0}, y, Z) : Q_2(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \otimes \mathbb{R}^n \) are bounded and Lipshitz continuous, where

\[
\partial_\mu^2 g(P_{\mathcal{X}_0}, y, Z) = \partial_\mu \left[ \partial_\mu g(\cdot, y) \right] (P_{\mathcal{X}_0}, Z).
\]
Main results. Assumptions

**Assumption (H1)** The coefficients $f, \sigma, \ell, h$ are measurable in all variables. Moreover, for all $(u(t), \eta(t)) \in U_1 \times U_2$, $f(\cdot, \cdot, u)$, $\sigma(\cdot, \cdot, u)$, $\ell(\cdot, \cdot, u) \in C_b^{1,1}(\mathbb{R} \times Q_2(\mathbb{R}^d); \mathbb{R})$, $h(\cdot, \cdot) \in C_b^{1,1}(\mathbb{R} \times Q_2(\mathbb{R}^n); \mathbb{R})$. More precisely, for each $u(t) \in U_1$, denoting $\varphi(x, \mu) = f(t, x, \mu, u)$, $\sigma(t, x, \mu, u)$, $f(t, x, \mu, u)$, $h(x, \mu)$, the function $\varphi(\cdot, \cdot)$ enjoys the following properties:

1. For fixed $\mu \in Q_2(\mathbb{R})$, $\varphi(\cdot, \mu)$ continuously differentiable with respect to $x$;
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1. For fixed $\mu \in Q_2(\mathbb{R})$, $\varphi(\cdot, \mu)$ continuously differentiable with respect to $x$;
2. For fixed $x \in \mathbb{R}$, $\varphi(x, \cdot) \in C^{1,1}_b(Q_2(\mathbb{R}))$;
3. All the derivatives $\partial_x \varphi$ and $\partial_\mu \varphi : \varphi = f, \sigma, \ell, h$, are bounded and Lipschitz continuous, with Lipschitz constants independent of $(u(t), \eta(t))$. 
**Assumption (H2)** The coefficients $f, \sigma, \ell, h$ satisfy assumption (H1). Furthermore, for all $u(t) \in U_1$, $f(t, \cdot, \cdot, u)$, $\sigma(t, \cdot, \cdot, u)$, $\ell(t, \cdot, \cdot, u) \in C_b^{2,1}(\mathbb{R} \times Q_2(\mathbb{R}); \mathbb{R})$, $h(\cdot, \cdot) \in C_b^{2,1}(\mathbb{R} \times Q_2(\mathbb{R}); \mathbb{R})$. More precisely, for each $u(t) \in U_1$, the derivatives of $f, \sigma, \ell, h$, denoted by a generic function $\varphi(t, x, \mu)$, enjoy the following properties:

1. $\partial_x \varphi(t, \cdot, \cdot) \in C_b^{1,1}(\mathbb{R} \times Q_2(\mathbb{R}))$;
**Main results. Assumptions**

**Assumption (H2)** The coefficients $f, \sigma, \ell, h$ satisfy assumption (H1). Furthermore, for all $u(t) \in U_1$, $f(t, \cdot, \cdot, u), \sigma(t, \cdot, \cdot, u), \ell(t, \cdot, \cdot, u) \in C^{2,1}_b(\mathbb{R} \times Q_2(\mathbb{R}); \mathbb{R})$, $h(\cdot, \cdot) \in C^{2,1}_b(\mathbb{R} \times Q_2(\mathbb{R}); \mathbb{R})$. More precisely, for each $u(t) \in U_1$, the derivatives of $f, \sigma, \ell, h$, denoted by a generic function $\varphi(t, x, \mu)$, enjoy the following properties:

1. $\partial_x \varphi(t, \cdot, \cdot) \in C^{1,1}_b(\mathbb{R} \times Q_2(\mathbb{R}))$;
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1. $\partial_x \varphi(t, \cdot, \cdot) \in C^{1,1}_b(\mathbb{R} \times Q_2(\mathbb{R}))$;
2. $\partial_\mu \varphi(t, \cdot, \cdot) \in C^{1,1}_b(\mathbb{R} \times Q_2(\mathbb{R}) \times \mathbb{R})$;
3. All the second-order derivatives of $f, \sigma, \ell, h$, are bounded and Lipschitz continuous with Lipschitz constants independent of $(u(t), \eta(t))$. 

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Main results. Assumptions

**Assumption (H3)** The functions $G(\cdot) : [0, T] \to \mathbb{R}$, and $M(\cdot) : [0, T] \to \mathbb{R}^+$ are continuous and bounded.
Main results. Adjoint equation

**First-order adjoint equation.** We consider the first-order adjoint equation, which is the following McKean-Vlasov linear BSDE:

\[
\begin{aligned}
-dp(t) &= \left[ f_x(t)p(t) + \hat{E}\left[ f^*_\mu(t)(t)p(t) \right] \right. \\
&\quad+ \sigma_x(t)q(t) + \hat{E}\left[ \sigma^*_\mu(t)q(t) \right] \\
&\quad- \ell_x(t) - \hat{E}\left[ \ell^*_\mu(t)(t) \right] \right] dt - q(t)dB(t), \\
p(T) &= h_x(T) + \hat{E}\left[ h^*_\mu(T) \right].
\end{aligned}
\]

(7)

Here, from (??), \( t \in [0, T] \), for \( \varphi = f, \sigma, \ell \), we obtain

\[
\hat{E}\left[ \partial_\mu \varphi^*(t) \right] = \hat{E}\left[ \partial_\mu \varphi(t, \hat{X}(t), P_{\hat{X}^*(t)}, \hat{u}^*(t); z) \right] \bigg|_{z=X^*(t)}
\]

(8)

\[
= \int_{\hat{\Omega}} \partial_\mu \varphi(t, \hat{X}(t, \hat{w}), P_{\hat{X}^*(t, \hat{w})}, \hat{u}^*(t, \hat{w}); X^*(t, \hat{w}))d\hat{P}(\hat{w}),
\]
Main results. Adjoint equation

Second-order adjoint equation. Consider the following standard linear BSDE

\[
\begin{align*}
\text{d}P(t) &= - \left\{ 2(b_x(t) + \hat{E}[\hat{b}_\mu^*(t)])P(t) + [\sigma_x(t) + \hat{E}(\hat{\sigma}_\mu^*(t))]^2 P(t) \\
&\quad + 2(\sigma_x(t) + \hat{E}[\hat{\sigma}_\mu^*(t)]) Q(t) + (H_{xx}(t) + \hat{E}[\hat{H}_{\mu y}^*(t)]) \right\} \text{d}t \\
&\quad + Q(t) \text{d}B(t), \\
P(T) &= -(h_{xx}(T) + \hat{E}[\hat{h}_{\mu y}^*(T)]).
\end{align*}
\]  

(9)

Similar to (8) and (??), we have

\[
\hat{E}[\hat{H}_{\mu y}^*(t))] = \hat{E} \left[ \partial_\mu \partial_y H(t, \hat{X}(t), P_{\hat{X}^*(t)}, \hat{u}^*(t), \hat{p}(t), \hat{q}(t); y) \right] \bigg|_{y=\hat{X}^*(t)}
= \int_{\hat{\Omega}} \partial_\mu \partial_y H(t, \hat{X}(t, \hat{\omega}), P_{\hat{X}^*(t)}, \hat{u}^*(t, \hat{\omega}), \hat{p}(t), \hat{q}(t); \hat{X}^*(t)) \text{d}\hat{P}(\hat{\omega})
\]
Main results. Hamiltonian

Let us define the Hamiltonian associated to our continuous-singular control problem. For any \((t, x, \mu, u, p, q) \in [0, T] \times \mathbb{R} \times Q_2(\mathbb{R}) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}\)

\[
H(t, x, \mu, u, p, q) = f(t, x, \mu, u)p + \sigma(t, x, \mu, u)q - \ell(t, x, \mu, u). \quad (10)
\]

where \((p(\cdot), q(\cdot))\) be a pair of adapted processes, solution of the first-order adjoint equation
Main results. Necessary conditions

Let \((u^*(\cdot), \eta^*(\cdot), X^*(\cdot))\) is an optimal solution of the McKean-Vlasov control problem (2)-(1). We introduce the following variational equations for our continuous-singular control problem. Let \(Y^{u, \eta}(\cdot)\) and \(Z^{\epsilon}(\cdot)\) be the solutions of (11), (12) associated to \((u^*(\cdot), \eta^*(\cdot))\) respectively.

**First-order variational equation:** let \(E_{\epsilon} = [0, \epsilon]\), \(t \in [0, T]\)

\[
\begin{aligned}
    dY^{u, \eta}(t) &= \left[ f_x(t) Y^{u, \eta}(t) + \hat{E}[\hat{f}_\mu(t) \hat{Y}^{u, \eta}(t)] + \delta f(t) 1_{E_{\epsilon}}(t) \right] dt \\
    &+ \left[ \sigma_x(t) Y^{\epsilon}(t) + \hat{E}[\hat{\sigma}_\mu(t) \hat{Y}^{u, \eta}(t)] + \delta \sigma(t) 1_{E_{\epsilon}}(t) \right] dB(t) \\
    &+ G(t) d(\eta^{\epsilon} - \eta^*)(t), \\
    Y^{u, \eta}(0) &= 0.
\end{aligned}
\]  

(11)

Here the process \(Y^{u, \eta}(\cdot)\) is called the *first-order variational process*, associated to \((u^{\epsilon}(\cdot), \eta^{\epsilon}(\cdot))\) which is depend explicitly to singular control. The process \(\eta^{\epsilon}(\cdot)\) is the convex perturbed control given by \(\eta^{\epsilon}(t) = \eta^*(t) + \epsilon (\eta(t) - \eta^*(t))\).
Main results. Necessary conditions

Second-order variational equation:

\[
\begin{align*}
\text{d}Z^\varepsilon(t) &= \left[ f_x(t)Z^\varepsilon(t) + \hat{E}[f_\mu(t)\hat{Z}^\varepsilon(t)] + L_{xx}(t, f, Y^\varepsilon) + L_{\mu x}(t, \hat{f}, \hat{Y}^\varepsilon) \right] dt \\
&+ \left[ \sigma_x(t)Z^\varepsilon(t) + \hat{E}[\hat{\sigma}_\mu(t)\hat{Z}^\varepsilon(t)] + L_{xx}(t, \sigma, Y^\varepsilon) + L_{\mu x}(t, \hat{\sigma}, \hat{Y}^\varepsilon) \right] dB(t), \\
&+ \left[ \delta f_x(t)Y^\varepsilon(t) + \hat{E}[\delta f_\mu(t)\hat{Y}^\varepsilon(t)] \right] 1_{E^\varepsilon}(t) dt \\
&+ \left[ \delta \sigma_x(t)Y^\varepsilon(t) + \hat{E}[\delta \sigma_\mu(t)\hat{Y}^\varepsilon(t)] \right] 1_{E^\varepsilon}(t) dB(t), \\
Z^\varepsilon(0) &= 0.
\end{align*}
\] (12)

Here the process \( Z^\varepsilon(\cdot) \) is called the second-order variational process.
Main results. Estimations

Lemma

Let $X^\varepsilon(\cdot) = X^{u^\varepsilon, \eta^\varepsilon}(\cdot)$ be the solutions of (2) corresponding to continuous-singular control $(u^\varepsilon(\cdot), \eta^\varepsilon(\cdot))$. Let assumptions (H1) and (H2) hold. Then we have

$$\lim_{\varepsilon \to 0} E\left( \sup_{t \in [0,T]} |X^\varepsilon(t) - X^*(t)|^2 \right) = 0.$$

Let $X^{u^\varepsilon, \eta^*}(\cdot)$ be the solution of (2), corresponding to $(u^\varepsilon(\cdot), \eta^*(\cdot))$. Let $Y^\varepsilon(\cdot)$ be the solution of (??), corresponding to $(u^\varepsilon(\cdot), \eta^*(\cdot))$, then the following estimation holds
Main results. Estimations

**Lemma**

\[
\lim_{\varepsilon \to 0} E \left[ \sup_{0 \leq t \leq T} \left| X^{u^{\varepsilon}, \eta^*(t)}(t) - X^*(t) \right|^2 \right] = 0. \quad (13)
\]

\[
\lim_{\varepsilon \to 0} E \left[ \sup_{0 \leq t \leq T} \left| X^{\varepsilon}(t) - X^{u^{\varepsilon}, \eta^*(t)}(t) \right|^2 \right] = 0. \quad (14)
\]

\[
\lim_{\varepsilon \to 0} E \left[ \sup_{0 \leq t \leq T} \left| X^{u^{\varepsilon}, \eta^*(t)}(t) - X^*(t) - Y^{\varepsilon}(t) \right|^2 \right] = 0. \quad (15)
\]

**Proposition**

Let \( Y^{\varepsilon}(t) \) solution (12) associated to \( (u^{\varepsilon}(\cdot), \eta^*(\cdot)) \). Under assumption \( H1 \), the following estimate holds

\[
\lim_{\varepsilon \to 0} E \left[ \sup_{0 \leq t \leq T} \left| X^{u^{\varepsilon}, \eta^*(t)}(t) - X^*(t) - Y^{\varepsilon}(t) - Z^{\varepsilon}(t) \right|^2 \right] = 0. \quad (16)
\]
The following theorem constitutes the main contribution of this paper.

**Theorem 3.1 (Stochastic maximum principle)** Let \( (u^*(\cdot), \eta^*(\cdot), X^*(\cdot)) \) is an optimal solution of the McKean-Vlasov control problem (2)-(1). Let assumptions (H1), (H2) and (H3) hold. Then there are two pairs of \( F_t \)-adapted processes \( (p(\cdot), q(\cdot)) \) and \( (P(\cdot), Q(\cdot)) \) that satisfy (7) and (9) respectively, such that for all \( (u(t), \eta(t)) \in U_1 \times U_2 \), we have

\[
0 \leq H(t, X^*(t), P_{X^*(t)}, u^*(t), p^*(t), q^*(t)) - H(t, x^*(t), P_{X^*(t)}, u(t), p^*(t), q^*(t)) - \frac{1}{2} P(t) \left( \sigma(t, X^*(t), P_{X^*(t)}, u(t)) \right)^2 \\
+ E \int_{[0,T]} (M(t) + G(t)p(t)) \mathrm{d}(\eta - \eta^*) (t).
\]

\( P \)-a.s., a.e. \( t \in [0, T] \).
Main results. Stochastic maximum principle
Main results. Stochastic maximum principle

We derive the variational inequality (18) in several steps. From the optimality of \((u^*(\cdot), \eta^*(\cdot))\), we have

\[
J(\mu^\varepsilon(\cdot), \eta^\varepsilon(\cdot)) - J(u^*(\cdot), \eta^*(\cdot)) \geq 0. \tag{18}
\]

Now, we separate the above inequality into two parts

\[
J_1^\varepsilon = J(\mu^\varepsilon(\cdot), \eta^\varepsilon(\cdot)) - J(\mu^\varepsilon(\cdot), \eta^*(\cdot)), \tag{19}
\]

\[
J_2^\varepsilon = J(\mu^\varepsilon(\cdot), \eta^*(\cdot)) - J(u^*(\cdot), \eta^*(\cdot)), \tag{20}
\]

where \(J(\mu^\varepsilon(\cdot), \eta^\varepsilon(\cdot)) - J(u^*(\cdot), \eta^*(\cdot)) = J_1^\varepsilon + J_2^\varepsilon\). The variational inequality will be derived from the fact that

\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (J_1^\varepsilon + J_2^\varepsilon) \geq 0. \tag{21}
\]
**Future developments** We have discussed a general Peng’s type necessary conditions in the form of Pontryagin stochastic maximum principle of optimal continuous-singular control for nonlinear controlled McKean-Vlasov stochastic differential equation. If the coefficients of the singular parts $G(t) = M(t) = 0$, our stochastic maximum principle (Theorem 3.1) coincides with maximum principle developed in Buckdahn et al. [3, Theorem 3.5].

Apparently, there are many problems left unsolved such as:

A. One possible problem is to study the general Peng’s type maximum principle for optimal control for SDE, the coefficients of the singular parts $G(\cdot)$ and $M(\cdot)$ depend explicitly to the state of the solution process $X^{u,\eta}$ of the form

\[
\begin{align*}
    dX^{u,\eta}(t) &= f(t, X^{u,\eta}(t), u(t)) \, dt + \sigma(t, X^{u,\eta}(t), u(t)) \, dW(t) \\
    &+ G(t, X^{u,\eta}) \, d\eta(t), \\
    X^{u,\eta}(0) &= x_0,
\end{align*}
\]
and the cost functional of the form

\[
J(u(\cdot), \eta(\cdot)) = E \left[ \int_0^T f(t, X_u, \eta(t), u(t)) dt + h(X_u, \eta(T)) + \int_{[0,T]} M(t, X_u, \eta) d\eta(t) \right].
\]
B. It would be interesting to investigate the McKean-Vlasov maximum principle (local version via Bensoussan’s convex method and general Peng’s maximum principle) for optimal continuous-singular control for McKean-Vlasov SDE, the coefficients of the singular parts $G(\cdot)$ and $M(\cdot)$ of the state equation depend on the state of the solution process as well as of its probability law and the control variable.

The state equation has the form

$$
\begin{cases}
    dX^{u,\eta}(t) = f\left(t, X^{u,\eta}(t), P_{X^{u,\eta}(t)}, u(t)\right) dt + \sigma\left(t, X^{u,\eta}(t), P_{X^{u,\eta}(t)}, u(t)\right) dW(t), \\
    + G(t, X^{u}, P_{X^{u,\eta}(t)}) d\eta(t), \\
    X^{u,\eta}(0) = x_0,
\end{cases}
$$

and the expected cost has the form

$$
J(u(\cdot), \eta(\cdot)) = E \left[ \int_{0}^{T} f(t, X^{u,\eta}(t), P_{X^{u,\eta}(t)}, u(t)) dt + h(X^{u,\eta}(T), P_{X^{u,\eta}(T)}) \\
+ \int_{[0,T]} M(t, X^{u,\eta}, P_{X^{u,\eta}(t)}) d\eta(t) \right].
$$
Another challenging problem left unsolved is to derive various maximum principles in the case where the coefficients $f, \sigma, \ell, G$ and $M$ depend on the state of the solution process $X^{u,\eta}(\cdot)$, the continuous control variable $u(\cdot)$ as well as of probability law of the pair $P(X^{u,\eta}(t), u(t))$. So we investigate the problem:

$$
\begin{cases}
    dX^{u,\eta}(t) = f(t, X^{u,\eta}, u(t), P(X^{u,\eta}(t), u(t))) dt + \sigma(t, X^{u,\eta}(t), u(t), P(X^{u,\eta}(t), u(t))) d\eta(t), \\
    X^{u,\eta}(0) = x_0,
\end{cases}
$$

and the cost functional has the general form

$$
J(u(\cdot), \eta(\cdot)) = E \left[ \int_0^T f(t, X^{u,\eta}, u(t), P(X^{u,\eta}(t), u(t))) dt + h(X^{u,\eta}(T), P(X^{u,\eta}(T))) \\
    + \int_{[0,T]} M(t, X^{u,\eta}, u(t), P(X^{u,\eta}(t), u(t))) d\eta(t) \right].
$$

We hope to study these interesting new problems in forthcoming works.


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Thank you for your attention

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