On optimal stochastic control problem of McKean-Vlasov type with some applications via the derivative with respect the law of probability

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Novelty in this work

Our work distinguishes itself from the above ones in the following aspects.

1. We study the more general controlled nonlinear McKean-Vlasov type system, where the coefficients of the equation depend on the state of the solution process $X^{u,\eta}$ as well as of its probability measures $P_{X^{u,\eta}(t)}$.
2. Second, we apply the first and second-order derivatives with respect to probability measures to establish our Peng's type necessary optimality conditions.
3. Third, we study the general continuous-singular control problem, where the control domain is not assumed to be convex.

Novelty in this work

4. Forth, the second-order derivative with respect to probability measures in *Wasserstein space* is applied to establish our result without convexity conditions.
5. Our McKean-Vlasov control problem occur naturally in the probabilistic analysis of financial optimization problems. Moreover, the above mathematical McKean-Vlasov approaches play an important role in different fields of economics, finance, physics, chemistry and game theory.

Stochastic singular control

Definition

An admissible continuous-singular control is a pair $(u(\cdot), \eta(\cdot))$ of measurable $\mathbb{U}_1 \times \mathbb{U}_2$-valued, $\mathcal{F}_t-$adapted processes, such that
(1) $\eta(\cdot)$ is of bounded variation process, nondecreasing, continuous on the left with right limits and $\eta(0) = 0$.
(2) $E\left[\sup_{t\in[0,T]} |u(t)|^2 + |\eta(T)|^2\right] < \infty$.
where
$\mathbb{U}_1$: is a a non empty subset of $\mathbb{R}^n$,
$\mathbb{U}_2 = ([0, +\infty))^m$.

Admissible continuous-singular control

Definition

Let $\mathcal{U}_1$: the class of measurable, adapted processes $u(\cdot) : [0, T] \times \Omega \to \mathbb{U}_1$. Let $\mathcal{U}_2$: the class of measurable, adapted processes $\eta(\cdot) : [0, T] \times \Omega \to \mathbb{U}_2$ such that $\eta(\cdot)$ is of bounded variation, nondecreasing continuous on the left with right limits and $\eta(0) = 0$. We denote by $\mathcal{U}_1 \times \mathcal{U}_2 ([0, T])$, the set of all admissible continuous-singular controls.

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$$J(u(\cdot), \eta(\cdot)) = E\Big[\int_0^T f(t, X^{u,\eta}(t), P_{X^{u,\eta}(t)}, u(t))\mathrm{d}t + h(X^{u,\eta}(T), P_{X^{u,\eta}(t)}) + \int_{[0,T]} M(t)\mathrm{d}\eta(t)\Big], \tag{1}$$

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3. Such that $X^{u,\eta}(t)$ solution of the following McKean-Vlasov SDEs:

$$\begin{cases} dX^{u,\eta}(t) = f\left(t, X^{u,\eta}(t), P_{X^{u,\eta}(t)}, u(t)\right) dt \\ \qquad + \sigma\left(t, X^{u,\eta}(t), P_{X^{u,\eta}(t)}, u(t)\right) \mathrm{d}B(t) \\ \qquad + G(t)\mathrm{d}\eta(t), \\ X^{u,\eta}(0) = x_0, \end{cases} \tag{2}$$

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Differentiability with respect to measure

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3. We assume that probability space $(\Omega, \mathcal{F}, P)$ is *rich-enough* in the sense that for every $\mu \in Q_2(\mathbb{R}^n)$, there is a random variable $X \in \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n)$ such that $\mu = P_X$.

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4. We suppose that there is a sub-$\sigma-$field $\mathcal{F}_0 \subset \mathcal{F}$ such that $\mathcal{F}_0$ is *rich-enough i.e,*

$$Q_2(\mathbb{R}^n) \triangleq \left\{ P_X : X \in \mathbb{L}^2(\mathcal{F}_0, \mathbb{R}^n) \right\}. \tag{3}$$

Differentiability with respect to measure, Lift function

Definition

for any function $f : Q_2(\mathbb{R}^n) \rightarrow \mathbb{R}$ we define a function $\widetilde{f} : \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n) \rightarrow \mathbb{R}$ such that

$$\widetilde{f}(X) = f(P_X), \ X \in \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n). \tag{4}$$

Clearly, the function $\widetilde{f}$, called the *lift* of f, depends only on the law of $X \in \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n)$ and is independent of the choice of the representative X. (see [3])

Differentiability with respect to measure, Lift function

Definition

A function $g : Q_2(\mathbb{R}^n) \to \mathbb{R}$ is said to be differentiable at a distribution $\mu_0 \in Q_2(\mathbb{R}^n)$ if there exists $X_0 \in \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n)$, with $\mu_0 = P_{X_0}$ such that its lift $\widetilde{g}$ is *Fréchet-differentiable* at X_0.

- More precisely, there exists a continuous linear functional $\mathcal{D}\widetilde{g}(X_0) : \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n) \to \mathbb{R}$ such that

$$\begin{aligned} \widetilde{g}(X_0 + \zeta) - \widetilde{g}(X_0) &= \langle \mathcal{D}\widetilde{g}(X_0) \cdot \zeta \rangle + o(\|\zeta\|_2) \\ &= \mathcal{D}_\zeta g(\mu_0) + o(\|\zeta\|_2), \end{aligned} \tag{5}$$

where $\langle . \cdot . \rangle$ is the dual product on $\mathbb{L}^2(\mathcal{F}, \mathbb{R}^n)$.

Differentiability with respect to measure

- We called $\mathcal{D}_\zeta g(\mu_0)$ the *Fréchet-derivative* of g at μ_0 in the direction ξ. In this case we have

$$\mathcal{D}_\zeta g(\mu_0) = \langle \mathcal{D}\widetilde{g}(X_0) \cdot \zeta \rangle = \frac{\mathrm{d}}{\mathrm{d}t}\widetilde{g}\left(X_0 + t\zeta\right)\Big|_{t=0}, \text{ with } \mu_0 = P_{X_0}. \tag{6}$$

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- By applying *Riesz representation theorem*, there is a unique random variable $\Theta_0 \in \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n)$ such that $\langle \mathcal{D}\widetilde{g}(X_0) \cdot \zeta \rangle = (\Theta_0 \cdot \zeta)_2 = E\left[(\Theta_0 \cdot \zeta)_2\right]$ where $\zeta \in \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n)$.

Differentiability with respect to measure

Definition

We say that the function $g \in \mathbb{C}_b^{1,1}(Q_2(\mathbb{R}^n))$ if for all $X \in \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n)$ there exists a P_X-modification of $\partial_\mu g\,(P_X, \cdot)$ (denoted by $\partial_\mu g$) such that

1. $\partial_\mu g : Q_2\,(\mathbb{R}^n) \times \mathbb{R}^n \to \mathbb{R}^n$ is bounded and Lipschitz continuous. That is for some $C > 0$, it holds that

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2. $\left|\partial_\mu g(\mu, x)\right| \leq C$, $\forall \mu \in Q_2(\mathbb{R}^n)$, $\forall x \in \mathbb{R}^n$.
3. The derivatives $\partial_\mu g$ satisfied the following

$$\left|\partial_\mu g(\mu, x) - \partial_\mu g(\mu', x')\right| \leq C\left[\mathbb{T}\left(\mu, \mu'\right) + \left|x - x'\right|\right], \quad \forall \mu, \mu' \in Q_2(\mathbb{R}^n), \forall x, x' \in \mathbb{R}^n.$$

Second-order derivatives with respect to measure

Definition

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3. The mapps $\partial_x \partial_\mu g(\cdot, \cdot) : Q_2(\mathbb{R}^n) \times \mathbb{R}^n \to \mathbb{R}^n \otimes \mathbb{R}^n$ and $\partial_\mu^2 g(P_{X_0}, y, Z) : Q_2(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \otimes \mathbb{R}^n$ are bounded and Lipshitz continuous, where

$$\partial_\mu^2 g(P_{X_0}, y, Z) = \partial_\mu \left[\partial_\mu g(\cdot, y)\right] (P_{X_0}, Z) .$$

Main results. Assumptions

Assumption (H1) The coefficients f, σ, ℓ, h are measurable in all variables. Moreover, for all $(u(t), \eta(t)) \in \mathbb{U}_1 \times \mathbb{U}_2$, $f(\cdot,\cdot,u)$, $\sigma(\cdot,\cdot,u)$, $\ell(\cdot,\cdot,u) \in \mathbb{C}_b^{1,1}(\mathbb{R} \times Q_2(\mathbb{R}^d);\mathbb{R})$, $h(\cdot,\cdot) \in \mathbb{C}_b^{1,1}(\mathbb{R} \times Q_2(\mathbb{R}^n);\mathbb{R})$. More precisely, for each $u(t) \in \mathbb{U}_1$, denoting $\varphi(x,\mu) = f(t,x,\mu,u)$, $\sigma(t,x,\mu,u)$, $f(t,x,\mu,u)$, $h(x,\mu)$, the function $\varphi(\cdot,\cdot)$ enjoys the following properties:

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2. For fixed $x \in \mathbb{R}$, $\varphi(x,\cdot) \in \mathbb{C}_b^{1,1}(Q_2(\mathbb{R}))$;
3. All the derivatives $\partial_x \varphi$ and $\partial_\mu \varphi : \varphi = f, \sigma, \ell, h$, are bounded and Lipschitz continuous, with Lipschitz constants independent of $(u(t), \eta(t))$.

Main results. Assumptions

Assumption (H2) The coefficients f, σ, ℓ, h satisfy assumption (H1). Furthermore, for all $u(t) \in \mathbb{U}_1$, $f(t, \cdot, \cdot, u)$, $\sigma(t, \cdot, \cdot, u)$, $\ell(t, \cdot, \cdot, u) \in \mathbb{C}_b^{2,1}(\mathbb{R} \times Q_2(\mathbb{R}); \mathbb{R})$, $h(\cdot, \cdot) \in \mathbb{C}_b^{2,1}(\mathbb{R} \times Q_2(\mathbb{R}); \mathbb{R})$. More precisely, for each $u(t) \in \mathbb{U}_1$, the derivatives of f, σ, ℓ, h, denoted by a generic function $\varphi(t, x, \mu)$, enjoy the following properties:

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1. $\partial_x \varphi(t,\cdot,\cdot) \in \mathbb{C}_b^{1,1}(\mathbb{R} \times Q_2(\mathbb{R}))$;
2. $\partial_\mu \varphi(t,\cdot,\cdot) \in \mathbb{C}_b^{1,1}(\mathbb{R} \times Q_2(\mathbb{R}) \times \mathbb{R})$;
3. All the second-order derivatives of f, σ, ℓ, h, are bounded and Lipschitz continuous with Lipschitz constants independent of $(u(t), \eta(t))$.

Main results. Assumptions

Assumption (H3) The functions $G(\cdot) : [0, T] \to \mathbb{R}$, and $M(\cdot) : [0, T] \to \mathbb{R}^{+}$ are continuous and bounded.

Main results. Adjoint equation

First-order adjoint equation. We consider the first-order adjoint equation, which is the following McKean-Vlasov linear BSDE:

$$\begin{cases} -\mathrm{d}p(t) = \Big[f_x(t)p(t) + \widehat{E}\Big[\widehat{f_\mu^*}(t)(t)\widehat{p}(t)\Big] + \sigma_x(t)q(t) + \widehat{E}\Big[\widehat{\sigma}_\mu^*(t)\widehat{q}(t)\Big] \\ \qquad\qquad -\ell_x(t) - \widehat{E}\Big[\widehat{\ell_\mu^*}(t)(t)\Big]\Big]\,\mathrm{d}t - q(t)\mathrm{d}B(t), \\ p(T) = h_x(T) + \widehat{E}[\widehat{h_\mu^*}(T)]. \end{cases} \tag{7}$$

Here, from (**??**), $t \in [0, T]$, for $\varphi = f, \sigma, \ell$, we obtain

$$\begin{aligned} \widehat{E}\left[\partial_\mu \widehat{\varphi^*}(t)\right] &= \widehat{E}\left[\partial_\mu \varphi(t, \widehat{X}(t), P_{X^*(t)}, \widehat{u}^*(t); z)\right]\Bigg|_{z=X^*(t)} \qquad (8) \\ &= \int_{\widehat{\Omega}} \partial_\mu \varphi(t, \widehat{X}(t,\widehat{w}), P_{X^*(t,w)}, \widehat{u}^*(t,\widehat{w}); X^*(t,w))\mathrm{d}\widehat{P}(\widehat{w}), \end{aligned}$$

Main results. Adjoint equation

Second-order adjoint equation. Consider the following standard linear BSDE

$$\begin{cases} \mathrm{d}P(t) = -\Big\{2(b_x(t) + \widehat{E}[\widehat{b}^*_\mu(t)])P(t) + [\sigma_x(t) + \widehat{E}(\widehat{\sigma}^*_\mu(t))]^2 P(t) \\ \qquad\qquad + 2(\sigma_x(t) + \widehat{E}[\widehat{\sigma}^*_\mu(t)])Q(t) + (H_{xx}(t) + \widehat{E}[\widehat{H}^*_{\mu y}(t)])\Big\}\mathrm{d}t \\ \qquad\qquad + Q(t)\mathrm{d}B(t), \\ P(T) = -(h_{xx}(T) + \widehat{E}[\widehat{h}^*_{\mu y}(T)]). \end{cases} \tag{9}$$

Similar to (8) and (**??**), we have

$$\widehat{E}[\widehat{H}^*_{\mu y}(t)]) = \widehat{E}\left[\partial_\mu \partial_y H(t, \widehat{X}(t), P_{X^*(t)}, \widehat{u}^*(t), \widehat{p}(t), \widehat{q}(t); y)\right]\Big|_{y=X^*(t)}$$

$$= \int_{\widehat{\Omega}} \partial_\mu \partial_y H(t, \widehat{X}(t, \widehat{w}), P_{X^*(t)}, \widehat{u}^*(t, \widehat{w}), \widehat{p}(t), \widehat{q}(t); X^*(t))\mathrm{d}\widehat{P}(\widehat{w}$$

Main results. Hamiltonian

Let us define the Hamiltonian associated to our continuous-singular control problem. For any $(t, x, \mu, u, p, q) \in [0, T] \times \mathbb{R} \times Q_2(\mathbb{R}) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$

$$H(t, x, \mu, u, p, q) = f(t, x, \mu, u)p + \sigma(t, x, \mu, u)q - \ell(t, x, \mu, u). \quad (10)$$

where $(p(\cdot), q(\cdot))$ be a pair of adapted processes, solution of the first-order adjoint equation

Main results. Necessary conditions

Let $(u^*(\cdot), \eta^*(\cdot), X^*(\cdot))$ is an optimal solution of the McKean-Vlasov control problem (2)-(1). We introduce the following variational equations for our continuous-singular control problem. Let $Y^{u^\varepsilon,\eta^\varepsilon}(\cdot)$ and $Z^\varepsilon(\cdot)$ be the solutions of (11), (12) associated to $(u^*(\cdot), \eta^*(\cdot))$ respectively.

First-order variational equation: let $E_\varepsilon = [0, \varepsilon]$, $t \in [0, T]$

$$\begin{cases} \mathrm{d}Y^{u^\varepsilon,\eta^\varepsilon}(t) = \Big[f_x(t)Y^{u^\varepsilon,\eta^\varepsilon}(t) + \widehat{E}[\widehat{f}_\mu(t)\widehat{Y}^{u^\varepsilon,\eta^\varepsilon}(t)] + \delta f(t)1_{E_\varepsilon}(t)\Big]\mathrm{d}t \\ \qquad + \Big[\sigma_x(t)Y^\varepsilon(t) + \widehat{E}[\widehat{\sigma}_\mu(t)\widehat{Y}^{u^\varepsilon,\eta^\varepsilon}(t)] + \delta\sigma(t)1_{E_\varepsilon}(t)\Big]\mathrm{d}B(t) \\ \qquad + G(t)\mathrm{d}(\eta^\varepsilon - \eta^*)(t), \\ Y^{u^\varepsilon,\eta^\varepsilon}(0) = 0. \end{cases} \tag{11}$$

Here the process $Y^{u^\varepsilon,\eta^\varepsilon}(\cdot)$ is called the *first-order variational process*, associated to $(u^\varepsilon(\cdot), \eta^\varepsilon(\cdot))$ which is depend explicitly to singular control. The process $\eta^\varepsilon(\cdot)$ is the convex perturbed control given by $\eta^\varepsilon(t) = \eta^*(t) + \varepsilon(\eta(t) - \eta^*(t))$.

Main results. Necessary conditions

Second-order variational equation:

$$\begin{cases} \mathrm{d}Z^{\varepsilon}(t)= \Big[f_x(t)Z^{\varepsilon}(t)+\widehat{E}[\widehat{f_\mu}(t)\widehat{Z}^{\varepsilon}(t)]+\mathcal{L}_{xx}(t,f,Y^{\varepsilon})+\mathcal{L}_{\mu x}(t,\widehat{f},\widehat{Y}^{\varepsilon})\Big]\,dt \\ +\Big[\sigma_x(t)Z^{\varepsilon}(t)+\widehat{E}[\widehat{\sigma}_\mu(t)\widehat{Z}^{\varepsilon}(t)]+\mathcal{L}_{xx}(t,\sigma,Y^{\varepsilon})+\mathcal{L}_{\mu x}(t,\widehat{\sigma},\widehat{Y}^{\varepsilon})\Big]\,dB(t), \\ +\Big[\delta f_x(t)Y^{\varepsilon}(t)+\widehat{E}[\delta\widehat{f_\mu}(t)\widehat{Y}^{\varepsilon}(t)]\Big]\,1_{E_\varepsilon}(t)\,\mathrm{d}t \\ +\Big[\delta\sigma_x(t)Y^{\varepsilon}(t)+\widehat{E}[\delta\widehat{\sigma}_\mu(t)\widehat{Y}^{\varepsilon}(t)]\Big]\,1_{E_\varepsilon}(t)\,\mathrm{d}B(t), \\ Z^{\varepsilon}(0)=0. \end{cases} \tag{12}$$

Here the process $Z^{\varepsilon}(\cdot)$ is called the *second-order variational process.*

Main results. Estimations

Lemma

Let $X^{\varepsilon}(\cdot) = X^{u^{\varepsilon},\eta^{\varepsilon}}(\cdot)$ be the solutions of (2) corresponding to continuous-singular control $(u^{\varepsilon}(\cdot), \eta^{\varepsilon}(\cdot))$. Let assumptions (H1) and (H2) hold. Then we have

$$\lim_{\varepsilon \to 0} E\left(\sup_{t \in [0,T]} |X^{\varepsilon}(t) - X^{*}(t)|^{2} \right) = 0.$$

Let $X^{u^{\varepsilon},\eta^{*}}(\cdot)$ be the solution of (2), corresponding to $(u^{\varepsilon}(\cdot), \eta^{*}(\cdot))$. Let $Y^{\varepsilon}(\cdot)$ be the solution of (**??**), corresponding to $(u^{\varepsilon}(\cdot), \eta^{*}(\cdot))$, then the following estimation holds

Main results. Estimations

Lemma

$$\lim_{\varepsilon\to 0} E\left[\sup_{0\leq t\leq T}\left|X^{u^{\varepsilon},\eta^{*}}(t)-X^{*}(t)\right|^{2}\right]=0. \quad (13)$$

$$\lim_{\varepsilon\to 0} E\left[\sup_{0\leq t\leq T}\left|X^{\varepsilon}(t)-X^{u^{\varepsilon},\eta^{*}}(t)\right|^{2}\right]=0. \quad (14)$$

$$\lim_{\varepsilon\to 0} E\left[\sup_{0\leq t\leq T}\left|X^{u^{\varepsilon},\eta^{*}}(t)-X^{*}(t)-Y^{\varepsilon}(t)\right|^{2}\right]=0. \quad (15)$$

Proposition

Let $Y^{\varepsilon}(t)$ solution (12) associated to $\left(u^{\varepsilon}\left(\cdot\right),\eta^{*}\left(\cdot\right)\right)$. Under assumption H1, the following estimate holds

$$\lim_{\varepsilon\to 0} E\left[\sup_{0\leq t\leq T}\left|X^{u^{\varepsilon},\eta^{*}}(t)-X^{*}(t)-Y^{\varepsilon}(t)-Z^{\varepsilon}(t)\right|^{2}\right]=0. \quad (16)$$

Main results. Stochastic maximum principle

The following theorem constitutes the main contribution of this paper.

Theorem

Theorem 3.1 *(Stochastic maximum principle) Let* $(u^*(\cdot), \eta^*(\cdot), X^*(\cdot))$ *is an optimal solution of the McKean-Vlasov control problem (2)-(1). Let assumptions (H1), (H2) and (H3) hold. Then there are two pairs of* F_t-*adapted processes* $(p(\cdot), q(\cdot))$ *and* $(P(\cdot), Q(\cdot))$ *that satisfy (7) and (9) respectively, such that for all* $(u(t), \eta(t)) \in U_1 \times U_2$*, we have*

$$0 \leq H(t, X^*(t), P_{X^*(t)}, u^*(t), p^*(t), q^*(t)) - H(t, x^*(t), P_{X^*(t)}, u(t), p^*(t), \tag{17}$$

$$-\frac{1}{2}P(t)\left(\sigma(t, X^*(t), P_{X^*(t)}, u(t))) - \sigma\left(t, X^*(t), P_{X^*(t)}, u^*(t)\right)\right)^2$$

$$+ E\int_{[0,T]} (M(t) + G(t)p(t))\mathrm{d}\left(\eta - \eta^*\right)(t).$$

$$P-a.s., \ a.e. \ t \in [0, T].$$

Main results. Stochastic maximum principle

Main results. Stochastic maximum principle

We derive the variational inequality (**??**) in several steps. From the optimality of $\left(u^{*}(\cdot),\eta^{*}(\cdot)\right)$, we have

$$J\left(u^{\varepsilon}(\cdot),\eta^{\varepsilon}(\cdot)\right)-J\left(u^{*}(\cdot),\eta^{*}(\cdot)\right)\geq 0. \tag{18}$$

Now, we separate the above inequality into two parts

$$J_1^{\varepsilon}=J\left(u^{\varepsilon}(\cdot),\eta^{\varepsilon}(\cdot)\right)-J\left(u^{\varepsilon}(\cdot),\eta^{*}(\cdot)\right), \tag{19}$$
$$J_2^{\varepsilon}=J\left(u^{\varepsilon}(\cdot),\eta^{*}(\cdot)\right)-J\left(u^{*}(\cdot),\eta^{*}(\cdot)\right), \tag{20}$$

where $J\left(u^{\varepsilon}(\cdot),\eta^{\varepsilon}(\cdot)\right)-J\left(u^{*}(\cdot),\eta^{*}(\cdot)\right)=J_1^{\varepsilon}+J_2^{\varepsilon}$. The variational inequality will be derived from the fact that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(J_1^{\varepsilon}+J_2^{\varepsilon}\right)\geq 0. \tag{21}$$

Future developments We have discussed a general Peng's type necessary conditions in the form of Pontryagin stochastic maximum principle of optimal continuous-singular control for nonlinear controlled McKean-Vlasov stochastic differential equation. If the coefficients of the singular parts $G(t) = M(t) = 0$, our stochastic maximum principle (Theorem 3.1) coincides with maximum principle developed in Buckdahn et al. [3, Theorem 3.5].
Apparently, there are many problems left unsolved such as:

A. One possible problem is to study the general Peng's type maximum principle for optimal control for SDE, the coefficients of the singular parts $G(\cdot)$ and $M(\cdot)$ depend explicitly to the state of the solution process $X^{u,\eta}$ of the form

$$\begin{cases} dX^{u,\eta}(t) = f\left(t, X^{u,\eta}(t), u(t)\right) dt + \sigma\left(t, X^{u,\eta}(t), u(t)\right) dW(t) \\ \qquad\qquad + G(t, X^{u,\eta}) d\eta(t), \\ X^{u,\eta}(0) = x_0, \end{cases}$$

and the cost functional of the form

$$J\left(u(\cdot),\eta(\cdot)\right)=E\left[\int_0^T f(t,X^{u,\eta}(t),u(t))dt+h(X^{u,\eta}(T))\right.$$
$$\left.+\int_{[0,T]} M(t,X^{u,\eta})d\eta(t)\right].$$

Future developments

B. It would be interesting to investigate the McKean-Vlasov maximum principle (local version via Bensoussan's convex method and general Peng's maximum principle) for optimal continuous-singular control for McKean-Vlasov SDE, the coefficients of the singular parts $G(\cdot)$ and $M(\cdot)$ of the state equation depend on the state of the solution process as well as of its probability law and the control variable.of the form

$$\begin{cases} dX^{u,\eta}(t) = f\left(t, X^{u,\eta}(t), P_{X^{u,\eta}(t)}, u(t)\right) dt + \sigma\left(t, X^{u,\eta}(t), P_{X^{u,\eta}(t)}, \right. \\ \qquad\qquad + G(t, X^{u}, P_{X^{u,\eta}(t)})\mathrm{d}\eta(t), \\ X^{u,\eta}(0) = x_0, \end{cases}$$

and the expected cost has the form

$$J(u(\cdot), \eta(\cdot)) = E\left[\int_0^T f(t, X^{u,\eta}(t), P_{X^{u,\eta}(t)}, u(t))\mathrm{d}t + h(X^{u,\eta}(T), P_{X^{u} \right.$$
$$\left. + \int_{[0,T]} M(t, X^{u,\eta}, P_{X^{u,\eta}(t)})\mathrm{d}\eta(t)\right].$$

Future developments

C. Another challenging problem left unsolved is to derive a various maximum principles in the case where the coefficients f, σ, ℓ, G and M depend on the state of the solution process $X^{u,\eta}(\cdot)$, the continuous control variable $u(\cdot)$ as well as of probability law of the pair $P_{(X^{u,\eta}(t),u(t))}$. So we investigate the problem:

$$\begin{cases} dX^{u,\eta}(t) = f(t, X^{u,\eta}, u(t), P_{(X^{u,\eta}(t),u(t))})\mathrm{d}t + \sigma(t, X^{u,\eta}(t), u(t), P_{(X \\ \qquad\qquad + G(t, X^{u,\eta}, u(t), P_{(X^{u,\eta}(t),u(t))})\mathrm{d}\eta(t), \\ X^{u,\eta}(0) = x_0, \end{cases}$$

and the cost functional has the general form

$$J(u(\cdot), \eta(\cdot)) = E\left[\int_0^T f(t, X^{u,\eta}, u(t), P_{(X^{u,\eta}(t),u(t))})\mathrm{d}t + h(X^{u,\eta}(T), P \right. \\ \left. + \int_{[0,T]} M(t, X^{u,\eta}, u(t), P_{(X^{u,\eta}(t),u(t))})\mathrm{d}\eta(t)\right].$$

We hope to study these interesting new problems in forthcoming works.

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Thank you for your attention

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