The One Step Malliavin scheme

new discretization of BSDEs implemented with deep learning regressions

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9th Colloquium on Backward Stochastic Differential Equations and Mean Field Systems

Annecy

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Overview

Contents

Forward-Backward Stochastic Differential Equations

BSDE – non-linear extension to the **martingale representation theorem**

$$X_t = x_0 + \int_0^t \mu(s, X_s)\mathrm{d}s + \int_0^t \sigma(s, X_s)\mathrm{d}W_s$$

$$Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s.$$

Semi-linear PDEs with terminal boundaries

$$\frac{\partial u}{\partial t} + \langle \mu | \nabla u \rangle + \frac{1}{2}\operatorname{Tr}\left[\sigma\sigma^T \operatorname{Hess} u\right] + f(u, \nabla u) = 0,$$

$$u(T, x) = g(x),$$

General Feynman–Kac relation, (Pardoux and Peng, 1992)

Under certain regularity conditions the solutions coincide $\mathbb{P}$-a.s.

$$Y_t = u(t, X_t), \quad Z_t = (\nabla u \sigma)(t, X_t).$$

Discrete time approximations

- Discretize $\pi^N := \{0 = t_0 < t_1 < \cdots < t_N = T\}$, e.g. $h := T/N, t_n = nh$
- SDE: well-understood, e.g. **Euler–Maruyama** scheme, $n = 0, \ldots, N-1$

$$X_0^\pi = x_0, \qquad X_{n+1}^\pi = X_n^\pi + \mu(t_n, X_n^\pi)\Delta t_n + \sigma(t_n, X_n^\pi)\Delta W_n.$$

- Itô isometry + discretized time integrals

$$Y_N^\pi = g(X_N^\pi), \quad Z_N^\pi = (\sigma\nabla g)(t_N, X_N^\pi),$$

$$Z_n^\pi = \frac{1}{\Delta t_n}\mathbb{E}\left[Y_{n+1}^\pi \Delta W_n \middle| \mathcal{F}_{t_n}\right]$$

$$Y_n^\pi = \mathbb{E}\left[Y_{n+1}^\pi + \Delta t_n f(t_n, X_n^\pi, Y_{n+1}^\pi, Z_n^\pi) \middle| \mathcal{F}_{t_n}\right]$$

one step vs multi step schemes ... implicit schemes require Picard iterations

Take-away

- The main difficulty is the **approximation of Z**
- A standard convergence analysis, e.g. (Bouchard and Touzi, 2004), shows

$$\limsup_{|\pi|\to 0} \frac{1}{|\pi|} \max_{0\le n\le N} \mathbb{E}\left[|\Delta Y_n^\pi|^2\right] + \sum_{n=0}^{N-1} \mathbb{E}\left[\int_{t_n}^{t_{n+1}} |Z_r - Z_{n+1}^\pi|^2 \mathrm{d}r\right] < \infty$$

Contents

WARNING!

The following content is controversial and might be disturbing for some audiences (mostly those of an analysis heavy background)

Viewer discretion is advised!

Malliavin Calculus in Scare Quotes

Differentiation on a Wiener space
Let $W(h) = \int_0^T h^T(t)\mathrm{d}W_t$ with some $h \in L^2([0,T];\mathbb{R}^n)$
Put $\mathcal{R} \subseteq L^2(\Omega,\mathbb{P};\mathbb{R})$ for $\Phi = \varphi(W(h_1),\dots,W(h_d))$ with $\varphi \in C_p^\infty(\mathbb{R}^d;\mathbb{R})$
Define the **Malliavin derivative** of such *smooth* random variables by

$$D_s\Phi := \sum_{i=1}^d \partial_i\varphi(W(h_1),\dots,W(h_d))h_i(s)$$

The derivative operator can be extended to $\mathbb{D}^{1,p}(\Omega,\mathbb{P};\mathbb{R}) \subseteq L^p\ (\Omega,\mathbb{P};\mathbb{R})$ by the closure with respect to the following norm

$$\|\Phi\|_{\mathbb{D}^{1,p}} := \left(|\Phi|^p + \left(\int_0^T |D_s\Phi|^2\mathrm{d}s\right)^{p/2}\right)^{1/p}.$$

Clark-Ocone formula: the predictable adapted process in the martingale representation theorem is the Malliavin derivative itself

$$X(W) = \mathbb{E}\left[X(W)\right] + \int_0^T \mathbb{E}\left[D_tX|\mathcal{F}_t\right]\mathrm{d}W_t$$

Connection with (F)BSDEs

It can be shown that for Itô processes the Malliavin derivative satisfies

$$D_s X_t = \sigma(s, X_s) + \int_s^t (\nabla\mu)(r, X_r)\mathrm{d}r + \int_s^t (\nabla\sigma)(r, X_r)\mathrm{d}W_r.$$

Feynman-Kac $Z_t \sim (\nabla u)(t, X_t) \sim$ **sensitivity** + Clark-Ocone & martingale representation $\implies$ Malliavin derivative? **(Yes.)**

Malliavin Derivative's BSDE, e.g. (Geiss and Steinicke, 2016; Mastrolia et al., 2017)

Under certain regularity conditions $Y \in \mathbb{D}^{1,2}(\mathbb{R}^q), Z \in \mathbb{D}^{1,2}(\mathbb{R}^{q\times d})$

$$D_s Y_t = D_s\xi + \int_t^T [\nabla_x \mathbf{f_r} D_s X_r + \nabla_y \mathbf{f_r} D_s Y_r + \nabla_z \mathbf{f_r} D_s Z_r]\,\mathrm{d}r - \int_t^T D_s Z_r \mathrm{d}W_r, \quad (1)$$

$$D_s Y_t = 0, D_s Z_t = 0, t < s.$$

where $\mathbf{f_r} := f(r, \mathbf{X_r}, \mathbf{Y_r}, \mathbf{Z_r})$. There is a continuous version such that $Z_s = D_s Y_s$.
The control process satisfies a linear BSDE itself.

Malliavin chain rule

Malliavin Chain Rule

Let $\psi \in C_b^1(\mathbb{R}^d; \mathbb{R}^q)$ and $X \in \mathbb{D}^{1,p}(\mathbb{R}^q)$. Then $\psi(F) \in \mathbb{D}^{1,p}(\mathbb{R}^q)$ and for all $0 \leq s \leq T$

$$D_s\psi(X) = \nabla_x\psi(X)D_sX.$$

Recall: Feynman-Kac relations $\implies$ $Y_t = u(t, X_t), Z_t = (\sigma\nabla_x u)(t, X_t)$.

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Objective I

Back to FBSDE systems

$$X_t = \eta + \int_0^t \mu(r, X_r)\mathrm{d}r + \int_0^t \sigma(r, X_r)\mathrm{d}W_r,$$

$$Y_t = g(X_T) + \int_t^T f(r, X_r, Y_r, Z_r)\mathrm{d}r - \int_t^T Z_r \mathrm{d}W_r.$$

But under suitable assumptions also $X \in \mathbb{D}^{1,2}(\mathbb{R}^d)$, $Y \in \mathbb{D}^{1,2}(\mathbb{R}^q)$, $Z \in \mathbb{D}^{1,2}(\mathbb{R}^{q\times n})$ **and** $s \leq t$

$$D_s X_t = \sigma(s, X_s) + \int_s^t \nabla_x \mu(r, X_r) D_s X_r \mathrm{d}r + \int_s^t \nabla_x \sigma(r, X_r) D_s X_r \mathrm{d}W_r,$$

$$D_s Y_t = \nabla_x g(X_T) D_s X_T + \int_t^T \Big[\nabla_x f(r, \mathbf{X}_r) D_s X_r + \nabla_y f(r, \mathbf{X}_r) D_s Y_r + \nabla_z f(r, \mathbf{X}_r) D_s Z_r\Big]\mathrm{d}r - \int_t^T D_s Z_r \mathrm{d}W_r.$$

Objective II

Simultaneous discrete time approximation to the pair of solution triples $\{(X_t, Y_t, Z_t)\}_{0\le t\le T}$, $\{(D_sX_t, D_sY_t, D_sZ_t)\}_{0\le s\le t\le T}$ to the pair of FBSDE systems

Main ingredients

- associate the corresponding Malliavin derivatives in the Malliavin BSDE with the solution pair of the original – Malliavin chain rule

$$D_sY_t = \nabla_x y(t, X_t)D_sX_t, \quad D_sZ_t = \nabla_x z(t, X_t)D_sX_t =: \gamma(t, X_t)D_sX_t$$

- combine this with the non-linear Feynman-Kac formulae

$$\nabla_x y(t, X_t)\sigma(t, X_t) = z(t, X_t)$$

After a suitable time discretization, discrete time estimates read as follows

$$D_nY^\pi_{n+1} := Z^\pi_{n+1}\sigma(t_{n+1}, X^\pi_{n+1})^{-1}D_nX^\pi_{n+1}, \quad D_nZ^\pi_n :=:= \Gamma^\pi_n D_nX^\pi_n$$

OSM discretization

$$D_n Y^\pi_{n+1} := Z^\pi_{n+1}\sigma^{-1}(t_{n+1}, X^\pi_{n+1})D_n X^\pi_{n+1}, \quad D_n Z^\pi_n :=:= \Gamma^\pi_n D_n X^\pi_n$$

Approximate the forward SDEs with Euler-Maruyama approximations and

$$Y^\pi_N = g(X^\pi_N), \quad Z^\pi_N = \nabla_x g(X^\pi_N)\sigma(T, X^\pi_n),$$

$$\begin{aligned}\Gamma^\pi_n \sigma(t_n, X^\pi_n) = D_n Z^\pi_n = \frac{1}{\Delta t_n}\mathbb{E}_n\Big[\Delta W_n\Big\{ & D_n Y^\pi_{n+1} \\ & + \Delta t_n \nabla_x f(t_{n+1}, \mathbf{X}^\pi_{n+1})D_n X^\pi_{n+1} \\ & + \Delta t_n \nabla_y f(t_{n+1}, \mathbf{X}^\pi_{n+1})D_n Y^\pi_{n+1} \\ & + \Delta t_n \nabla_z f(t_{n+1}, \mathbf{X}^\pi_{n+1})D_n Z^\pi_n \Big\}\Big],\end{aligned}$$

$$\begin{aligned}Z^\pi_n = \mathbb{E}_n\Big[& D_n Y^\pi_{n+1} + \Delta t_n \nabla_x f(t_{n+1}, \mathbf{X}^\pi_{n+1})D_n X^\pi_{n+1} \\ & + \Delta t_n \nabla_y f(t_{n+1}, \mathbf{X}^\pi_{n+1})D_n Y^\pi_{n+1} \\ & + \Delta t_n \nabla_z f(t_{n+1}, \mathbf{X}^\pi_{n+1})D_n Z^\pi_n \Big],\end{aligned}$$

$$\begin{aligned}Y^\pi_n = & \vartheta_y \Delta t_n f(t_n, \mathbf{X}^\pi_n) \\ & + \mathbb{E}_n\Big[Y^\pi_{n+1} + (1-\vartheta_y)\Delta t_n f(t_{n+1}, X^\pi_{n+1})\Big]\end{aligned}$$

Discrete time approximation error analysis

Main difficulty is the presence of Γs and their corresponding estimates

1. to make sure of Malliavin differentiability
2. **additive noise**
3. to guarantee uniformly bounded Malliavin derivatives
4. suitable **Lipschitz** (in space) and (1/2)-Hölder (in time) assumptions

Main result, Negyesi et al., 2021

Under suitable assumptions

$$\limsup_{|\pi|\to 0} \frac{1}{|\pi|}\mathcal{E}(|\pi|) < \infty, \tag{3}$$

where

$$\mathcal{E}(|\pi|) := \max_{0\le n\le N} \mathbb{E}\left[|\Delta Y_n^\pi|^2\right] + \max_{0\le n\le N} \mathbb{E}\left[|\Delta Z_n^\pi|^2\right] + \sum_{n=0}^{N-1} \mathbb{E}\left[\int_{t_n}^{t_{n+1}} |\Gamma_r - \Gamma_n^\pi|^2 \mathrm{d}r\right]$$

Sketch of the proof

$$\limsup_{|\pi|\to 0} \frac{1}{|\pi|}\mathcal{E}(|\pi|) < \infty,$$

1. SDEs: $\mathcal{O}(|\pi|^{1/2})$ ✓
2. standard mean-squared continuity result for Y; similar estimates for Z via Malliavin BSDE
3. estimate for the best $L^2(\Omega, \mathbb{P}; \mathbb{R}^{d\times d})$ projections of DZ given π

$$\varepsilon^{DZ}(|\pi|) := \sum_{n=0}^{N-1} \mathbb{E}\left[\int_{t_n}^{t_{n+1}} \left|D_{t_n} Z_r - \widetilde{DZ}_n^{n+1}\right|^2 \mathrm{d}r\right],$$

   with $\widetilde{DZ}_n^{n+1} := \frac{1}{\Delta t_n}\mathbb{E}_n\left[\int_{t_n}^{t_{n+1}} D_{t_n} Z_r \mathrm{d}r\right]$
4. Malliavin chain rule, etc. estimates: (recursive) upper bounds for $\widetilde{DZ}_n^{n+1} - D_n Z_n^\pi$, ΔZ_n^π, ΔY_n^π – uniform boundedness
5. **Grönwall** type estimate for the first two terms
6. Γ: step 3 + Malliavin chain rule estimates

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Two fully-implementable schemes

1. **BCOS**: Fourier cosine expansion methods given analytical conditional characteristic function of the Markov transitions $\Phi_{X_{n+1}^\pi|X_n^\pi=x}(u|x)$ (Ruijter and Oosterlee, 2015) – small d, benchmark in the scalar setting
2. **Deep BSDE**: neural network regression Monte Carlo – similar to Huré et al., 2020. (Y, Z, Γ) are parametrized by (separate) DNNs at each time instance.

$$\begin{aligned}
\mathcal{L}_n^{z,\gamma}(\theta^z,\theta^\gamma) := \mathbb{E}\Big[\Big|&(1+\Delta t_n \nabla_y f(t_{n+1},\widehat{\mathbf{X}}_{n+1}^\pi))D_n\widehat{Y}_{n+1}^\pi \\
&+\Delta t_n \nabla_x f(t_{n+1},\widehat{\mathbf{X}}_{n+1}^\pi)D_n X_{n+1}^\pi - \psi(X_n^\pi|\theta^z) \\
&+\Delta t_n \nabla_z f(t_{n+1},\widehat{\mathbf{X}}_{n+1}^\pi)\chi(X_n^\pi|\theta^\gamma)\sigma(t_n,X_n^\pi) \\
&-\chi(X_n^\pi|\theta^\gamma)\sigma(t_n,X_n^\pi)\Delta W_n\Big|^2\Big],
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_n^{y}(\theta^y) := \mathbb{E}\Big[\Big|&\widehat{Y}_{n+1}^\pi + (1-\vartheta_y)\Delta t_n f(t_{n+1},\widehat{\mathbf{X}}_{n+1}^\pi) - \varphi(X_n^\pi|\theta^y) \\
&+\vartheta_y \Delta t_n f(t_n,X_n^\pi,\varphi(X_n^\pi|\theta^y),\widehat{Z}_n^\pi) - \widehat{Z}_n^\pi \Delta W_n\Big|^2\Big]
\end{aligned}$$

Full error analysis

Assumption: **additive noise**, C_b^2 coefficients

$$\left\|\Phi_t - \widehat{\Phi}^{\pi}_{t_n}\right\|_{H_\Phi} \lesssim \overbrace{\left\|\Phi_t - \Phi_{t_n}\right\|_{H_\Phi}}^{\text{regularity}} + \overbrace{\left\|\Phi_{t_n} - \Phi^{\pi}_n\right\|_{H_\Phi}}^{\text{discretization}} + \overbrace{\left\|\Phi^{\pi}_n - \widehat{\Phi}^{\pi}_n\right\|_{H_\Phi}}^{\text{approximation}}$$
$$+ \overbrace{\text{"}\mathbb{E} - \widehat{\mathbb{E}}\text{"}}^{\text{simulation}} + \overbrace{\text{"}\phi \approx \sum \alpha_k \varphi_k\text{"}}^{\text{regression bias}}$$

In $L^2(\Omega, \mathbb{P}; \cdot)$ related norms

- regularity: $\mathcal{O}(|\pi|^{1/2})$ – *in different norms for* Y, Z, Γ ✓
- discretization: $\mathcal{O}(|\pi|^{1/2})$ – *same as Euler*
- approximation: empirically
- simulation: generally intertwined with regression biases, in ML applications less troublesome due to re-simulation for each SGD iteration
- regression bias: asymptotic result for the cumulative regression bias via a **UAT argument**

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$$+ \overbrace{"\mathbb{E} - \widehat{\mathbb{E}}"}^{\text{simulation}} + \overbrace{"\phi \approx \sum \alpha_k \varphi_k"}^{\text{regression bias}} \leq C(|\pi| + N \sum_{n=0}^{N-1} \{\epsilon^y_n + \epsilon^z_n\} + \sum_{n=0}^{N-1} \epsilon^\gamma_n)$$

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Convergence of total approximation errors

$$\mu = \mathbf{0}_d, \qquad \sigma = I_d, \qquad f(t,x,y,z) = \frac{\omega(t,\lambda x)}{[1+\omega(t,\lambda x)]^2}\left[\lambda^2 d(y-\gamma) - 1 - \frac{\lambda^2}{2}d\right]$$

$$g(x) = \gamma + \frac{\omega(T,\lambda x)}{1+\omega(T,\lambda x)}, \qquad \omega(t,x) = \exp\left(t + \sum_{i=1}^{d} x_i\right)$$

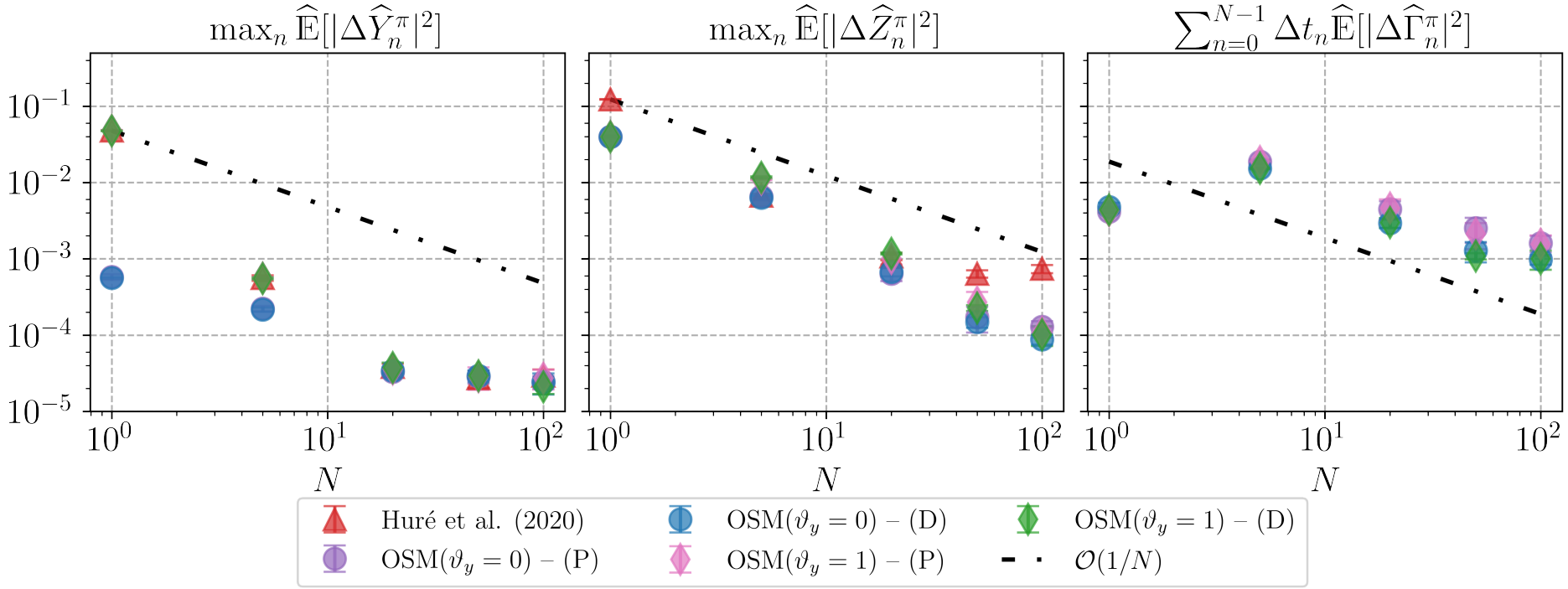

Figure: $d = 10$, Fig.1b in Negyesi et al., 2021.

BCOS benchmarked regression errors

$$\mu = \mathbf{0}_d, \qquad \sigma = \sqrt{2} I_d, \qquad f(t,x,y,z) = |z|^2, \qquad g(x) = x^T A x + v^T x + c,$$

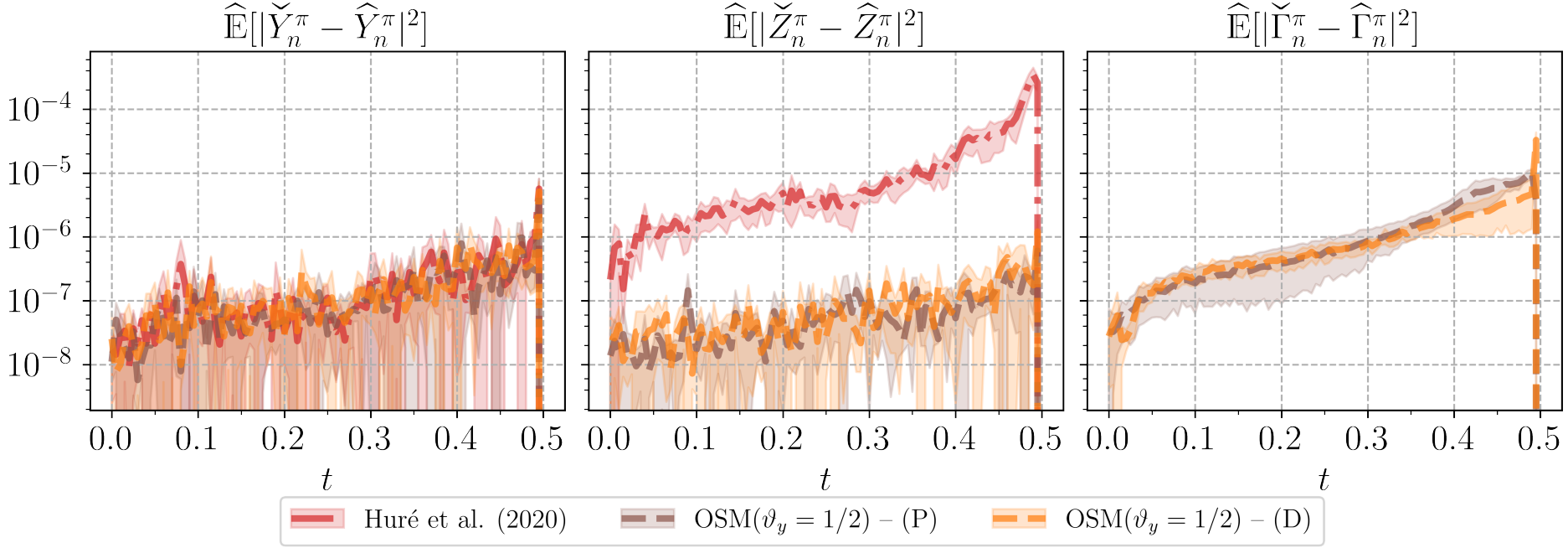

Figure: $d = 1, N = 100$, Fig.2a in Negyesi et al., 2021.

Cumulative regression errors "convergence"

$$\mu = \mathbf{0}_d, \qquad \sigma = \sqrt{2} I_d, \qquad f(t,x,y,z) = |z|^2, \qquad g(x) = x^T A x + v^T x + c,$$

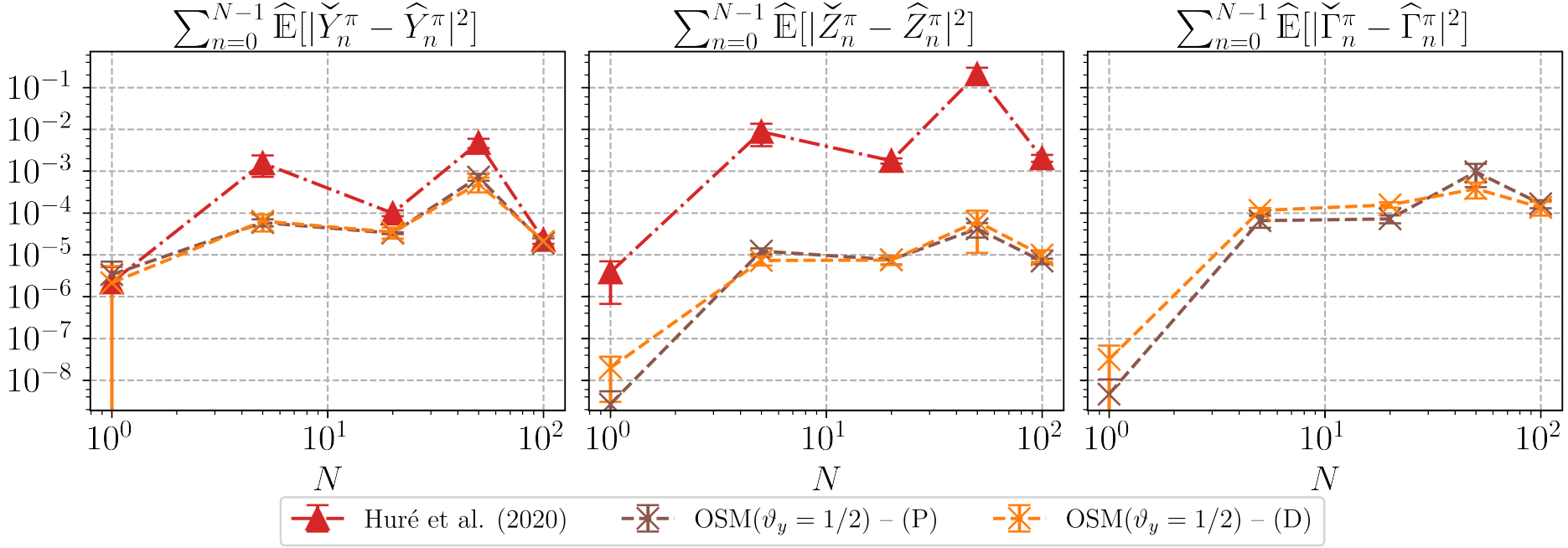

Figure: $d = 1$, Fig.2b in Negyesi et al., 2021.

Total relative approximation errors over time

$$\mu = \mathbf{0}_d, \qquad \sigma = \sqrt{2} I_d, \qquad f(t,x,y,z) = |z|^2, \qquad g(x) = x^T A x + v^T x + c,$$

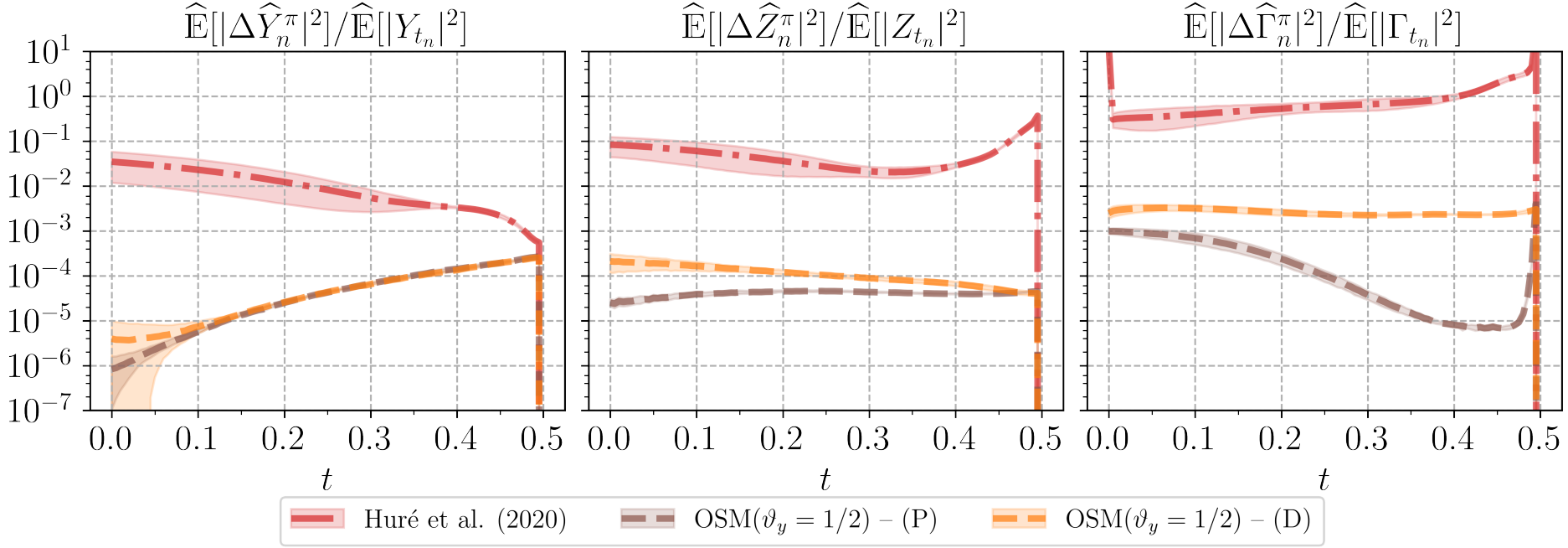

Figure: $d = 50$, $N = 100$, Fig.3a in Negyesi et al., 2021.

Comparison with the Deep BSDE

Comparison with Huré et al., 2020 – whose Γ estimates are computed via naive automatic differentiation.

Table: $d = 50, N = 100$.

	OSM($\vartheta_y = 1/2$) (P)	(D)	Huré et al. (2020)
$\lvert\Delta \widehat{Y}_0^\pi\rvert/\lvert Y_0\rvert$	$8 \times 10^{-4}\,(5 \times 10^{-4})$	$1 \times 10^{-3}\,(1 \times 10^{-3})$	$1.7 \times 10^{-1}\,(8 \times 10^{-2})$
$\lvert\Delta \widehat{Z}_0^\pi\rvert/\lvert Z_0\rvert$	$5.0 \times 10^{-3}\,(5 \times 10^{-4})$	$1.4 \times 10^{-2}\,(3 \times 10^{-3})$	$2.8 \times 10^{-1}\,(7 \times 10^{-2})$
$\lvert\Delta \widehat{\Gamma}_0^\pi\rvert/\lvert \Gamma_0\rvert$	$3.1 \times 10^{-2}\,(2 \times 10^{-3})$	$4.9 \times 10^{-2}\,(7 \times 10^{-3})$	$3.5\,(1 \times 10^{-1})$
$\max_n \widehat{\mathbb{E}}[\lvert\Delta \widehat{Y}_n^\pi\rvert^2]$	$2.7\,(1 \times 10^{-1})$	$2.5\,(3 \times 10^{-1})$	$7 \times 10^{1}\,(4 \times 10^{1})$
$\max_n \widehat{\mathbb{E}}[\lvert\Delta \widehat{Z}_n^\pi\rvert^2]$	$3.4 \times 10^{-2}\,(1 \times 10^{-3})$	$3.1 \times 10^{-2}\,(3 \times 10^{-3})$	$2.8 \times 10^{2}\,(1 \times 10^{1})$
$\sum_{n=0}^{N-1} \Delta t_n \widehat{\mathbb{E}}[\lvert\Delta \widehat{\Gamma}_n^\pi\rvert^2]$	$4.1 \times 10^{-4}\,(6 \times 10^{-5})$	$3.3 \times 10^{-3}\,(2 \times 10^{-4})$	$2.9\,(2 \times 10^{-1})$
runtime (s)	$1.36 \times 10^{3}\,(1 \times 10^{1})$	$1.62 \times 10^{3}\,(4 \times 10^{1})$	$6.16 \times 10^{2}\,(1 \times 10^{1})$

Contents

Summary

- The main difficulty in the numerical approximation of BSDEs lies in the Z process, whose conditional variance diverges in the classical backward Euler scheme
- The One-Step Malliavin (OSM) scheme is built on a linear BSDE representation of the control process given by Malliavin calculus
- The discrete time approximations of OSM follow from a merged formulation of the Malliavin chain rule and the non-linear Feynman-Kac formulae
- OSM includes second-order sensitivities, Γs, via $\{D_s Z_t\}_{s \leq t}$
- The OSM scheme exhibits *"optimal"* convergence rate $\mathcal{O}(|\pi|^{1/2})$
- A Deep BSDE approach yields orders of magnitude better approximations than in the classical discretization framework

. . . future self. . .

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Malliavin BSDE in the literature I

Alternative approaches related to (1)

- Turkedjiev, 2015: (Malliavin) integration by parts formulas leading to

$$Z_t = \mathbb{E}\left[g(X_T) H_T^t + \int_t^T f(s, X_s, Y_s, Z_s) H_s^t \mathrm{d}s \middle| \mathcal{F}_t \right],$$

  with **Malliavin weights** defined by

$$H_r^s := \frac{1}{r-s} \int_s^r \sigma^{-1}(t, X_t) D_s X_t \mathrm{d}W_t$$

- Hu et al., 2011: linear BSDEs admit to a representation formula
- Briand and Labart, 2014: Wiener chaos expansion on Y, differentiable estimates giving control estimates via the identity $Z_t = D_t Y_t$

General diffusions I

Remarks pointing in the direction of general diffusion coefficients

- $D_n X^\pi_{n+1}$ converge with the same order as the Euler scheme

$$D_n X^\pi_m := \begin{cases} \mathbb{1}_{m=n}\sigma(t_n, X^\pi_n), & 0 \le m \le n \le N, \\ D_n X^\pi_{m-1} + \nabla_x \mu(t_{m-1}, X^\pi_{m-1}) D_n X^\pi_{m-1} \Delta t_{m-1} & \\ \qquad + \nabla_x \sigma(t_{m-1}, X^\pi_{m-1}) D_n X^\pi_{m-1} \Delta W_{m-1}, & 0 \le n < m \le N. \end{cases}$$

  guarantees $\limsup_{|\pi|\to 0} \frac{1}{|\pi|}\mathbb{E}\left[\left|D_{t_n} X_{t_{n+1}} - D_n X^\pi_{n+1}\right|^2\right] < \infty$.

- Girsanov theorem allows to treat a wide range of non-constant drifts by changing to an appropriate probability measure: with some suitable H the Doléans-Dade exponential is defined as follows

$$\mathcal{E}^H_t := \exp\left(\int_0^t H_r \mathrm{d}W_r - \frac{1}{2}\int_0^t |H_r|^2 \mathrm{d}r\right),$$

General diffusions II

defining a new probability measure $\mathbb{Q}^H$ by the Radon-Nikodym derivative

$$\mathrm{d}\mathbb{Q}^H = \mathcal{E}_t^H \mathrm{d}\mathbb{P}.$$

Then $B_t^H := W_t - \int_0^t H_r^T \mathrm{d}r$ is a Brownian motion under $\mathbb{Q}^H$.

- The main difficulty with the diffusion component is induced by the Malliavin chain rule approximation *(uniform boundedness, . . .)*. Preliminary **empirical** results can be found in Negyesi et al., 2021 suggesting that under suitable differentiability assumptions the convergence rate of (3) is preserved.

$$\begin{aligned} D_{t_n} Y_{t_{n+1}} - D_n Y_{n+1}^\pi = & \left[Z_{t_{n+1}} \sigma^{-1}(t_{n+1}, X_{t_{n+1}}) - Z_{n+1}^\pi \sigma^{-1}(t_{n+1}, X_{n+1}^\pi) \right] D_{t_n} X_{t_{n+1}} \\ & + Z_{n+1}^\pi \sigma^{-1}(t_{n+1}, X_{n+1}^\pi) \left[D_{t_n} X_{t_{n+1}} - D_n X_{n+1}^\pi \right]. \end{aligned}$$

uniform boundedness

General diffusions III

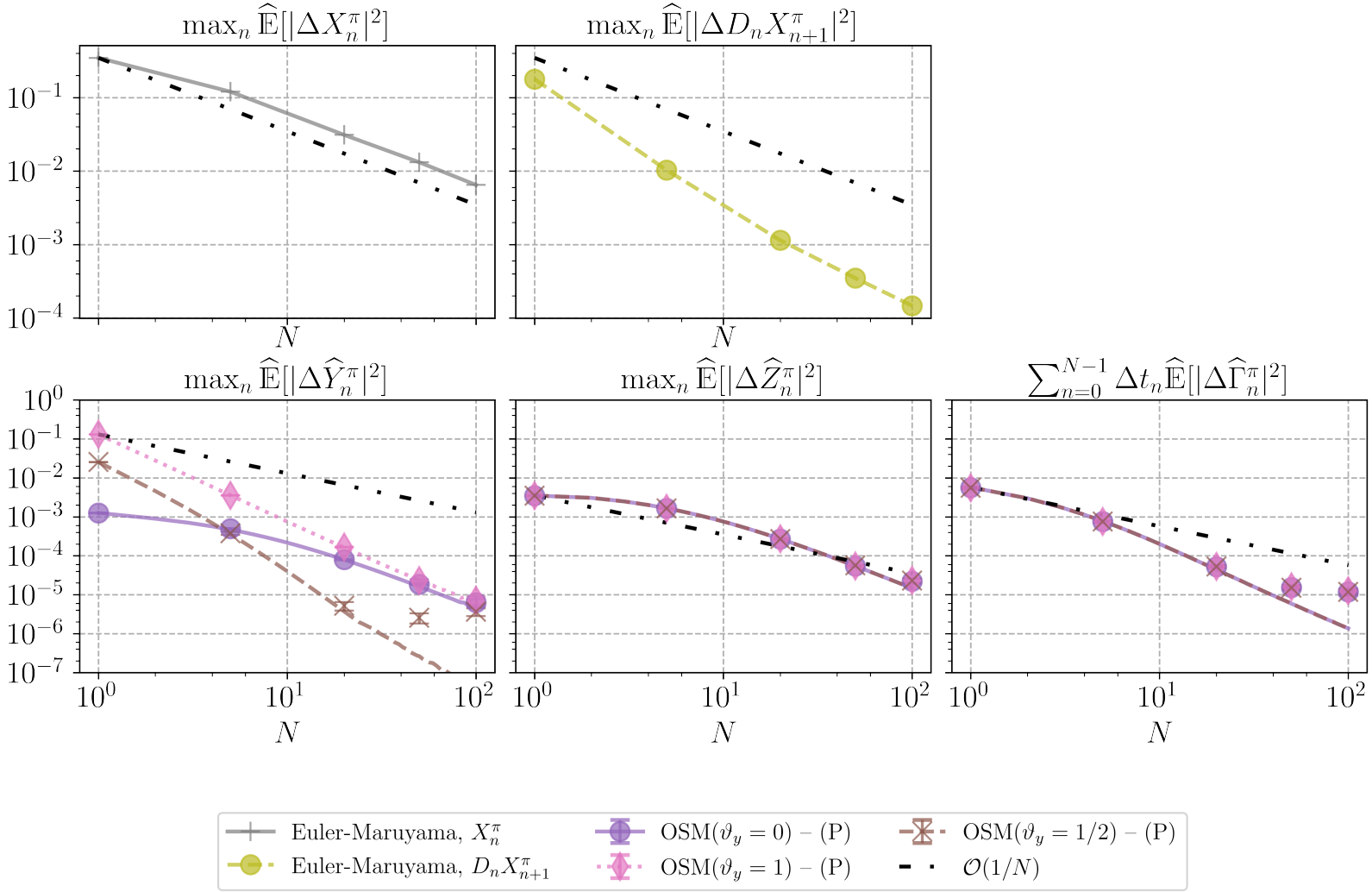

Figure: $d = 50$, $N = 100$, Fig.4 in Negyesi et al., 2021.

OSM-(D) vs OSM-(P)

Two versions of Deep BSDE, depending on whether $\Gamma \leftarrow \chi(\cdot|\theta^\gamma)$ or if it's fixed as the derivative of $\nabla_x \psi(\cdot|\theta^z)$ – similar to Huré et al., 2020
In case of the latter the loss function needs to be adjusted accordingly

$$\begin{aligned}\mathcal{L}_n^{z,\nabla z}(\theta^z) := \mathbb{E}\Big[\Big|&(1+\Delta t_n \nabla_y f(t_{n+1}, \widehat{\mathbf{X}}_{n+1}^\pi))D_n \widehat{Y}_{n+1}^\pi + \Delta t_n \nabla_x f(t_{n+1}, \widehat{\mathbf{X}}_{n+1}^\pi) D_n X_{n+1}^\pi \\ &- \psi(X_n^\pi|\theta^z) + \Delta t_n \nabla_z f(t_{n+1}, \widehat{\mathbf{X}}_{n+1}^\pi) \nabla_x \psi(X_n^\pi|\theta^z)\sigma(t_n, X_n^\pi) \\ &- \nabla_x \psi(X_n^\pi|\theta^z)\sigma(t_n, X_n^\pi)\Delta W_n\Big|^2\Big].\end{aligned}$$

Error analysis: extra term for the Jacobian condition, depending on the bounds of the network and its derivatives.