The One Step Malliavin scheme
new discretization of BSDEs implemented with deep learning regressions

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Annecy

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Overview

1. Discrete time approximation of BSDEs
2. Malliavin calculus in scare quotes
3. One Step Malliavin scheme
4. Fully-implementable schemes
5. Summary
Forward-Backward Stochastic Differential Equations

BSDE – non-linear extension to the martingale representation theorem

\[ X_t = x_0 + \int_0^t \mu(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s \]

\[ Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s. \]

Semi-linear PDEs with terminal boundaries

\[ \frac{\partial u}{\partial t} + \langle \mu | \nabla u \rangle + \frac{1}{2} \text{Tr} \left[ \sigma \sigma^T \text{Hess} \ u \right] + f(u, \nabla u) = 0, \]

\[ u(T, x) = g(x), \]

General Feynman–Kac relation, (Pardoux and Peng, 1992)

Under certain regularity conditions the solutions coincide \( P \)-a.s.

\[ Y_t = u(t, X_t), \quad Z_t = (\nabla u \sigma)(t, X_t). \]
Discrete time approximations

- Discretize $\pi^N := \{0 = t_0 < t_1 < \cdots < t_N = T\}$, e.g. $h := T/N, t_n = nh$

- SDE: well-understood, e.g. Euler–Maruyama scheme, $n = 0, \ldots, N - 1$

$$X_0^\pi = x_0, \quad X_{n+1}^\pi = X_n^\pi + \mu(t_n, X_n^\pi)\Delta t_n + \sigma(t_n, X_n^\pi)\Delta W_n.$$ 

- Itô isometry + discretized time integrals

$$Y_N^\pi = g(X_N^\pi), \quad Z_N^\pi = (\sigma \nabla g)(t_N, X_N^\pi),$$

$$Z_n^\pi = \frac{1}{\Delta t_n} \mathbb{E} \left[ Y_{n+1}^\pi \Delta W_n | \mathcal{F}_{t_n} \right],$$

$$Y_n^\pi = \mathbb{E} \left[ Y_{n+1}^\pi + \Delta t_n f(t_n, X_n^\pi, Y_n^\pi, Z_n^\pi) | \mathcal{F}_{t_n} \right]$$

one step vs multi step schemes ... implicit schemes require Picard iterations

Take-away

- The main difficulty is the approximation of $Z$

- A standard convergence analysis, e.g. (Bouchard and Touzi, 2004), shows

$$\limsup_{|\pi| \to 0} \frac{1}{|\pi|} \max_{0 \leq n \leq N} \mathbb{E} \left[ |\Delta Y_n^\pi|^2 \right] + \sum_{n=0}^{N-1} \mathbb{E} \left[ \int_{t_n}^{t_{n+1}} |Z_r - Z_{n+1}^\pi|^2 dr \right] < \infty$$
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WARNING!

The following content is controversial and might be disturbing for some audiences (mostly those of an analysis heavy background)

Viewer discretion is advised!
Malliavin Calculus in Scare Quotes

Differentiation on a Wiener space

Let $W(h) = \int_0^T h^T(t) dW_t$ with some $h \in L^2([0, T]; \mathbb{R}^n)$

Put $\mathcal{R} \subseteq L^2(\Omega, \mathbb{P}; \mathbb{R})$ for $\Phi = \varphi(W(h_1), \ldots, W(h_d))$ with $\varphi \in C^\infty_{p} (\mathbb{R}^d; \mathbb{R})$

Define the Malliavin derivative of such smooth random variables by

$$D_s \Phi := \sum_{i=1}^{d} \partial_i \varphi(W(h_1), \ldots, W(h_d)) h_i(s)$$

The derivative operator can be extended to $\mathcal{D}^{1,p}(\Omega, \mathbb{P}; \mathbb{R}) \subseteq L^p (\Omega, \mathbb{P}; \mathbb{R})$ by the closure with respect to the following norm

$$\|\Phi\|_{\mathcal{D}^{1,p}} := \left( |\Phi|^p + \left( \int_0^T |D_s \Phi|^2 ds \right)^{p/2} \right)^{1/p}.$$

Clark-Ocone formula: the predictable adapted process in the martingale representation theorem is the Malliavin derivative itself

$$X(W) = \mathbb{E} [X(W)] + \int_0^T \mathbb{E} [D_t X | \mathcal{F}_t] dW_t$$
Connection with (F)BSDEs

It can be shown that for Itô processes the Malliavin derivative satisfies

\[ D_s X_t = \sigma(s, X_s) + \int_s^t (\nabla \mu)(r, X_r)dr + \int_s^t (\nabla \sigma)(r, X_r)dW_r. \]

Feynman-Kac \( Z_t \sim (\nabla u)(t, X_t) \sim \text{sensitivity} \) + Clark-Ocone & martingale representation \( \Rightarrow \) Malliavin derivative? (Yes.)

Malliavin Derivative’s BSDE, e.g. (Geiss and Steinicke, 2016; Mastrolia et al., 2017)

Under certain regularity conditions \( Y \in D^{1,2}(\mathbb{R}^q), Z \in D^{1,2}(\mathbb{R}^{q \times d}) \)

\[
D_s Y_t = D_s \xi + \int_t^T \left[ \nabla_x f_r D_s X_r + \nabla_y f_r D_s Y_r + \nabla_z f_r D_s Z_r \right] dr - \int_t^T D_s Z_r dW_r, \tag{1}
\]

\[ D_s Y_t = 0, D_s Z_t = 0, t < s. \]

where \( f_r := f(r, X_r, Y_r, Z_r) \). There is a continuous version such that \( Z_s = D_s Y_s \).

The control process satisfies a linear BSDE itself.
Malliavin chain rule

Let $\psi \in C^1_b(\mathbb{R}^d; \mathbb{R}^q)$ and $X \in \mathbb{D}^{1,p}(\mathbb{R}^q)$. Then $\psi(F) \in \mathbb{D}^{1,p}(\mathbb{R}^q)$ and for all $0 \leq s \leq T$

$$D_s \psi(X) = \nabla_x \psi(X) D_s X.$$ 

Recall: Feynman-Kac relations $\implies Y_t = u(t, X_t)$, $Z_t = (\sigma \nabla_x u)(t, X_t)$. 
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Back to FBSDE systems

\[ X_t = \eta + \int_0^t \mu(r, X_r)dr + \int_0^t \sigma(r, X_r)dW_r, \]

\[ Y_t = g(X_T) + \int_t^T f(r, X_r, Y_r, Z_r)dr - \int_t^T Z_r dW_r. \]

But under suitable assumptions also \( X \in D^{1,2}(\mathbb{R}^d), \ Y \in D^{1,2}(\mathbb{R}^q), \ Z \in D^{1,2}(\mathbb{R}^{q \times n}) \) and \( s \leq t \)

\[ D_sX_t = \sigma(s, X_s) + \int_s^t \nabla_x \mu(r, X_r)D_sX_rdr + \int_s^t \nabla_x \sigma(r, X_r)D_sX_r dW_r, \]

\[ D_sY_t = \nabla_x g(X_T)D_sX_T + \int_t^T \left[ \nabla_x f(r, X_r)D_sX_r + \nabla_y f(r, X_r)D_sY_r + \nabla_z f(r, X_r)D_sZ_r \right]dr - \int_t^T D_sZ_r dW_r. \]
Simultaneous discrete time approximation to the pair of solution triples
\{(X_t, Y_t, Z_t)\}_{0 \leq t \leq T}, \{(D_s X_t, D_s Y_t, D_s Z_t)\}_{0 \leq s \leq t \leq T} to the pair of FBSDE systems

Main ingredients

- associate the corresponding Malliavin derivatives in the Malliavin BSDE with the solution pair of the original – Malliavin chain rule

\[ D_s Y_t = \nabla_x y(t, X_t) D_s X_t, \quad D_s Z_t = \nabla_x z(t, X_t) D_s X_t =: \gamma(t, X_t) D_s X_t \]

- combine this with the non-linear Feynman-Kac formulae

\[ \nabla_x y(t, X_t) \sigma(t, X_t) = z(t, X_t) \]

After a suitable time discretization, discrete time estimates read as follows

\[ D_n Y_{n+1}^\pi := Z_{n+1}^\pi \sigma(t_{n+1}, X_{n+1}^\pi)^{-1} D_n X_{n+1}^\pi, \quad D_n Z_n^\pi := :\Gamma_n^\pi D_n X_n^\pi \]
\[ D_n Y_{n+1}^{\pi} := Z_{n+1}^{\pi} \sigma^{-1}(t_{n+1}, X_{n+1}^{\pi}) D_n X_{n+1}^{\pi}, \quad D_n Z_{n}^{\pi} := \Gamma_{n}^{\pi} D_n X_{n}^{\pi} \]

Approximate the forward SDEs with Euler-Maruyama approximations and

\[ Y_{N}^{\pi} = g(X_{N}^{\pi}), \quad Z_{N}^{\pi} = \nabla_{x} g(X_{N}^{\pi}) \sigma(T, X_{n}^{\pi}), \]

\[ \Gamma_{n}^{\pi} \sigma(t_{n}, X_{n}^{\pi}) = D_{n} Z_{n}^{\pi} = \frac{1}{\Delta t_{n}} \mathbb{E}_{n} \left[ \Delta W_{n} \left\{ D_{n} Y_{n+1}^{\pi} + \Delta t_{n} \nabla_{x} f(t_{n+1}, X_{n+1}^{\pi}) D_{n} X_{n+1}^{\pi} + \Delta t_{n} \nabla_{y} f(t_{n+1}, X_{n+1}^{\pi}) D_{n} Y_{n+1}^{\pi} + \Delta t_{n} \nabla_{z} f(t_{n+1}, X_{n+1}^{\pi}) D_{n} Z_{n}^{\pi} \right\} \right], \]

\[ Z_{n}^{\pi} = \mathbb{E}_{n} \left[ D_{n} Y_{n+1}^{\pi} + \Delta t_{n} \nabla_{x} f(t_{n+1}, X_{n+1}^{\pi}) D_{n} X_{n+1}^{\pi} + \Delta t_{n} \nabla_{y} f(t_{n+1}, X_{n+1}^{\pi}) D_{n} Y_{n+1}^{\pi} + \Delta t_{n} \nabla_{z} f(t_{n+1}, X_{n+1}^{\pi}) D_{n} Z_{n}^{\pi} \right], \]

\[ Y_{n}^{\pi} = \vartheta_{y} \Delta t_{n} f(t_{n}, X_{n}^{\pi}) + \mathbb{E}_{n} \left[ Y_{n+1}^{\pi} + (1 - \vartheta_{y}) \Delta t_{n} f(t_{n+1}, X_{n+1}^{\pi}) \right] \]
Discrete time approximation error analysis

Main difficulty is the presence of $\Gamma$s and their corresponding estimates

1. to make sure of Malliavin differentiability
2. additive noise
3. to guarantee uniformly bounded Malliavin derivatives
4. suitable Lipschitz (in space) and $(1/2)$-Hölder (in time) assumptions

Main result, Negyesi et al., 2021

Under suitable assumptions

$$\lim_{|\pi| \to 0} \sup_{|\pi|} \frac{1}{|\pi|} \mathcal{E}(|\pi|) < \infty,$$

where

$$\mathcal{E}(|\pi|) := \max_{0 \leq n \leq N} \mathbb{E} \left[ |\Delta Y_n^\pi|^2 \right] + \max_{0 \leq n \leq N} \mathbb{E} \left[ |\Delta Z_n^\pi|^2 \right] + \sum_{n=0}^{N-1} \mathbb{E} \left[ \int_{t_n}^{t_{n+1}} |\Gamma_r - \Gamma_n^\pi|^2 dr \right]$$
Sketch of the proof

\[
\limsup_{|\pi| \to 0} \frac{1}{|\pi|} \mathcal{E}(|\pi|) < \infty,
\]

1. **SDEs:** $O(|\pi|^{1/2}) \checkmark$

2. Standard mean-squared continuity result for $Y$; similar estimates for $Z$ via Malliavin BSDE

3. Estimate for the best $L^2(\Omega, \mathbb{P}; \mathbb{R}^{d \times d})$ projections of $DZ$ given $\pi$

   \[
   \mathcal{E}^{DZ}(|\pi|) := \sum_{n=0}^{N-1} \mathbb{E} \left[ \int_{t_n}^{t_{n+1}} \left| D_{t_n} Z_r - \tilde{DZ}_{n+1}^n \right|^2 dr \right],
   \]

   with $\tilde{DZ}_{n+1}^n := \frac{1}{\Delta t_n} \mathbb{E}_n \left[ \int_{t_n}^{t_{n+1}} D_{t_n} Z_r dr \right]$

4. Malliavin chain rule, etc. estimates: (recursive) upper bounds for $\tilde{DZ}_{n+1}^n - D_n Z_n^\pi, \Delta Z_n^\pi, \Delta Y_n^\pi$ – uniform boundedness

5. **Grönwall** type estimate for the first two terms

6. $\Gamma$: step 3 + Malliavin chain rule estimates
Two fully-implementable schemes

1. **BCOS**: Fourier cosine expansion methods given analytical conditional characteristic function of the Markov transitions $\Phi_{X_{n+1}|X_n = x}(u|x)$ (Ruijter and Oosterlee, 2015) – small $d$, benchmark in the scalar setting

2. **Deep BSDE**: neural network regression Monte Carlo – similar to Huré et al., 2020. $(Y, Z, \Gamma)$ are parametrized by (separate) DNNs at each time instance.

$$
\mathcal{L}^{Z,\gamma}(\theta^Z, \theta^\gamma) := \mathbb{E} \left[ (1 + \Delta t_n \nabla_y f(t_{n+1}, \hat{X}_{n+1}^{\pi}))D_n \hat{Y}_{n+1}^{\pi} \\
+ \Delta t_n \nabla_x f(t_{n+1}, \hat{X}_{n+1}^{\pi})D_n X_{n+1}^{\pi} - \psi(X_n^{\pi}|\theta^Z) \\
+ \Delta t_n \nabla_z f(t_{n+1}, \hat{X}_{n+1}^{\pi})\chi(X_n^{\pi}|\theta^\gamma)\sigma(t_n, X_n^{\pi}) \\
- \chi(X_n^{\pi}|\theta^\gamma)\sigma(t_n, X_n^{\pi})\Delta W_n \right]^2,
$$

$$
\mathcal{L}^Y(\theta^Y) := \mathbb{E} \left[ \hat{Y}_{n+1}^{\pi} + (1 - \vartheta_Y)\Delta t_n f(t_{n+1}, \hat{X}_{n+1}^{\pi}) - \varphi(X_n^{\pi}|\theta^Y) \\
+ \vartheta_Y \Delta t_n f(t_n, X_n^{\pi}, \varphi(X_n^{\pi}|\theta^Y), \hat{Z}_n^{\pi}) - \hat{Z}_n^{\pi} \Delta W_n \right]^2,
$$
Full error analysis

Assumption: **additive noise**, $C^2_b$ coefficients

$$\left\| \Phi_t - \hat{\Phi}_{t_n} \right\|_{H_{\Phi}} \leq \left\| \Phi_t - \Phi_{t_n} \right\|_{H_{\Phi}} + \left\| \Phi_{t_n} - \Phi_n \right\|_{H_{\Phi}} + \left\| \Phi_n - \hat{\Phi}_n \right\|_{H_{\Phi}}$$

- **regularity**: $O(|\pi|^{1/2})$ – in different norms for $Y, Z, \Gamma$
- **discretization**: $O(|\pi|^{1/2})$ – same as Euler
- **approximation**: empirically
- **simulation**: generally intertwined with regression biases, in ML applications less troublesome due to re-simulation for each SGD iteration
- **regression bias**: asymptotic result for the cumulative regression bias via a UAT argument

In $L^2(\Omega, \mathbb{P}; \cdot)$ related norms
Full error analysis

Assumption: **additive noise**, $C^2_b$ coefficients

\[
\|\Phi_t - \Phi_{t_n}\|_{H_{\Phi}} \lesssim \|\Phi_t - \Phi_{t_n}\|_{H_{\Phi}} + \|\Phi_{t_n} - \Phi_n\|_{H_{\Phi}} + \|\Phi_n - \Phi_{\pi n}\|_{H_{\Phi}}
\]

- **regularity**
- **discretization**
- **approximation**

\[
\text{simulation} + \text{"E} - \hat{\text{E}}" + \text{"} \phi \approx \sum \alpha_k \varphi_k" \]

In $L^2(\Omega, P; \cdot)$ related norms

- regularity: $O(|\pi|^{1/2})$ – *in different norms* for $Y, Z, \Gamma$
- discretization: $O(|\pi|^{1/2})$ – *same as Euler*
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\[
\| \Phi_t - \hat{\Phi}_{t_n} \|_{H_\Phi} \lesssim \| \Phi_t - \Phi_{t_n} \|_{H_\Phi} + \| \Phi_{t_n} - \Phi_n^{\pi} \|_{H_\Phi} + \| \Phi_n^{\pi} - \hat{\Phi}_n^{\pi} \|_{H_\Phi} \\
+ "E - \hat{E}" + "\phi \approx \sum \alpha_k \varphi_k" \\
\]

In $L^2(\Omega, \mathbb{P}; \cdot)$ related norms

- **regularity**: $O(|\pi|^{1/2})$ – in different norms for $Y, Z, \Gamma$
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\[
\|\Phi_t - \hat{\Phi}_{t_n}\|_{H_\Phi} \lesssim \left\{ \begin{array}{l}
\text{regularity} \\
\text{discretization} \\
\text{approximation} \\
\text{simulation} \\
\text{regression bias}
\end{array} \right.
\]

+ $\|\Phi_t - \Phi_{t_n}\|_{H_\Phi}$
+ $\|\Phi_{t_n} - \Phi_n\|_{H_\Phi}$
+ $\|\Phi_n - \hat{\Phi}_n\|_{H_\Phi}$
+ $\|E - \hat{E}\|$
+ $\|\phi \approx \sum \alpha_k \varphi_k\|$

In $L^2(\Omega, \mathbb{P}; \cdot)$ related norms

- **regularity**: $O(|\pi|^{1/2})$ — **in different norms for $Y, Z, \Gamma$**
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Assumption: **additive noise**, $C^2_b$ coefficients

\[
\begin{align*}
\left\| \Phi_t - \Phi_{\pi t_n} \right\|_{H_\Phi} & \lesssim \left\| \Phi_t - \Phi_{t_n} \right\|_{H_\Phi} + \left\| \Phi_{t_n} - \Phi_{\pi n} \right\|_{H_\Phi} + \left\| \Phi_{\pi n} - \Phi_{\pi n} \right\|_{H_\Phi} \\
& \quad + \left\| \psi - \hat{\phi} \right\| + \left\| \phi \approx \sum \alpha_k \varphi_k \right\|
\end{align*}
\]

In $L^2(\Omega, \mathbb{P}; \cdot)$ related norms

- regularity: $O(|\pi|^{1/2})$ – *in different norms for $Y, Z, \Gamma$*
- discretization: $O(|\pi|^{1/2})$ – *same as Euler*
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Assumption: **additive noise**, $C^2_b$ coefficients

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\]

- **regularity**
- **discretization**
- **approximation**
- simulation: \"E - \hat{E}\"
- regression bias: \"$\phi \approx \sum \alpha_k \varphi_k$\"

In $L^2(\Omega, \mathbb{P}; \cdot)$ related norms

- **regularity**: $O(|\pi|^{1/2})$ – *in different norms for $Y$, $Z$, $\Gamma$*
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\left\| \Phi_t - \Phi_{t_n}^\pi \right\|_{H_\Phi} \lesssim \left\| \Phi_t - \Phi_{t_n} \right\|_{H_\Phi} + \left\| \Phi_{t_n} - \Phi_n^\pi \right\|_{H_\Phi} + \left\| \Phi_n^\pi - \Phi_n \right\|_{H_\Phi}
\]

\[
= \underbrace{\left\| \Phi_t - \phi \right\|_{H_\Phi}}_{\text{simulation}} + \underbrace{\left\| \phi \right\|_{H_\Phi}}_{\text{regression bias}} = \underbrace{\left\| \Phi_t - \phi \right\|_{H_\Phi}}_{\text{simulation}} + \underbrace{\left\| \phi \right\|_{H_\Phi}}_{\text{regression bias}} \leq C(|\pi| + N \sum_{n=0}^{N-1} \{\epsilon_n^Y + \epsilon_n^Z\} + \sum_{n=0}^{N-1} \epsilon_n^\gamma)}
\]

In $L^2(\Omega, \mathbb{P}; \cdot)$ related norms

- **regularity**: $O(|\pi|^{1/2})$ – in different norms for $Y, Z, \Gamma$
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Convergence of total approximation errors

\[ \mu = 0_d, \quad \sigma = I_d, \quad f(t, x, y, z) = \frac{\omega(t, \lambda x)}{[1 + \omega(t, \lambda x)]^2} \left[ \lambda^2 d(y - \gamma) - 1 - \frac{\lambda^2}{2} d \right] \]

\[ g(x) = \gamma + \frac{\omega(T, \lambda x)}{1 + \omega(T, \lambda x)}, \quad \omega(t, x) = \exp \left( t + \sum_{i=1}^{d} x_i \right) \]

\[ \max_n \mathbb{E}[|\Delta \hat{Y}_{ni}|^2] \quad \max_n \mathbb{E}[|\Delta \hat{Z}_{ni}|^2] \quad \sum_{n=0}^{N-1} \Delta t_n \mathbb{E}[|\Delta \hat{\Gamma}_{ni}|^2] \]

Figure: \( d = 10 \), Fig.1b in Negyesi et al., 2021.
BCOS benchmarked regression errors

$$\mu = 0_d, \quad \sigma = \sqrt{2} I_d, \quad f(t, x, y, z) = |z|^2, \quad g(x) = x^T A x + v^T x + c,$$

Figure: $d = 1, N = 100$, Fig.2a in Negyesi et al., 2021.
Cumulative regression errors "convergence"

\[ \mu = 0_d, \quad \sigma = \sqrt{2l_d}, \quad f(t, x, y, z) = |z|^2, \quad g(x) = x^T A x + v^T x + c, \]

\[
\sum_{n=0}^{N-1} \mathbb{E}[|\hat{Y}^\pi_n - \hat{Y}^\pi|^2] \\
\sum_{n=0}^{N-1} \mathbb{E}[|\hat{Z}^\pi_n - \hat{Z}^\pi|^2] \\
\sum_{n=0}^{N-1} \mathbb{E}[|\hat{\Gamma}_n - \hat{\Gamma}_n|^2]
\]

Figure: \( d = 1 \), Fig.2b in Negyesi et al., 2021.
Total relative approximation errors over time

\[ \mu = 0_d, \quad \sigma = \sqrt{2}l_d, \quad f(t, x, y, z) = |z|^2, \quad g(x) = x^T A x + \nu^T x + c, \]

Figure: \( d = 50, N = 100, \) Fig.3a in Negyesi et al., 2021.
Comparison with the Deep BSDE

Comparison with Huré et al., 2020 – whose $\Gamma$ estimates are computed via naive automatic differentiation.

Table: $d = 50$, $N = 100$.

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<td>$</td>
<td>\Delta \hat{Y}_0^\pi</td>
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<td>\Delta \hat{Z}_0^\pi</td>
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<td>max$_n \hat{E}[</td>
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<td>^2]$</td>
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<td>$\sum_{n=0}^{N-1} \Delta t_n \hat{E}[</td>
<td>\Delta \hat{\Gamma}_n</td>
<td>]$</td>
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<tr>
<td>runtime (s)</td>
<td>$1.36 \times 10^3 \ (1 \times 10^1)$</td>
<td>$1.62 \times 10^3 \ (4 \times 10^1)$</td>
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The main difficulty in the numerical approximation of BSDEs lies in the $Z$ process, whose conditional variance diverges in the classical backward Euler scheme.

The One-Step Malliavin (OSM) scheme is built on a linear BSDE representation of the control process given by Malliavin calculus.

The discrete time approximations of OSM follow from a merged formulation of the Malliavin chain rule and the non-linear Feynman-Kac formulae.

OSM includes second-order sensitivities, $Γ_s$, via $\{D_sZ_t\}_{s≤t}$.

The OSM scheme exhibits "optimal" convergence rate $O(|π|^{1/2})$.

A Deep BSDE approach yields orders of magnitude better approximations than in the classical discretization framework.

...future self...


Alternative approaches related to (1)

- Turkedjiev, 2015: (Malliavin) integration by parts formulas leading to

\[ Z_t = \mathbb{E} \left[ g(X_T)H_T^t + \int_t^T f(s, X_s, Y_s, Z_s)H_s^t ds \bigg| \mathcal{F}_t \right] , \]

with **Malliavin weights** defined by

\[ H_s^r := \frac{1}{r-s} \int_s^r \sigma^{-1}(t, X_t)D_sX_t dW_t \]

- Hu et al., 2011: linear BSDEs admit to a representation formula

- Briand and Labart, 2014: Wiener chaos expansion on \( Y \), differentiable estimates giving control estimates via the identity \( Z_t = D_t Y_t \)
Remarks pointing in the direction of general diffusion coefficients

- $D_n X_{n+1}^\pi$ converge with the same order as the Euler scheme

\[
D_n X_m^\pi := \begin{cases}
1_{m=n} \sigma(t_n, X_n^\pi), & 0 \leq m \leq n \leq N,
D_n X_{m-1}^\pi + \nabla_x \mu(t_{m-1}, X_{m-1}^\pi) D_n X_{m-1}^\pi \Delta t_{m-1} \\
+ \nabla_x \sigma(t_{m-1}, X_{m-1}^\pi) D_n X_{m-1}^\pi \Delta W_{m-1}, & 0 \leq n < m \leq N.
\end{cases}
\]

Guarantees

\[
\limsup_{|\pi| \to 0} \frac{1}{|\pi|} \mathbb{E} \left[ \left| D_t X_{t_{n+1}} - D_n X_{n+1}^\pi \right|^2 \right] < \infty.
\]

- Girsanov theorem allows to treat a wide range of non-constant drifts by changing to an appropriate probability measure: with some suitable $H$ the Doléans-Dade exponential is defined as follows

\[
\mathcal{E}_t^H := \exp \left( \int_0^t H_r dW_r - \frac{1}{2} \int_0^t |H_r|^2 dr \right),
\]
defining a new probability measure $Q^H$ by the Radon-Nikodym derivative
\[ dQ^H = E^H_t dP. \]

Then $B^H_t := W_t - \int_0^t H_r^T dr$ is a Brownian motion under $Q^H$.

The main difficulty with the diffusion component is induced by the Malliavin chain rule approximation (uniform boundedness, . . .). Preliminary empirical results can be found in Negyesi et al., 2021 suggesting that under suitable differentiability assumptions the convergence rate of (3) is preserved.

\[
D_t Y_{t+1} - D_n Y_{n+1}^{\pi} = \left[ Z_{t+1} \sigma^{-1}(t_{n+1}, X_{t_{n+1}}) - Z_{n+1} \sigma^{-1}(t_{n+1}, X_{n+1}^{\pi}) \right] D_t X_{t_{n+1}} \\
+ Z_{n+1} \sigma^{-1}(t_{n+1}, X_{n+1}^{\pi}) \left[ D_t X_{t_{n+1}} - D_n X_{n+1}^{\pi} \right].
\]
Figure: $d = 50, N = 100$, Fig.4 in Negyesi et al., 2021.
Two versions of Deep BSDE, depending on whether \( \Gamma \leftarrow \chi(\cdot|\theta^\gamma) \) or if it’s fixed as the derivative of \( \nabla_x \psi(\cdot|\theta^z) \) – similar to Huré et al., 2020

In case of the latter the loss function needs to be adjusted accordingly

\[
L_n^{z,\nabla z}(\theta^z) := \mathbb{E} \left[ \left| (1 + \Delta t_n \nabla_y f(t_{n+1}, \hat{X}^\pi_{n+1})) D_n \hat{Y}^\pi_{n+1} + \Delta t_n \nabla_x f(t_{n+1}, \hat{X}^\pi_{n+1}) D_n X^\pi_{n+1} \right. \right.
\]
\[
- \psi(X^\pi_n|\theta^z) + \Delta t_n \nabla z f(t_{n+1}, \hat{X}^\pi_{n+1}) \nabla_x \psi(X^\pi_n|\theta^z) \sigma(t_n, X^\pi_n) \]
\[
- \nabla_x \psi(X^\pi_n|\theta^z) \sigma(t_n, X^\pi_n) \Delta W_n \left| \right|_2^2. \]

Error analysis: extra term for the Jacobian condition, depending on the bounds of the network and its derivatives.