Differentiability of Quadratic Forward-Backward Stochastic Differential Equations (QFBSDE) with rough drift

Rhoss B. Likibi Pellat
Visiting researcher at Humboldt University of Berlin
joint work with O. Menoukeu Pamen (U.Liverpool/AIMS-Ghana) and P. Imkeller(HU-Berlin)

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Outlines

Main results:

Consider the Markovian Forward-Backward SDE

$$\begin{cases} X_t^x(\omega) & = x + \int_0^t b(s, X_s^x(\omega))\mathrm{d}s + W_t, \\ Y_t^x(\omega) & = \phi(X_T^x(\omega)) + \int_t^T g(s, X_s^x(\omega), Y_s, Z_s)\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s \end{cases} \tag{1}$$

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(iii) The explicit convergence rate holds for any $p \geq 2$

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |Y_t^n - Y_t|^{2p} + \Big(\int_0^T |Z_t^n - Z_t|^2 \mathrm{d}t\Big)^p\Big] \leq C(p,\kappa)n^{\frac{-\kappa}{4q}}.$$

where $\kappa > 0$.

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  $\rightarrow$ $|g(t,x,y,z)| \leq C(1+|y|+(1+|y|^k)|z|+|z|^2), \quad k \in \mathbb{N}$.

Regularity of the coefficients

Assume that (b, σ, ϕ, g) satisfy:

→ $|(b,\sigma)(t,x)| \leq C(1+|x|)$ $b, \sigma \in C_b^1$ and b', σ' are **Lipschitz** in x

→ ϕ, g continuously differentiable with bounded derivatives.

Then the Malliavin derivatives of (Y^x, Z^x) solution to (2) satisfies:

$$D_\theta Y_t^x = \nabla_x \phi(X_T^x) D_\theta X_T^x - \int_t^T D_\theta Z_s^x \mathrm{d}B_s + \int_t^T \nabla_x g(s, X_s^x, Y_s^x, Z_s^x) D_\theta X_s^x \mathrm{d}s$$

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IMPORTANT:

Good regularities of $(b, \sigma) + (\phi, g) \implies$ well posedness of (3)

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Bounded and measurable drift

The forward equation in (1) has a unique strong solution $X_t^x \in L^2(\Omega; W^{1,2}_{loc}(\mathbb{R}^d))$ *O. Menoukeu Pamen et al. (2013), S.E.A. Mohammed et al. (2015)* s.t. $\forall p \geq 1$ $\sup_{x\in\mathbb{R}} \sup_{s\in[0,t]} \mathbb{E}\left[|D_s X_t^x|^p\right] < \infty$

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Standing assumptions

- Given $T > 0$ and $d \in \mathbb{N}\backslash\{0\}$,

$$Y_t = \xi + \int_t^T g(s, \omega, Y_s, Z_s)\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s. \tag{4}$$

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(A2) $g : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is $\mathfrak{F}$-predictable and continuous in (y, z) and $\|g(t, 0, 0)\|_{L^\infty} \leq \Lambda_0$, $\alpha \in (0, 1)$ $\mathbb{P}$-a.s.

$$\begin{aligned}|g(t, \cdot, y, z) - g(t, \cdot, y', z')| \leq \Lambda_y \Big(1 + |z|^\alpha + |z'|^\alpha\Big)|y - y'| \\ + \Lambda_z \Big(1 + (f(|y|) + f(|y'|))(|z| + |z'|)\Big)|z - z'|\end{aligned}$$

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- Under (A1) and (A2) g satisfies:

$$|g(t, \cdot, y, z)| \leq \Lambda_0 + \Lambda_y|y| + \Lambda_z(|z| + f(|y|)|z|^2) \text{ a.s.}$$

Solvability and bounds

→ Bahlali(2019)

Theorem (Solvability)

Under (A1) and (A2), the BSDE (4) *has a unique solution* $(Y, Z) \in \mathcal{S}^\infty(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^d)$.

- Explicit bounds for (Y, Z)

Lemma (Bounds)

Under (A1) and (A2) we have

$$\|Y\|_{\mathcal{S}^\infty} \leq \Upsilon^{(1)} := (\|\xi\|_{L^\infty} + \Lambda_0 T)e^{\Lambda_y T},$$

$$\|Z * B\|_{BMO} \leq \Upsilon^{(2)} := 2\Upsilon^{(1)}\Big(\Upsilon^{(1)} + T(\Lambda_0 + \Lambda_z + \Lambda_y \Upsilon^{(1)})\Big) \exp(4\|(1 + \Lambda_z f)\|_{L^1[0,\Upsilon}$$

- $\|M\|_{BMO} = \sup_\tau \|\mathbb{E}[\langle M\rangle_T - \langle M\rangle_\tau]/\mathfrak{F}_\tau\|_\infty^{1/2} < \infty$,

Additional assumptions

(B1) $\xi \in \mathbb{D}^{1,\infty}$, g is continuously differentiable in (y,z), $\alpha \in (0,1)$

$$|\nabla_y g(t,y,z)| \leq \Lambda_y(1+|z|^{\alpha}) \text{ a.s.},$$
$$|\nabla_z g(t,y,z)| \leq \Lambda_z(1+f(|y|)|z|) \text{ a.s.},$$

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(B1) $\xi \in \mathbb{D}^{1,\infty}$, g is continuously differentiable in (y,z), $\alpha \in (0,1)$

$$|\nabla_y g(t,y,z)| \leq \Lambda_y(1+|z|^{\alpha}) \text{ a.s.},$$
$$|\nabla_z g(t,y,z)| \leq \Lambda_z(1+f(|y|)|z|) \text{ a.s.},$$

(B2) $(g(t,y,z))_{t\in[0,T]} \in \mathbb{L}_{1,2p}(\mathbb{R})$, $\exists$ $(K_u(t))_{u,t\in[0,T]}$, $(\tilde{K}_u(t))_{u,t\in[0,T]}$ s.t.

$$\int_0^T \sup_{0\leq t\leq T} \mathbb{E}|K_u(t)|^{2p}\mathrm{d}u + \|\tilde{K}_u(t)\|^{2p}_{\mathcal{S}^{2p}} < \infty,$$

$$|D_u g(t,y,z)| \leq K_u(t)(1+|y|+f(|y|)|z|^{\alpha}) + \tilde{K}_u(t)(1+|z|^{\alpha}+f(|y|)|z|) \text{ a.s.}$$

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Theorem

Let (A1),(A2), (B1) and (B2) be in force. Set $\Theta = (X,Y,Z)$, then, if $t \in [\theta, T]$,

$$D_\theta Y_t = D_\theta \xi - \int_t^T D_\theta Z_s \mathrm{d}B_s + \int_t^T (D_\theta g)(s,\Theta_s)\mathrm{d}s + \int_t^T \langle (\nabla g)(s,\Theta_s), D_\theta \Theta_s\rangle \mathrm{d}s$$

Moreover, $\{D_t Y_t : 0 \leq t \leq T\}$ is a version of $\{Z_t : 0 \leq t \leq T\}$.

Consider $(Y^n, Z^n)_{n\geq 1}$ satisfying the BSDE

$$Y_t^n = \xi + \int_t^T g_n(s, Y_s^n, Z_s^n)\mathrm{d}s - \int_t^T Z_s^n \mathrm{d}B_s, \tag{5}$$

where $(g_n)_{n\in\mathbb{N}}$:

$$g_n(t, y, z) := g(t, \tilde{\rho}_n(y), \rho_n(z)) \tag{6}$$

$\rho_n : \mathbb{R}^d \to \mathbb{R}^d, z \mapsto \rho_n(z) = (\tilde{\rho}_n(z_1), \dots, \tilde{\rho}_n(z_d)), n \in \mathbb{N}.$

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$$\tilde{\rho}_n(x) = \begin{cases} n+1, & x > n+2, \\ x, & |x| \leq n, \\ -(n+1), & x < -(n+2). \end{cases} \tag{7}$$

$\nabla\tilde{\rho}_n$ uniformly bounded by 1, and converges to 1 locally uniformly.

- Uniform bounds of $(Y^n, Z^n)_{n\in\mathbb{N}}$

Lemma

For each $n \in \mathbb{N}$, *the BSDE* (5) *has a unique solution* $(Y^n, Z^n) \in \mathcal{S}^\infty(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^d)$. *In addition, the process* $Z^n \in \mathcal{H}_{BMO}$, *and* $\sup_{n\in\mathbb{N}} \|\mathcal{E}(Z^n * B)\|_{BMO} \leq \Upsilon^{(2)}$. *There exists* $r > 1$ *independent of* n *such that* $\sup_{n\in\mathbb{N}} \|\mathcal{E}(Z^n * B)\|_{L^r} < \infty$.

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- Uniform $\mathbb{L}_{1,2} \times (\mathbb{L}_{1,2})^d$ norms of $(Y^n, Z^n)_{n\in\mathbb{N}}$.

Lemma

Suppose $\xi \in \mathbb{D}^{1,\infty}$. Then the solution $\Theta^n = (Y^n, Z^n)_{n\in\mathbb{N}}$ to BSDE (5) belongs to $\mathbb{L}_{1,2} \times (\mathbb{L}_{1,2})^d$. A version of $\{(D_u Y_t^n, D_u Z_t^n) : 0 \leq u, t \leq T\}$ is given by

$$\begin{aligned} D_u Y_t^n =& 0 \text{ and } D_u Z_t^n = 0, \text{ if } t \in [0, u), \\ D_u Y_t^n =& D_u\xi - \int_t^T D_u Z_s^n \mathrm{d}B_s \\ & + \int_t^T [(D_u g_n)(s, \Theta_s^n) + \langle (\nabla g_n)(s, \Theta_s^n), D_u \Theta_s^n \rangle] \,\mathrm{d}s, \text{ if } t \in [u, T]. \end{aligned} \tag{8}$$

$$\begin{cases} X_t^x(\omega) & = x + \int_0^t b(s, X_s^x(\omega))\mathrm{d}s + W_t, \\ Y_t^x(\omega) & = \phi(X_T^x(\omega)) + \int_t^T g(s, X_s^x(\omega), Y_s, Z_s)\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s \end{cases}$$

(AX): b is bounded and measurable.

(AY): $\phi : \mathbb{R} \to \mathbb{R}$ is continuous, measurable and uniformly bounded; g is a measurable function: $\|g(t,0,0,0)\|_\infty \leq \Lambda_0$

$$|g(t,x,y,z) - g(t,x',y,z)| \leq \Lambda_x(1 + |y| + [f(|y|)|z|]^\alpha)|x - x'|,$$
$$|g(t,x,y,z) - g(t,x,y',z')| \leq \Lambda_y(1 + (|z| + |z'|))|y - y'| + \Lambda_z(1 + (f(|y|) + f(|y'|))(|z| + |z'|))|z - z'|),$$

where $f \in L^1_{loc}(\mathbb{R}, \mathbb{R}_+)$ is locally bounded and non-decreasing.

AY1): ϕ and g are continuously differentiable in (x,y,z) and for $\alpha \in (0,1)$

$$|\nabla_x g(t,x,y,z)| \leq \Lambda_x(1 + |y| + [f(|y|)|z|]^\alpha),$$
$$|\nabla_y g(t,x,y,z)| \leq \Lambda_y(1 + |z|^\alpha),$$
$$|\nabla_z g(t,x,y,z)| \leq \Lambda_z(1 + f(|y|)|z|),$$
$$|\nabla_x \phi| \leq \Lambda_\phi(1 + |x|),$$

Theorem (R.L.P.,P. Imkeller & O.Menoukeu-Pamen)

Under (AX), (AY) and (AY1) a version of $(D_u Y_t^x, D_u Z_t^x)_{u,t\in[0,T]}$ *is the unique solution to the BSDE*

$$D_u Y_t^x = \nabla_x \phi(X_T^x) D_u X_T^x - \int_t^T D_u Z_s^x \mathrm{d}B_s + \int_t^T \nabla_x g(s, X_s^x, Y_s^x, Z_s^x) D_u X_s^x \mathrm{d}s + \int_t^T \nabla_y g(s, X_s^x, Y_s^x, Z_s^x) D_u Y_s^x \mathrm{d}s + \int_t^T \nabla_z g(s, X_s^x, Y_s^x, Z_s^x) D_u Z_s^x \mathrm{d}s,$$

where $D_u X^x$ *is the Malliavin derivative of the process* X^x*.*

- We assume now

(AX1) $\exists \beta \in (0,1)$ s.t. $b \in L^\infty([0,T]; C_b^\beta(\mathbb{R}^d;\mathbb{R}^d))$,

$\rightarrow$ $L^\infty([0,T]; C_b^\beta(\mathbb{R}^d;\mathbb{R}^d))$ is the space of vector fields $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ having all components in $L^\infty([0,T]; C_b^\beta(\mathbb{R}^d))$ and $L^\infty([0,T]; C_b^\beta(\mathbb{R}^d))$ stands for the set of all bounded Borel functions $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ such that

$$[b]_{\beta,T} = \sup_{t\in[0,T]} \sup_{x\neq y\in\mathbb{R}^d} \frac{|b(t,x) - b(t,y)|}{|x-y|^\beta} < \infty$$

Lemma

Under Assumption (AX1), the solution (X_t^x) to the forward equation is Malliavin differentiable and for any $p \geq 2$

$$\sup_{0\leq s\leq t} \mathbb{E}\Big[\sup_{s\leq t\leq T} |D_s X_t^x|^p\Big] < \infty. \tag{9}$$

Sketch of proof

- Fix $\lambda > 0$ $u_\lambda \in L^\infty([0,\infty); C_b^{2+\beta}(\mathbb{R}^d))$ (*F.Flandoli et al. (2009)*) solves:

$$\partial_t u_\lambda + \mathcal{L}u_\lambda - \lambda u_\lambda = -b, \quad (t,x) \in [0,+\infty) \times \mathbb{R}^d,$$

  where $\mathcal{L}u_\lambda = 1/2\Delta u_\lambda + b \cdot Du_\lambda$.
- For λ large enough, $\Psi_\lambda(t,x) = x + u_\lambda(t,x)$ is C^2-diffeomorphism
- Let us consider the following SDE

$$\tilde{X}_t = y + \int_s^t \tilde{b}(v,\tilde{X}_v)\mathrm{d}v + \int_s^t \tilde{\sigma}(v,\tilde{X}_v)\mathrm{d}B_v \quad t \in [s,T] \tag{10}$$

  where $\tilde{b}(t,y) = -\lambda u_\lambda(t,\Psi_\lambda^{-1}(t,y))$ and $\tilde{\sigma}(t,y) = D\Psi_\lambda(t,\Psi_\lambda^{-1}(t,y))$.
- From Theorem 2.2.1 in Nualart we have $\forall p \geq 2$

$$\sup_{0\leq s\leq t} \mathbb{E}[\sup_{s\leq t\leq T} |D_s\tilde{X}_t|^p] < \infty.$$

- The chain rule for Malliavin calculus

$$\mathbb{E}[\sup_{s\leq t\leq T} |D_sX_t|^p] \leq C\mathbb{E}[\sup_{s\leq t\leq T} |D_s\tilde{X}_t|^p] < \infty.$$

Representation of the derivatives

- Under (AX) or (AX1), the representation of $D_sX_t^x$ is missing in the literature. Indeed,

$$D_sX_t^x = 1_{s\leq t}\left(I_d + \int_s^t b'(s,X_s)D_sX_v^x \mathrm{d}v\right)$$

ill-posed!! since b' is just a distribution

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ill-posed!! since b' is just a distribution

- In one dimension: Under (AX), the following holds for $s \leq t$

$$D_sX_t^x = \exp\left(-\int_s^t \int_{\mathbb{R}} b(s,y)L^X(\mathrm{d}s,\mathrm{d}y)\right)$$

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- In multi-dimension: Under (AX1),for any $\beta' \in (0,\beta)$ we have: *K. Lê (2020)*

$$(D_s^iX_t^x)^j = \delta_{i,j} + \sum_{k=1}^d \int_s^t (D_s^iX_v^x)^j \mathrm{d}V_v^{k,j}(b,X), \quad \forall i,j \in \{1,\cdots,d\}.$$

where $V_t^{k,j}(b,X) = \mathcal{A}_t^X[\partial_k b^j]$ for every $t \ge 0$ and $j,k \in \{1,\cdots,d\}$

Theorem (R.L.P., P. Imkeller & O. Menoukeu-Pamen)

Suppose (AX1), (AY) and (AY1) a version of $(D_u Y_t^x, D_u Z_t^x)_{u,t\in[0,T]}$ *is the unique solution to the BSDE*

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\begin{aligned}
D_u Y_t^x = \nabla_x \phi(X_T^x) D_u X_T^x - \int_t^T D_u Z_s^x \mathrm{d}B_s + \int_t^T \nabla_x g(s, X_s^x, Y_s^x, Z_s^x) D_u X_s^x \mathrm{d}s,\\
+ \int_t^T \nabla_z g(s, X_s^x, Y_s^x, Z_s^x) D_u Z_s^x \mathrm{d}s + \int_t^T \nabla_y g(s, X_s^x, Y_s^x, Z_s^x) D_u Y_s^x \mathrm{d}s,
\end{aligned}
$$

Moreover, $\{D_t Y_t^x : 0 \le t \le T\}$ *is continuous a version of* $\{Z_t^x : 0 \le t \le T\}$ *and*

$$
\begin{aligned}
D_u Y_t^x (\nabla_x X_u^x) &= \nabla_x Y_t^x,\\
Z_t^x (\nabla_x X_t^x) &= \nabla_x Y_t^x,\\
D_u Z_t^x (\nabla_x X_t^x) &= \nabla_x Z_t^x
\end{aligned}
$$

Zhang's path regularity

- $\delta_N = \{t_i : 0 = t_0 < \cdots < t_N = T\}$ and $|\delta_N| = \max_{0\leq i\leq N} |t_{i+1} - t_i|$

Lemma

Under (AX1), (AY) and (AY1) it holds: For all $p \geq 2$,

$$\sum_{i=0}^{N-1} \mathbb{E}\Big[\Big(\int_{t_i}^{t_{i+1}} |Z_t - Z_{t_i}|^2 \mathrm{d}t\Big)^{p/2}\Big] \leq C(p)|\delta_N|^{p/2}$$

where

$$\tilde{Z}_{t_i}^{\delta_N} = \frac{1}{t_{i+1} - t_i}\mathbb{E}\Big[\int_{t_i}^{t_{i+1}} Z_s \mathrm{d}s \Big/ \mathfrak{F}_{t_i}\Big],$$

Convergence rate

- We consider this family of truncated BSDE

$$Y_t^n = \phi(X_T) + \int_t^T g_n(s, X_s, Y_s^n, Z_s^n)\mathrm{d}s - \int_t^T Z_s^n \mathrm{d}B_s, \tag{11}$$

where: $g_n(t,x,y,z) = g(t,x,\tilde{\rho}_n(y),\rho_n(z))$ and $\tilde{\rho}_n$ is given by (7)

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where: $g_n(t,x,y,z) = g(t,x,\tilde{\rho}_n(y),\rho_n(z))$ and $\tilde{\rho}_n$ is given by (7)

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{t\in[0,T]} |Y_t^n - Y_t|^{2p} + \Big(\int_0^T |Z_t^n - Z_t|^2 \mathrm{d}t\Big)^p\Big] \\
&\leq \Pi\Big[\mathbb{E}^{\mathbb{Q}^n}\Big(\sup_{t\in[0,T]} |Y_t^n - Y_t|^{2pq} + \Big(\int_0^T |Z_t^n - Z_t|^2 \mathrm{d}t\Big)^{pq}\Big)\Big]^{\frac{1}{q}}, \\
&\leq C\Big(\mathbb{E}^{\mathbb{Q}^n}\Big[\int_0^T \Big(|\tilde{\rho}(Y_s^n) - Y_s^n| + |\rho(Z_s^n) - Z_s^n|\Big)^2 \mathrm{d}s\Big]^{2pq}\Big)^{\frac{1}{2q}}
\end{aligned}$$

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where: $g_n(t, x, y, z) = g(t, x, \tilde{\rho}_n(y), \rho_n(z))$ and $\tilde{\rho}_n$ is given by (7)

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |Y_t^n - Y_t|^{2p} + \Big(\int_0^T |Z_t^n - Z_t|^2 \mathrm{d}t\Big)^p\Big]$$

$$\leq \Pi\Big[\mathbb{E}^{\mathbb{Q}^n}\Big(\sup_{t\in[0,T]} |Y_t^n - Y_t|^{2pq} + \Big(\int_0^T |Z_t^n - Z_t|^2 \mathrm{d}t\Big)^{pq}\Big)\Big]^{\frac{1}{q}},$$

$$\leq C\Big(\mathbb{E}^{\mathbb{Q}^n}\Big[\int_0^T \Big(|\tilde{\rho}(Y_s^n) - Y_s^n| + |\rho(Z_s^n) - Z_s^n|\Big)^2 \mathrm{d}s\Big]^{2pq}\Big)^{\frac{1}{2q}}$$

- From properties of $\tilde{\rho}$ and the uniform boundedness of Y^n we have:

$$\Big(|\tilde{\rho}(Y_s^n) - Y_s^n| + |\rho(Z_s^n) - Z_s^n|\Big)^2 \leq 8((\Upsilon^{(1)})^2 \mathbb{I}_{\{|Y_s^n| > n\}} + |Z_s^n|^2 \mathbb{I}_{\{|Z_s^n| > n\}})$$

Thanks for your attention!
Merci pour votre attention!