Differentiability of Quadratic Forward-Backward Stochastic Differential Equations (QFBSDE) with rough drift

Rhoss B. Likibi Pellat
Visiting researcher at Humboldt University of Berlin
joint work with O. Menoukeu Pamen (U.Liverpool/AIMS-Ghana) and P. Imkeller(HU-Berlin)

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Outlines

1. Main results
2. Motivation
3. Existence and Uniqueness
4. Malliavin Differentiability of abstract BSDEs
   - Family of truncated generators
5. Differentiability of Markovian FBSDEs
   - The case of SDEs with $C^\beta_b$ drift
6. Applications
7. End
Main results:

Consider the Markovian Forward-Backward SDE

\[
\begin{align*}
X^x_t(\omega) &= x + \int_0^t b(s, X^x_s(\omega))ds + W_t, \\
Y^x_t(\omega) &= \phi(X^x_T(\omega)) + \int_t^T g(s, X^x_s(\omega), Y_s, Z_s)ds - \int_t^T Z_s dW_s
\end{align*}
\] (1)

Our aim is to show:
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\end{align*} \tag{1} \]

Our aim is to show:

(i) (1) is differentiable w.r.t \( \omega \in \Omega \) (Malliavin derivative):
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\[ \rightarrow \text{g allows this type of non-linearities: } f(|y|)|z|^2; \forall f \in L^1_{\text{Loc}}(\mathbb{R}^+), \]

where \( \kappa > 0 \).
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   \[
   \rightarrow \quad b \in L^\infty([0, T]; C^\beta_{\text{b}}(\mathbb{R}^d)), \beta \in (0,1)
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   \end{itemize}

(iii) The explicit convergence rate holds for any $p \geq 2$

\[
\mathbb{E} \left[ \sup_{t \in [0, T]} |Y_t^n - Y_t|^2p + \left( \int_0^T |Z_t^n - Z_t|^2dt \right)^p \right] \leq C(p, \kappa) n^{-\kappa/q}.
\]

where $\kappa > 0$. 

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Classical results

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→ Lipschitz framework
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→ \(|g(t, x, y, z)| \leq C(1 + |y| + |z|^2)\).
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✓ C. Frei & G. Dos Reis (2013)  
→ \( |g(t, x, y, z)| \leq C(1 + |y| + (1 + |y|^k)|z| + |z|^2), \quad k \in \mathbb{N} \).
Regularity of the coefficients

Assume that \((b, \sigma, \phi, g)\) satisfy:

\[
\rightarrow |(b, \sigma)(t, x)| \leq C(1 + |x|) \quad b, \sigma \in C^1_b \quad \text{and} \quad b', \sigma' \text{ are Lipschitz in } x
\]

\[
\rightarrow \phi, g \text{ continuously differentiable with bounded derivatives.}
\]

Then the Malliavin derivatives of \((Y^x, Z^x)\) solution to (2) satisfies:

\[
D_\theta Y^x_t = \nabla_x \phi(X^x_T)D_\theta X^x_T - \int_t^T D_\theta Z^x_s dB_s + \int_t^T \nabla_x g(s, X^x_s, Y^x_s, Z^x_s)D_\theta X^x_s ds
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+ \int_t^T \nabla_y g(s, X^x_s, Y^x_s, Z^x_s)D_\theta Y^x_s ds + \int_t^T \nabla_z g(s, X^x_s, Y^x_s, Z^x_s)D_\theta Z^x_s ds, \quad (3)
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\]
Assume that \((b, \sigma, \phi, g)\) satisfy:

\[ |(b, \sigma)(t, x)| \leq C(1 + |x|) \quad b, \sigma \in C_b^1 \text{ and } b', \sigma' \text{ are } \text{Lipschitz} \text{ in } x \]

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\]

**IMPORTANT:**

Good regularities of \((b, \sigma) + (\phi, g) \implies \text{well posedness of (3)}
Motivation

Non-smooth drift

The QUESTION??

What if $b$ is non-smooth? or non-Lipschitz continuous?

The results in the aforementioned papers (ref. therein) Do Not Cover It !!!

The equation (3) is Not Solved!!!

Bounded and measurable drift

The forward equation in (1) has a unique strong solution $X_t \in L^2(\Omega; W_1^2, \text{loc}(\mathbb{R}^d))$

O. Menoukeu Pamen et al. (2013), S.E.A. Mohammed et al. (2015)

$s.t.$

$$\sup_{x \in \mathbb{R}^d} \sup_{s \in [0, t]} E[|D_s X_t|^p] < \infty$$

✓

R.L.P. et al.

A class of quadratic forward-backward SDEs


→ Malliavin derivative of $(X_t, x, Y_t, x, Z_t, x)$ with measurable coefficients

→ Using the smoothness of the "weak decoupled field"

→ DO NOT PROVIDE a representation of the derivatives

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June 28, 2022
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$\sup_{x \in \mathbb{R}} \sup_{s \in [0,t]} \mathbb{E} \left[ |D_s X_t^x|^p \right] < \infty$
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\[ O.\ Menoukeu\ Pamen\ et\ al.\ (2013),\ S.E.A.\ Mohammed\ et\ al.\ (2015)\ s.t.\ \forall p \geq 1 \]
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Standing assumptions

Given $T > 0$ and $d \in \mathbb{N}\{0\}$,

$$Y_t = \xi + \int_t^T g(s, \omega, Y_s, Z_s)ds - \int_t^T Z_s dW_s. \quad (4)$$
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Given $T > 0$ and $d \in \mathbb{N}\backslash\{0\}$,

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(A1) $\xi$ is an $\mathfrak{F}_T$-measurable $\|\xi\|_{L^\infty} < \infty$;
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(A1) $\xi$ is an $\mathcal{F}_T$-measurable $\|\xi\|_{L^\infty} < \infty$;

(A2) $g : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is $\mathcal{F}$-predictable and continuous in $(y, z)$ and $\|g(t, 0, 0)\|_{L^\infty} \leq \Lambda_0$, $\alpha \in (0, 1)$ $\mathbb{P}$-a.s.

$$|g(t, \cdot, y, z) - g(t, \cdot, y', z')| \leq \Lambda_y \left(1 + |z|^\alpha + |z'|^\alpha\right)|y - y'|$$

$$+ \Lambda_z \left(1 + (f(|y|) + f(|y'|))(|z| + |z'|)\right)|z - z'|$$

$f \in L^1_{loc}(\mathbb{R})$ increasing and locally bounded.
Existence and Uniqueness

Standing assumptions

- Given $T > 0$ and $d \in \mathbb{N} \setminus \{0\}$,

$$Y_t = \xi + \int_t^T g(s, \omega, Y_s, Z_s) \, ds - \int_t^T Z_s \, dW_s.$$  \hfill (4)

(A1) $\xi$ is an $\mathcal{F}_T$-measurable $\|\xi\|_{L^\infty} < \infty$;

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$$|g(t, \cdot, y, z) - g(t, \cdot, y', z')| \leq \Lambda_y \left(1 + |z|^{\alpha} + |z'|^{\alpha}\right)|y - y'|$$

$$+ \Lambda_z \left(1 + (f(|y|) + f(|y'|))(|z| + |z'|))\right)|z - z'|$$

$f \in L^1_{\text{loc}}(\mathbb{R})$ increasing and locally bounded.

Under (A1) and (A2) $g$ satisfies:

$$|g(t, \cdot, y, z)| \leq \Lambda_0 + \Lambda_y |y| + \Lambda_z (|z| + f(|y|)|z|^2) \text{ a.s.}$$
Existence and Uniqueness

Solvability and bounds

→ Bahlali(2019)

Theorem (Solvability)

Under (A1) and (A2), the BSDE (4) has a unique solution $(Y, Z) \in S^\infty(\mathbb{R}) \times H^2(\mathbb{R}^d)$.

Explicit bounds for $(Y, Z)$

Lemma (Bounds)

Under (A1) and (A2) we have

$$\|Y\|_{S^\infty} \leq \gamma^{(1)} := (\|\xi\|_{L^\infty} + \Lambda_0 T) e^{\Lambda_y T},$$

$$\|Z \ast B\|_{BMO} \leq \gamma^{(2)} := 2\gamma^{(1)} \left( \gamma^{(1)} + T(\Lambda_0 + \Lambda_z + \Lambda_y \gamma^{(1)}) \right) \exp(4\|(1 + \Lambda_z f)\|_{L^1[0,T]})$$

$$\|M\|_{BMO} = \sup_{\tau} \mathbb{E}\left[ (\langle M \rangle_T - \langle M \rangle_\tau) / \mathcal{F}_\tau \right]^{1/2} < \infty,$$
Additional assumptions

(B1) \( \xi \in D^{1,\infty} \), \( g \) is continuously differentiable in \((y, z)\), \( \alpha \in (0, 1) \)

\[|\nabla_y g(t, y, z)| \leq \Lambda_y (1 + |z|^{\alpha}) \text{ a.s.},\]

\[|\nabla_z g(t, y, z)| \leq \Lambda_z (1 + f(|y|)|z|) \text{ a.s.},\]
Additional assumptions

(B1) $\xi \in D^{1,\infty}$, $g$ is continuously differentiable in $(y, z)$, $\alpha \in (0, 1)$

\[
|\nabla_y g(t, y, z)| \leq \Lambda_y (1 + |z|^\alpha) \text{ a.s.,} \\
|\nabla_z g(t, y, z)| \leq \Lambda_z (1 + f(|y|)|z|) \text{ a.s.,}
\]

(B2) $(g(t, y, z))_{t\in[0,T]} \in L_{1,2p}(\mathbb{R})$, $\exists (K_u(t))_{u,t\in[0,T]}$, $(\tilde{K}_u(t))_{u,t\in[0,T]}$ s.t.

\[
\int_0^T \sup_{0 \leq t \leq T} \mathbb{E}|K_u(t)|^{2p} \mathrm{d}u + \|\tilde{K}_u(t)\|_{S^2}^{2p} < \infty,
\]

\[
|D_u g(t, y, z)| \leq K_u(t)(1 + |y| + f(|y|)|z|^\alpha) + \tilde{K}_u(t)(1 + |z|^\alpha + f(|y|)|z|) \text{ a.s.}
\]
Additional assumptions

\((B1)\) \(\xi \in D^{1,\infty}\), \(g\) is continuously differentiable in \((y, z)\), \(\alpha \in (0, 1)\)

\[
|\nabla_y g(t, y, z)| \leq \Lambda_y (1 + |z|^\alpha) \text{ a.s.}, \\
|\nabla_z g(t, y, z)| \leq \Lambda_z (1 + f(|y|)|z|) \text{ a.s.},
\]

\((B2)\) \((g(t, y, z))_{t\in[0, T]} \in L_{1,2p}(\mathbb{R})\), \(\exists (K_u(t))_{u,t\in[0, T]}, (\tilde{K}_u(t))_{u,t\in[0, T]}\) s.t.

\[
\int_0^T \sup_{0 \leq t \leq T} \mathbb{E}|K_u(t)|^{2p}du + \|\tilde{K}_u(t)\|_{S^{2p}}^{2p} < \infty,
\]

\[
|D_{u}g(t, y, z)| \leq K_u(t)(1 + |y| + f(|y|)|z|^\alpha) + \tilde{K}_u(t)(1 + |z|^\alpha + f(|y|)|z|) \text{ a.s.}
\]

**Theorem**

Let \((A1),(A2), (B1)\) and \((B2)\) be in force. Set \(\Theta = (X, Y, Z)\), then, if \(t \in [\theta, T]\),

\[
D_\theta Y_t = D_\theta \xi - \int_t^T D_\theta Z_s dB_s + \int_t^T (D_\theta g)(s, \Theta_s)ds + \int_t^T \langle(\nabla g)(s, \Theta_s), D_\theta \Theta_s\rangle ds
\]

Moreover, \(\{D_t Y_t : 0 \leq t \leq T\}\) is a version of \(\{Z_t : 0 \leq t \leq T\}\).
Consider \((Y^n, Z^n)_{n \geq 1}\) satisfying the BSDE

\[
Y^n_t = \xi + \int_t^T g^n(s, Y^n_s, Z^n_s) \, ds - \int_t^T Z^n_s \, dB_s,
\]

where \((g^n)_{n \in \mathbb{N}}:\)

\[
g^n(t, y, z) := g(t, \tilde{\rho}_n(y), \rho_n(z))
\]

\[
\rho_n : \mathbb{R}^d \to \mathbb{R}^d, \quad z \mapsto \rho_n(z) = (\tilde{\rho}_n(z_1), \ldots, \tilde{\rho}_n(z_d)), \quad n \in \mathbb{N}.
\]
Consider \((Y^n, Z^n)_{n \geq 1}\) satisfying the BSDE

\[
Y^n_t = \xi + \int_t^T g_n(s, Y^n_s, Z^n_s)\,ds - \int_t^T Z^n_s\,dB_s,
\]  

where \((g_n)_{n \in \mathbb{N}}:\)

\[
g_n(t, y, z) := g(t, \tilde{\rho}_n(y), \rho_n(z))
\]

\[
\rho_n : \mathbb{R}^d \to \mathbb{R}^d, z \mapsto \rho_n(z) = (\tilde{\rho}_n(z_1), \ldots, \tilde{\rho}_n(z_d)), n \in \mathbb{N}.
\]

\[
\tilde{\rho}_n(x) = \begin{cases} 
  n + 1, & x > n + 2, \\
  x, & |x| \leq n, \\
  -(n + 1), & x < -(n + 2).
\end{cases}
\]

\(\nabla \tilde{\rho}_n\) uniformly bounded by 1, and converges to 1 locally uniformly.
Lemma

For each $n \in \mathbb{N}$, the BSDE (5) has a unique solution $(Y^n, Z^n) \in S^\infty(\mathbb{R}) \times H^2(\mathbb{R}^d)$. In addition, the process $Z^n \in \mathcal{H}_{BMO}$, and

$$\sup_{n \in \mathbb{N}} \|\mathcal{E}(Z^n \ast B)\|_{BMO} \leq \gamma(2).$$

There exists $r > 1$ independent of $n$ such that

$$\sup_{n \in \mathbb{N}} \|\mathcal{E}(Z^n \ast B)\|_{L^r} < \infty.$$
Lemma

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Lemma

Suppose $\xi \in D^{1,\infty}$. Then the solution $\Theta^n = (Y^n, Z^n)_{n \in \mathbb{N}}$ to BSDE (5) belongs to $L_{1,2} \times (L_{1,2})^d$. A version of $\{(D_u Y^n_t, D_u Z^n_t) : 0 \leq u, t \leq T\}$ is given by

$$D_u Y^n_t = 0 \text{ and } D_u Z^n_t = 0, \text{ if } t \in [0, u),$$

$$D_u Y^n_t = D_u \xi - \int_t^T D_u Z^n_s dB_s$$

$$+ \int_t^T [(D_u g^n)(s, \Theta^n_s) + \langle (\nabla g^n)(s, \Theta^n_s), D_u \Theta^n_s \rangle] ds, \text{ if } t \in [u, T].$$

\begin{align*}
\begin{cases}
X^x_t(\omega) & = x + \int_0^t b(s, X^x_s(\omega))ds + W_t, \\
Y^x_t(\omega) & = \phi(X^x_T(\omega)) + \int_t^T g(s, X^x_s(\omega), Y_s, Z_s)ds - \int_t^T Z_s dW_s
\end{cases}
\end{align*}

(AX): \textit{b} is bounded and measurable.

(AY): $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is continuous, measurable and uniformly bounded; $g$ is a measurable function: $\|g(t, 0, 0, 0)\|_\infty \leq \Lambda_0$

$$|g(t, x, y, z) - g(t, x', y, z)| \leq \Lambda_x (1 + |y| + [f(|y|)|z|]^{\alpha})|x - x'|,$$

$$|g(t, x, y, z) - g(t, x, y', z')| \leq \Lambda_y (1 + (|z| + |z'|))|y - y'|$$

$$+ \Lambda_z (1 + (f(|y|) + f(|y'|))(|z| + |z'|))|z - z'|,$$

where $f \in L^1_{loc}(\mathbb{R}, \mathbb{R}_+)$ is locally bounded and non-decreasing.

AY1): $\phi$ and $g$ are continuously differentiable in $(x, y, z)$ and for $\alpha \in (0, 1)$

$$|\nabla_x g(t, x, y, z)| \leq \Lambda_x (1 + |y| + [f(|y|)|z|]^{\alpha}),$$

$$|\nabla_y g(t, x, y, z)| \leq \Lambda_y (1 + |z|^{\alpha}),$$

$$|\nabla_z g(t, x, y, z)| \leq \Lambda_z (1 + f(|y|)|z|),$$

$$|\nabla_x \phi| \leq \Lambda_\phi (1 + |x|),$$
Theorem (R.L.P., P. Imkeller & O. Menoukeu-Pamen)

Under (AX), (AY) and (AY1) a version of \((D_u Y^x_t, D_u Z^x_t)\), \(u,t \in [0,T]\) is the unique solution to the BSDE

\[
D_u Y^x_t = \nabla_x \phi(X^x_T) D_u X^x_T - \int_t^T D_u Z^x_s dB_s + \int_t^T \nabla_x g(s, X^x_s, Y^x_s, Z^x_s) D_u X^x_s ds \\
+ \int_t^T \nabla_y g(s, X^x_s, Y^x_s, Z^x_s) D_u Y^x_s ds + \int_t^T \nabla_z g(s, X^x_s, Y^x_s, Z^x_s) D_u Z^x_s ds,
\]

where \(D_u X^x\) is the Malliavin derivative of the process \(X^x\).
We assume now

\((AX1)\) \(\exists \beta \in (0, 1)\) s.t. \(b \in L^\infty([0, T]; C^\beta_b(\mathbb{R}^d; \mathbb{R}^d))\),

- \(L^\infty([0, T]; C^\beta_b(\mathbb{R}^d; \mathbb{R}^d))\) is the space of vector fields \(b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d\) having all components in \(L^\infty([0, T]; C^\beta_b(\mathbb{R}^d))\) and \(L^\infty([0, T]; C^\beta_b(\mathbb{R}^d))\) stands for the set of all bounded Borel functions \(b : [0, T] \times \mathbb{R}^d \to \mathbb{R}\) such that

\[
[b]_{\beta,T} = \sup_{t \in [0,T]} \sup_{x \neq y \in \mathbb{R}^d} \frac{|b(t,x) - b(t,y)|}{|x - y|^\beta} < \infty
\]

Lemma

Under Assumption (AX1), the solution \((X^X_t)\) to the forward equation is Malliavin differentiable and for any \(p \geq 2\)

\[
\sup_{0 \leq s \leq t} \mathbb{E} \left[ \sup_{s \leq t \leq T} |D_s X^X_t|^p \right] < \infty.
\]
Sketch of proof

- Fix $\lambda > 0$, $u_\lambda \in L^\infty([0, \infty); C_b^{2+\beta}(\mathbb{R}^d))$ (F.Flandoli et al. (2009)) solves:
  \[ \partial_t u_\lambda + \mathcal{L}u_\lambda - \lambda u_\lambda = -b, \quad (t, x) \in [0, +\infty) \times \mathbb{R}^d, \]
  where $\mathcal{L}u_\lambda = 1/2 \Delta u_\lambda + b \cdot Du_\lambda$.
- For $\lambda$ large enough, $\Psi_\lambda(t, x) = x + u_\lambda(t, x)$ is $C^2$-diffeomorphism.
- Let us consider the following SDE
  \[ \tilde{X}_t = y + \int_s^t \tilde{b}(v, \tilde{X}_v)dv + \int_s^t \tilde{\sigma}(v, \tilde{X}_v)dB_v \quad t \in [s, T] \tag{10} \]
  where $\tilde{b}(t, y) = -\lambda u_\lambda(t, \Psi_\lambda^{-1}(t, y))$ and $\tilde{\sigma}(t, y) = D\Psi_\lambda(t, \Psi_\lambda^{-1}(t, y))$.
- From Theorem 2.2.1 in Nualart we have $\forall p \geq 2$
  \[ \sup_{0 \leq s \leq t} \mathbb{E}[ \sup_{s \leq t \leq T} |D_s \tilde{X}_t|^p ] < \infty. \]
- The chain rule for Malliavin calculus
  \[ \mathbb{E}[ \sup_{s \leq t \leq T} |D_s X_t|^p ] \leq C \mathbb{E}[ \sup_{s \leq t \leq T} |D_s \tilde{X}_t|^p ] < \infty. \]
Representation of the derivatives

Under (AX) or (AX1), the representation of $D_sX_t^x$ is missing in the literature. Indeed,

$$D_sX_t^x = 1_{s \leq t} \left( l_d + \int_s^t b'(s, X_s) D_sX_v^x dv \right)$$

ill-posed!! since $b'$ is just a distribution
Representation of the derivatives

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ill-posed!! since $b'$ is just a distribution

- In one dimension: Under (AX), the following holds for $s \leq t$

$$D_s X_t^x = \exp \left( - \int_s^t \int_{\mathbb{R}} b(s, y) L^X(ds, dy) \right)$$
Representation of the derivatives

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\[ D_s X_t^x = 1_{s \leq t} \left( I_d + \int_s^t b'(s, X_s) D_s X_v^x \, dv \right) \]

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- In one dimension: Under (AX), the following holds for $s \leq t$

\[ D_s X_t^x = \exp \left( -\int_s^t \int_{\mathbb{R}} b(s, y) L^X(ds, dy) \right) \]

- In multi-dimension: Under (AX1), for any $\beta' \in (0, \beta)$ we have: K. Lê (2020)

\[ (D_i^X X_t^x)_j = \delta_{i,j} + \sum_{k=1}^d \int_s^t (D_i^X X_v^x)_j \, dV_{v}^{k,j}(b, X), \quad \forall i, j \in \{1, \cdots, d\}. \]

where $V_t^{k,j}(b, X) = A_t^X[\partial_k b']$ for every $t \geq 0$ and $j, k \in \{1, \cdots, d\}$
The case of SDEs with $\mathcal{C}_b^\beta$ drift

**Theorem (R.L.P., P. Imkeller & O. Menoukeu-Pamen)**

Suppose (AX1), (AY) and (AY1) a version of $(D_u Y^x_t, D_u Z^x_t)_{u,t \in [0,T]}$ is the unique solution to the BSDE

\[
D_u Y^x_t = \nabla_x \phi(X^x_T) D_u X^x_T - \int_t^T D_u Z^x_s dB_s + \int_t^T \nabla_x g(s, X^x_s, Y^x_s, Z^x_s) D_u X^x_s ds,
\]

\[
+ \int_t^T \nabla_z g(s, X^x_s, Y^x_s, Z^x_s) D_u Z^x_s ds + \int_t^T \nabla_y g(s, X^x_s, Y^x_s, Z^x_s) D_u Y^x_s ds,
\]

Moreover, $\{D_t Y^x_t : 0 \leq t \leq T\}$ is continuous a version of $\{Z_t^x : 0 \leq t \leq T\}$ and

\[
D_u Y^x_t (\nabla_x X^x_u) = \nabla_x Y^x_t,
\]

\[
Z^x_t (\nabla_x X^x_t) = \nabla_x Y^x_t,
\]

\[
D_u Z^x_t (\nabla_x X^x_t) = \nabla_x Z^x_t.
\]
Zhang’s path regularity

\[ \delta_N = \{ t_i : 0 = t_0 < \cdots < t_N = T \} \text{ and } |\delta_N| = \max_{0 \leq i \leq N} |t_{i+1} - t_i| \]

**Lemma**

Under (AX1), (AY) and (AY1) it holds: For all \( p \geq 2 \),

\[
\sum_{i=0}^{N-1} \mathbb{E} \left[ \left( \int_{t_i}^{t_{i+1}} |Z_t - Z_{t_i}|^2 \, dt \right)^{p/2} \right] \leq C(p) |\delta_N|^{p/2}
\]

where

\[
\tilde{Z}_{\delta_N} = \frac{1}{t_{i+1} - t_i} \mathbb{E} \left[ \int_{t_i}^{t_{i+1}} Z_s \, ds / \mathcal{F}_{t_i} \right],
\]
Convergence rate

We consider this family of truncated BSDE

\[ Y_t^n = \phi(X_T) + \int_t^T g_n(s, X_s, Y_s^n, Z_s^n) \, ds - \int_t^T Z_s^n \, dB_s, \]

where: \( g_n(t, x, y, z) = g(t, x, \tilde{\rho}_n(y), \rho_n(z)) \) and \( \tilde{\rho}_n \) is given by (7)
Applications

Convergence rate

We consider this family of truncated BSDE

\[ Y_t^n = \phi(X_T) + \int_t^T g_n(s, X_s, Y_s^n, Z_s^n) \, ds - \int_t^T Z_s^n \, dB_s, \quad (11) \]

where: \( g_n(t, x, y, z) = g(t, x, \tilde{\rho}_n(y), \rho_n(z)) \) and \( \tilde{\rho}_n \) is given by (7)

\[
\mathbb{E} \left[ \sup_{t \in [0, T]} |Y_t^n - Y_t|^{2p} + \left( \int_0^T |Z_t^n - Z_t|^2 \, dt \right)^p \right] \\
\leq \Pi \left[ \mathbb{E}^{Q^n} \left( \sup_{t \in [0, T]} |Y_t^n - Y_t|^{2pq} + \left( \int_0^T |Z_t^n - Z_t|^2 \, dt \right)^{pq} \right)^{\frac{1}{q}} \right] \\
\leq C \left( \mathbb{E}^{Q^n} \left[ \int_0^T \left( |\tilde{\rho}(Y_s^n) - Y_s^n| + |\rho(Z_s^n) - Z_s^n| \right)^2 \, ds \right] \right)^{2pq \frac{1}{2q}}
\]
Applications

Convergence rate

- We consider this family of truncated BSDE

\[
Y^n_t = \phi(X_T) + \int_t^T g_n(s, X_s, Y^n_s, Z^n_s) ds - \int_t^T Z^n_s dB_s,
\]

where: \( g_n(t, x, y, z) = g(t, x, \tilde{\rho}_n(y), \rho_n(z)) \) and \( \tilde{\rho}_n \) is given by (7)

\[
\mathbb{E} \left[ \sup_{t \in [0, T]} |Y^n_t - Y_t|^{2p} + \left( \int_0^T |Z^n_t - Z_t|^2 dt \right)^p \right] \\
\leq \Pi \left[ \mathbb{E}_Q^n \left( \sup_{t \in [0, T]} |Y^n_t - Y_t|^{2pq} + \left( \int_0^T |Z^n_t - Z_t|^2 dt \right)^{pq} \right) \right]^{\frac{1}{q}},
\]

\[
\leq C \left( \int_0^T \left( |\tilde{\rho}(Y^n_s) - Y^n_s| + |\rho(Z^n_s) - Z^n_s| \right)^2 ds \right)^{2pq} \frac{1}{2q}
\]

- From properties of \( \tilde{\rho} \) and the uniform boundedness of \( Y^n \) we have:

\[
\left( |\tilde{\rho}(Y^n_s) - Y^n_s| + |\rho(Z^n_s) - Z^n_s| \right)^2 \leq 8((\mathcal{Y}^{(1)})^2 \mathbb{I}_{\{|Y^n_s| > n\}} + |Z^n_s|^2 \mathbb{I}_{\{|Z^n| > n\}})
\]
Thanks for your attention!
Merci pour votre attention!