Switching problems with controlled randomisation and associated obliquely reflected BSDEs

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Annecy
– **Switching with controlled randomisation**: examples, mathematical formulation, associated BSDE and verification theorem

– **Randomised switching with signed costs**: study of the geometry of the domain and an existence theorem.
Classical switching problems

– Classical literature: Hamadène and Jeanblanc (2005, two modes), Djehiche, Hamadène and Popier (2007, d modes), Hu and Tang (2010, controlled drift, switched BSDEs, driver $f^i(t, y^i, z^i)$), Elie, Kharroubi (2011, controlled volatility), Chassagneux, Elie, Kharroubi (2012, driver $f^i(t, y, z^i)$).

– Notations: Time horizon $0 < T < \infty$. Probability space $(\Omega, \mathcal{G}, \mathbb{P})$, Brownian motion $W$, $\mathbb{F}^0 = (\mathcal{F}_t^0)_{t \in [0, T]}$ its augmented filtration.

– Control problem, starting from mode $i \in \{1, \ldots, d\}$ at time $t \in [0, T]$,

$$\mathcal{V}^i_t = \text{ess sup}_{(\tau_n, \zeta_n)_{n \geq 1}} \mathbb{E} \left[ \int_t^T \psi_{a_s}(X_s) \, ds + g_{a_T}(X_T) - \sum_{n \geq 1} c_{\zeta_{n-1}, \zeta_n} 1_{\{\tau_n < T\}} |\mathcal{F}_t^0 \right],$$

where $X$ is an underlying stochastic process.

– A strategy is $(\tau_n, \zeta_n)_{n \geq 0}$ where $(\tau_n)$ is a sequence of stopping times (switching times) and $\zeta_n$ is the mode on $[\tau_n, \tau_{n+1})$.

– State process: $a_t = \sum_{n \geq 0} \zeta_n 1_{\tau_n \leq t < \tau_{n+1}}, \ t \in [0, T]$.

– Admissibility: strategy with $\mathbb{E} \left[ (\sum_{n \geq 0} 1_{\tau_n \leq T})^2 \right] < \infty$.

– Process $\mathcal{V}$ lives in a convex domain of $\mathbb{R}^d$ and solves a BSDE with oblique reflections.
Switching with controlled randomisation
Switching with (controlled) randomisation

– The agent do not directly choose the new mode when she decides to switch.

– Randomised switching: the new mode is decided randomly (independently of everything up to now), according to a (known) distribution on \( \{1, \ldots, d\} \).

\( \rightarrow \) strategy \( (\tau_n)_{n \geq 1} \) nondecreasing sequence of random times.

If actual mode is \( i \) and the agent decides to switch, she pays cost \( \bar{c}_i \).

– Controlled randomisation: the agent first chooses a distribution in \( \{P^u : u \in C\} \). The new mode is drawn according to this distribution.

\( \rightarrow \) strategy \( (\tau_n, \alpha_n)_{n \geq 1} \) where \( \alpha_n \) is the chosen distribution at time \( \tau_n \).

If actual mode is \( i \) and the agent decides to switch using law \( P^u, u \in C \), the cost is \( \bar{c}_i^u \).

– Remark: randomised switching is a particular case of controlled randomisation when \( C \) is a singleton.
Example – randomised switching

- Assume $d = 3$.

- Here the agent only decides to switch, do not control the distribution of the new mode.

- New mode decided independently with transition matrix and cost

$$P = \begin{pmatrix}
0 & 0.5 & 0.5 \\
0.5 & 0 & 0.5 \\
0.5 & 0.5 & 0
\end{pmatrix}, \quad \bar{c} = \begin{pmatrix}
0.5 \\
0.5 \\
0.5
\end{pmatrix}.$$

$\hookrightarrow$ When the agent wants to switch, the new mode is determined by throwing a fair coin.

For example, if the present mode is 1 and the agent wants to switch, the new mode is 2 with probability 0.5 and 3 with probability 0.5.
Example – switching with controlled randomisation

– Here the agent decides when to switch and chooses the distribution of the new mode.

She chooses \( u \in [0, 1] \), and the new mode is determined by

\[
P^u = \begin{pmatrix}
0 & u & 1-u \\
1-u & 0 & u \\
u & 1-u & 0
\end{pmatrix},
\bar{c}^u = \begin{pmatrix}
1-u(1-u) \\
1-u(1-u) \\
1-u(1-u)
\end{pmatrix}.
\]

– Example: current mode is 1 and switching with control \( u \)
\( \rightarrow \) new mode is 2 (resp. 3) with probability \( u \) (resp. \( 1-u \)).
\( \rightarrow \) To increase the probability to be in mode 2 after the switch, the agent should take \( u \) closer to 1.
\( \rightarrow \) Reducing uncertainty induces a higher cost as \( \bar{c}_1^u \) is higher with \( u \) closer to 1.

– Applications to risk aversion.
Classical switching

- For each $d \geq 2$, the classical switching problem is a particular case of switching with controlled randomisation.

- For $d = 3$ for example, the transition matrices are:

  \[ P^1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \bar{c}^1 = \begin{pmatrix} c_{1,2} \\ c_{2,3} \\ c_{3,1} \end{pmatrix}, \]

  \[ P^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \bar{c}^2 = \begin{pmatrix} c_{1,3} \\ c_{2,1} \\ c_{3,2} \end{pmatrix}. \]
Some comments

– When the agent decides to switch, the new mode is chosen with some extra and independent noise ↪ mathematical analysis must deal with enlargement of filtrations.

– The enlarged filtration depends on the switching times, hence on the control.

– Classical switching ↪ “triangular inequality” \( c_{i,j} + c_{j,k} > c_{i,k} \) ↪ no simultaneous switches. Here, in general, the question of simultaneous switches arises: the agent may not be satisfied with the randomly reached state.
Setup

– Control set: $\mathcal{C}$ an ordered compact metric space.

– Probability space: $(\Omega, \mathcal{G}, \mathbb{P})$ with $\mathcal{G} = \sigma(\mathcal{W}, (\mathcal{U}_n)_{n \geq 1})$. $\mathcal{W}$ is a $\kappa$-dimensional Brownian motion and $\mathbb{F}^0$ its augmented natural filtration.

$(\mathcal{U}_n)_{n \geq 1}$ i.i.d. family of uniform r.v.'s on $[0, 1]$, independent of $\mathcal{W}$ models extra-randomness at switching times.

– Switching: if present mode is $i \in \{1, \ldots, d\}$ and agent wants to switch with control $u \in \mathcal{C}$ to new mode $F(u, i, \mathcal{U}) \in \{1, \ldots, d\}$ with $\mathcal{U}$ uniform on $[0, 1]$ and cost $\bar{c}_i^u$, where $\bar{c}: \{1, \ldots, d\} \times \mathcal{C} \to \mathbb{R}$ is continuous. We set $P_{i,j}^u = \mathbb{P}(F(u, i, \mathcal{U}) = j)$.

– Data: (similar to Hu and Tang (2010))

terminal condition $\xi = (\xi^1, \ldots, \xi^d) \in L^2(\mathcal{F}_T^0)$,
driver $f : \Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^{d \times \kappa} \to \mathbb{R}^d$ satisfying to

– $f(\cdot, 0, 0) \in H^2(\mathbb{F}^0)$ and $f$ is progressive,

– For all $(t, y, z) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^{d \times \kappa}$, $f^i(t, y, z) = f^i(t, y_i, z_i)$.

– For all $(t, y^1, y^2, z^1, z^2) \in [0, T] \times (\mathbb{R}^d)^2 \times (\mathbb{R}^{d \times \kappa})^2$,

$$|f(t, y^1, z^1) - f(t, y^2, z^2)| \leq L(|y^1 - y^2| + |z^1 - z^2|).$$
Strategies

– **Strategy** for the problem starting at \( t \) in \( i \):

\[
\phi = (\zeta_0 = i, \tau_0 = t, (\tau_n, \alpha_n)_{n \geq 1})
\]

where

- \((\tau_n, \alpha_n)_{n \geq 1}\) is a sequence of \( G \)-random variables valued in \([t, \infty) \times \mathcal{C}\),
- \(\tau_n \leq \tau_{n+1}\) for all \( n \geq 0 \), and
- for \( n \geq 0 \), \( \tau_{n+1} \) is a \( \mathbb{F}^n \)-stopping time and \( \alpha_{n+1} \) is \( \mathcal{F}^n_{\tau_{n+1}} \)-measurable.

We then set \( \mathbb{F}^{n+1} = (\mathcal{F}^{n+1}_t)_{t \geq 0} \) with \( \mathcal{F}^{n+1}_t = \mathcal{F}^n_t \vee \sigma(\bigcup_{n+1} \{\tau_{n+1} \leq t\}) \) for all \( t \geq 0 \).

– For all \( n \geq 0 \) and \( s \in [t, T] \), we define:

- the state after \( n+1 \) switches as \( \zeta_{n+1} = F(\alpha_{n+1}, \zeta_n, \mathcal{U}_{n+1}) \),
- the state process as \( a_s = \sum_{n \geq 0} \zeta_n 1_{[\tau_n, \tau_{n+1})}(s) \) and
- the cumulative cost process as \( A^\phi_s = \sum_{n \geq 0} \tilde{c}^{\alpha_{n+1}} \zeta_n 1_{\tau_{n+1} \leq s} \).

– We define the filtration associated to the strategy as \( \mathbb{F}^\infty = (\mathcal{F}^\infty_t)_{t \geq 0} \) with \( \mathcal{F}^\infty_t = \bigvee_{n \geq 0} F^n_t \), \( t \geq 0 \).

– A strategy \( \phi \) is admissible \((\phi \in \mathcal{A}^i_t)\) if

\[
A^\phi_T - A^\phi_t \in L^2(\mathcal{F}^\infty_T) \text{ and } \mathbb{E} \left[ (A^\phi_T)^2 | \mathcal{F}^0_t \right] < +\infty.
\]
Switching problem with controlled randomisation

– Given an admissible strategy $\phi$, the associated reward is given by (see Hu and Tang (2010)):

$$
\mathbb{E}\left[U_t^\phi - A_t^\phi | \mathcal{F}_t^0\right],
$$

with $(U^\phi, V^\phi, M^\phi)$ being the solution in $\mathbb{F}^\infty$ to the following switched BSDE: for $s \in [t, T]$,

$$
U_s = \xi^{a_T} + \int_t^T f^{ar}(r, U_r, V_r)dr - \int_t^T V_r dW_r - \int_t^T dM_r - \int_t^T dA_r^\phi,
$$

– Proposition: For $\phi$ admissible, $\mathbb{F}^\infty$ is right-continuous and there exists a unique solution to the BSDE.

– $M^\phi$ is a $\mathbb{F}^\infty$-martingale $\mapsto$ we obtain a martingale representation theorem: $M^\phi$ jumps only at the switching times of the strategy associated to $\mathbb{F}^\infty$.

– Problem value, starting in mode $i \in \{1, \ldots, d\}$ at time $t \in [0, T]$:

$$
\mathcal{V}_t^i = \text{ess sup}_{\phi \in A_t^i} \mathbb{E}\left[U_t^\phi - A_t^\phi | \mathcal{F}_t^0\right].
$$
Particular case

- Assume the driver does not depend upon $U, V$: $f(\omega, t, u, v) = f(\omega, t)$.
- Then, for $\phi$ admissible,

$$
E \left[ U^\phi - A_t^\phi \right| F_t] = E \left[ \xi^{aT} + \int_t^T f^{as}(s)ds - A_T^\phi \right| F_t]
$$

$$
= E \left[ \xi^{aT} + \int_t^T f^{as}(s)ds - \sum_{n \geq 0} \bar{c}_{\zeta n}^{\alpha+1} 1_{\{\tau_{n+1} \leq T\}} \right| F_t]
$$

- The problem thus writes:

$$
\mathcal{V}_t^i = \text{ess sup} \ E \left[ \xi^{aT} + \int_t^T f^{as}(s)ds - \sum_{n \geq 0} \bar{c}_{\zeta n}^{\alpha+1} 1_{\{\tau_{n+1} \leq T\}} \right| F_t]
$$
The domain of reflections

- **Classical switching**: value $V$ linked to the solution of an obliquely reflected BSDE in some convex domain. Similar here with positive costs.

- Heuristically, the maximal profit is greater than the expected profit obtained by the strategy:
  1. Switching instantaneously with control $u \in C$, leading to mode $j$ with probability $P_{i,j}$.  
  2. Following the optimal strategy in the new mode.

Then $V_t^i \geq \mathbb{E}[V_t^\zeta] - \bar{c}_i^u$ with $\zeta$ the (random) mode after switching from $i$ with control $u$.

- Since this strategy is available for each $u \in C$, we obtain
  $$V_t^i \geq \sup_{u \in C} \left( \sum_{j=1}^{d} P_{i,j}^u V_t^j - \bar{c}_i^u \right).$$

- The problem value lies into the following convex domain of $\mathbb{R}^d$:
  $$D = \left\{ y \in \mathbb{R}^d \mid y_i \geq \sup_{u \in C} \left( \sum_{j=1}^{d} P_{i,j}^u y_j - \bar{c}_i^u \right), 1 \leq i \leq d \right\}.$$
Examples of domains

- **Easy lemma:** let $D_0 := \{y \in D \mid y_d = 0\}$. Then $D = D_0 \oplus \mathbb{R} \cdot (1, \ldots, 1)$. $\leftrightarrow D$ is obtained by translating $D_0$ along the axis $\mathbb{R} \cdot (1, \ldots, 1)$.

Graphs of $D_0 = D \cap \{y_3 = 0\}$ as a subset of $\{(y_1, y_2, 0)\} \simeq \mathbb{R}^2$.

- **Blue:** the usual switching domain with cost 1,
- **Red:** domain from example 1,
- **Green:** domain from example 2.
- **Heuristically**, as in the classical case, one expects that $\mathcal{V} = Y$, where $(Y, Z, K)$ is the solution to the following *Obliquely Reflected BSDE*

$$
Y_t^i = \xi^i + \int_t^T f^i(s, Y_s^i, Z_s^i)ds - \int_t^T Z_s^i dW_s + \int_t^T dK_s^i, \\
(1)
$$

$$
Y_t \in \mathcal{D} \text{ and } \int_0^T \left( Y_t^i - \sup_{u \in \mathcal{C}} \left( \sum_{j=1}^d P_{i,j}^u Y_s^j - \bar{c}_i^u \right) \right) dK_t^i = 0, \quad (2)
$$

and that an optimal strategy starting at $t = \tau_0 \in [0, T]$ and mode $\zeta_0 \in \{1, \ldots, d\}$ is given by

$$
\tau_{k+1}^* = \text{arg sup} \left\{ s \geq \tau_k^* \mid Y_s^{\zeta_k^*} = \sup_{u \in \mathcal{C}} \left( \sum_{j=1}^d P_{i,j}^u \ Y_s^j - \bar{c}_i^u \right) \right\} \wedge (T + 1),
$$

$$
\alpha_{k+1}^* = \text{arg sup}_{u \in \mathcal{C}} \left( \sum_{j=1}^d P_{i,j}^{\zeta_k^*} \ Y_s^j - \bar{c}_i^{\zeta_k^*} \right).
$$

- This is indeed true in a **positive costs setting**.
- One easily deduces **uniqueness of solutions** in a **signed costs setting**.
Randomised switching with signed costs
Study of the domain of reflection

- We now assume **randomised switching**: \( C = \{0\} \), i.e. the agent do not control the distribution of the new state.
- We set \( P = (P_{i,j})_{i,j} \) and \( \bar{c} = (\bar{c}_i)_i \in \mathbb{R}^d \) **(signed costs)**.
- A first issue is that it is not a priori guaranteed that the domain \( \mathcal{D} \) has non-empty interior, or at least is non-empty.

Domain \( \mathcal{D}_0 = \mathcal{D} \cap \{y_3 = 0\} \), for \( \bar{c}_1 \in \{0.5, 0, -0.5, -1\} \) in the example of randomised switching.
For \( \bar{c}_1 = -1 \), the domain has empty interior and for \( \bar{c}_1 < -1 \) the domain is empty!
Study of the domain of reflection

– We assume that $P$ is irreducible, and we let $\mu$ be its unique invariant probability measure.

– In this setting, the domain is

$$D = \left\{ y \in \mathbb{R}^d : y \succcurlyeq Py - \bar{c} \right\},$$

with $\succcurlyeq$ the component by component partial ordering.

– If $y \in D$, we have $\mu y \geq \mu Py - \mu \bar{c} = \mu y - \mu \bar{c}$, hence $\mu \bar{c} \geq 0$.

– Questions: Conversely, if $\mu \bar{c} \geq 0$, can we conclude that $D$ is non-empty? What about the condition $\mu \bar{c} > 0$? How to interpret the condition $\mu \bar{c} \geq 0$ in terms of the switching problem?

– Recall that $D = D_0 \oplus \mathbb{R} \cdot (1, \ldots, 1)$
$\iff D$ is non-empty (resp. has non-empty interior) iif $D_0$ is (resp. has non-empty interior) in $\{ y_d = 0 \} \simeq \mathbb{R}^{d-1}$.

– Randomised switching with irreducible transition matrix
$\iff D_0$ is a simplex.
$\iff$ what are the coordinates of its vertices?
Study of the domain of reflection

- If $V_t = y$, constraints $y_1 \geq Py - \bar{c}_1$ and $y_2 \geq Py - \bar{c}_2$ are both saturated, i.e. if the current mode is 1 or 2, it is optimal to switch.

  $\leftarrow$ optimally, if current mode is 1 (or 2), simultaneous switches are needed until mode 3 is reached. Then apply optimal strategy from mode 3 for optimal reward $V^3_t = y_3 = 0$.

  $\leftarrow y_1 = V^1_t = V^3_t - C_{1,3} = -C_{1,3}$ with $C_{i,j}$ = mean cost to reach $j$ from $i$.

  $\leftarrow y = (-C_{1,3}, -C_{2,3}, 0)$.

- Similar argument can be applied to $z = (z_1, z_2, 0)$: optimal to switch to mode 2, where optimal reward is $z_2$. Thus $z = (z_2 - C_{1,2}, z_2, z_2 - C_{3,2})$, and since $z_3 = 0$, one gets $z_2 = C_{3,2}$ and $z = (C_{3,2} - C_{1,2}, C_{3,2}, 0)$. 
Study of the domain of reflection

– For \((i,j) \in \{1, \ldots, d\}^2\), the key quantity is the expected cost along an excursion from state \(i\) to state \(j\): 

\[ C_{i,j} = \mathbb{E} \left[ \sum_{n=0}^{\tau_j-1} \bar{c}_{X_n} \bigg| X_0 = i \right] , \]

where \(X\) is the irreducible Markov chain with transition matrix \(P\) and \(\tau_j = \inf \{ n \geq 0 | X_n = j \}\).

– More technical \(\leftrightarrow\) Combining linear algebra and Markov Chain arguments, link between \(\mu \bar{c}\) and the \(C_{i,j}\)'s.

**Theorem**

The following conditions are equivalent:

1. The domain \(\mathcal{D}\) is non-empty (resp. has non-empty interior).
2. There exists \(1 \leq i \neq j \leq d\) such that \(C_{i,j} + C_{j,i} \geq 0\) (resp. \(C_{i,j} + C_{j,i} > 0\)).
3. The inequality \(\mu \bar{c} \geq 0\) is satisfied (resp. \(\mu \bar{c} > 0\))
4. For all \(1 \leq i \neq j \leq d\), we have \(C_{i,j} + C_{j,i} \geq 0\) (resp. \(C_{i,j} + C_{j,i} > 0\)).
Study of the domain of reflection

– We recover the triangular inequality with the $C_{i,j}$’s:

### Corollary

The following conditions are equivalent:

1. The domain $\mathcal{D}$ is non-empty.
2. For all $1 \leq i, j, k \leq d$, we have $C_{j,k} \leq C_{j,i} + C_{i,k}$.
3. For any round trip of length less that $d$, i.e. $1 \leq n \leq d$ and $1 \leq i_1 \neq \ldots \neq i_n \leq d$, we have $\sum_{k=1}^{n-1} C_{i_k,i_{k+1}} + C_{i_n,i_1} \geq 0$.

– **Remark:** In the case of classical switching problems, this triangular inequality is satisfied with the costs to switch from mode $i$ to mode $j$. Here, with randomised switching, we need to consider the expected cost to switch from mode $i$ to $j$. 
existence of solutions to the BSDE

- Chassagneux and Richou (2020): studied obliquely reflected BSDEs in general.

\[ Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s - \int_t^T H(s, Y_s, Z_s)\Phi_s ds, \]

\[ Y \in \mathcal{D}, \Phi \in n_D(Y), \int_0^T |\Phi_t| 1_{\{Y_t \notin \partial D\}} dt = 0, \]

where

- \( n_D(y) \) is the outward normal cone at \( Y \) for the convex domain \( \mathcal{D} \),
- \( H \in \mathbb{R}^{d \times d} \) is a given operator allowing for oblique reflections satisfying to technical assumptions.

- Our task: construct an operator \( H \) such that \( H(y)n_D(y) \subset C_o(y) \) the oblique cone for \( y \) for our problem, and check that \( H \) mets the technical assumptions to apply the existence results.

\[ \leftrightarrow \] compute the cones at each \( y \in \partial \mathcal{D}_0 \), define \( H = H(y) \) first on \( \partial \mathcal{D}_0 \), then extend it to \( \mathcal{D}_0 \) by convexity, to \( \mathcal{D} \) and finally to \( \mathbb{R}^d \) by projection.
Existence of solutions to the BSDE

- **Notations**: $Q := I_d - P$ and for each $1 \leq i \leq d$, we set $Q^{(i,i)}$ the square matrix of size $d - 1$ obtained from $Q$ by deleting row $i$ and column $i$.

- **Markovian framework**: $\xi = g(X^t_{T,x})$ and $f(\omega, s, y, z) = \psi(s, X^t_{s,x}(\omega), y, z)$ for some maps $g : \mathbb{R}^q \to \mathbb{R}^d$ and $\psi : [0, T] \times \mathbb{R}^q \times \mathbb{R}^d \times \mathbb{R}^{d \times \kappa}$ and $X$ a Itô diffusion.

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**Theorem**

Assume some technical conditions on the maps $g$, $f$ and the coefficients $b$ and $\sigma$ of the dynamics of $X$.
Assume $\mathcal{D}$ has non-empty interior.
Moreover, assume that for all $1 \leq i \leq d$, the matrix $Q^{(i,i)}$ satisfies the following **copositivity hypothesis**: for all $\mathbb{R}^{d-1} \ni x \succeq 0$, $x \neq 0$, we have

$$
x^T Q^{(i,i)} x > 0.
$$

Then
- we can construct a $H$ satisfying to the technical assumptions,
- the reflected BSDE (1)-(2) admits a solution.
Remarks

- The copositivity hypothesis is always satisfied when $d = 3$.

- In dimension $d \geq 3$, this hypothesis is satisfied for the randomised switching with transition matrix $P_{i,j} = \frac{1}{d-1} 1_{i \neq j}$.

- A counter-example in dimension $d = 4$:

$$P = \begin{pmatrix}
0 & \frac{\sqrt{3}}{2} & 0 & 1 - \frac{\sqrt{3}}{2} \\
1 - \frac{\sqrt{3}}{2} & 0 & \sqrt{3} - 1 & 1 - \frac{\sqrt{3}}{2} \\
0 & 1 & 0 & 0 \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0
\end{pmatrix}.$$
Conclusion and further work

– We defined a **new switching problem** with uncertainty on the new mode when the agent decides to switch.
– When the costs are positive, we obtained a **representation theorem** in terms of a BSDE with oblique reflection, which implies the uniqueness for the BSDE.
– When the costs are signed and in the setting of randomised switching, we obtain a **characterisation of the non-emptiness** (using the control problem data) for the domain of reflections.
– We obtain **existence** in a Markovian framework for the randomised switching problem. In the paper, we have examples of existence for a controlled randomisation, and in a non-Markovian framework.

– The general study of existence of the BSDEs associated to switching problems with controlled randomisation remains **open**.
– A representation theorem with signed costs is **not** proved.
– Extension to **time-dependent** and **random** costs and transition probabilities.
Thank you for your attention!