# Mean-field BDSDEs and associated nonlocal semi-linear backward stochastic partial differential equations

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# Outline

# 1. Objective of the talk

Backward doubly stochastic differential equations (BDSDEs for short):

1) Pardoux, Peng (1990): existence and uniqueness of solutions of BSDEs

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s,\ 0 \le t \le T;$$

2) Pardoux, Peng (1994): existence and uniqueness of solutions of BDSDEs

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)d\overleftarrow{B}_s - \int_t^T Z_s dW_s,\ 0 \le t \le T;$$

3) Such BDSDEs have been intensively studied:

+ Stochastic partial differential equations (Bally, Matoussi (2001); Zhang, Zhao (2013); Matoussi, Piozin, Popier (2017));
+ Pontryagin maximum principle (Han, Peng, Wu (2010));
+ Zakai equation in filtering (Liptser, Shiryaev (2001));
+ Stochastic viscosity solutions (Buckdahn, Ma (2001));
+ Stochastic Volterra integral equations (Shi, Wen, Xiong (2020));
+ Mean-field BDSDEs (Li, Xing (2022); Li, Xing, Peng (2021))...

# 1. Objective of the talk

Mean-field problems:

1) Study of mean-field stochastic differential equations: Li, Min (2016); Buckdahn, Li, Peng, Rainer (2017); Hao, L. (2016)...

2) Study of mean-field backward stochastic differential equations: Buckdahn, Li, Peng (2009); Li, Liang, Zhang (2018)...

3) Such Mean-Field SDEs/BSDEs have been intensively studied:
   + In the frame of Mean-Field Games and related topics since 2006-2007 by J.M.Lasry and P.L.Lions;
   + By P.L.Lions in the frame of his lectures at Collège de France; notes written by Cardaliaguet;
   + Mean-Field FBSDEs with jumps and related nonlocal PDEs: Li (2018);
   + Non-zero sum Mean-Field Games: Carmona, Delarue, 2012-2013;
   + Stochastic maximum principle:
     + Pontryagin maximum principle (Buckdahn, Djehiche, Li (2011));
     + Peng's maximum principle (Buckdahn, Li, Ma (2016))...

**Investigate backward stochastic partial differential equations** for a general type of mean-field backward doubly stochastic differential equations. Extends:

- Pardoux and Peng (PTRF, 1994)

**The novelties in our work**:

- We investigate mean-field BDSDEs, i.e., BDSDEs whose driving coefficients also depend on the joint law of the solution process as well as the solution of an associated mean-field forward SDE;
- We prove the the $L^2$-regularity of the value function $V(t,x,P_\xi):=Y_t^{t,x,P_\xi}$. In particular, Malliavin calculus will be used to prove some crucial estimates for $Z^{t,x,P_\xi}$ and its derivatives;
- However, we have to use the (mean-field) Itô formula. To overcome this problem the characterisation of $V=(V(t,x,P_\xi))$ as the unique solution of the associated mean-field backward stochastic PDE uses the $C_b^{1,2,2}$-functions $\Psi(t,x,P_\xi):=E[V(t,x,P_\xi)\cdot\eta]$ for suitable $\eta\in L^\infty(\mathcal{F};\mathbb{R})$;
- We extend the classical mean-field Itô formula to smooth functions of solutions of mean-field BDSDEs.

1 Objective of the talk

2 Preliminaries

3 Mean-field SDEs and mean-field BDSDEs

4 First and second order derivatives of $X^{t,x,P_\xi}$

5 First and second order derivatives of $(Y^{t,x,P_\xi}, Z^{t,x,P_\xi})$

6 Related backward SPDEs of mean-field type

(gallop through)

**Spaces we work with**: For $p \geq 1$, we denote

• $L^p(\Omega, \mathcal{F}_T, P; \mathbb{R}^d)$ is the set of $\mathcal{F}_T$-measurable random variables $\xi : \Omega \to \mathbb{R}^d$ such that $\|\xi\|_{L^p} := (E[|\xi|^p])^{\frac{1}{p}} < \infty$.

• $\mathcal{S}^p_{\mathcal{F}}(t, T; \mathbb{R}^d)$ is the set of $\{\mathcal{F}_{t,s}\}$-adapted measurable continuous processes $\eta : \Omega \times [t, T] \to \mathbb{R}^d$ with $\|\eta\|_{\mathcal{S}^p} := \big(E\big[\sup_{t \leq s \leq T} |\eta(s)|^p\big]\big)^{\frac{1}{p}} < \infty$.

• $\mathcal{H}^p_{\mathcal{F}}(t, T; \mathbb{R}^d)$ is the set of $\{\mathcal{F}_{t,s}\}$-adapted measurable processes $\eta : \Omega \times [t, T] \to \mathbb{R}^d$ with $\|\eta\|_{\mathcal{H}^p} := \big(E\big[(\int_t^T |\eta(s)|^2 ds)^{\frac{p}{2}}\big]\big)^{\frac{1}{p}} < \infty$.

• $C^k_b(\mathbb{R}^p, \mathbb{R}^q)$ is the set of functions of class $C^k$ from $\mathbb{R}^p$ into $\mathbb{R}^q$ whose partial derivatives of all order less than or equal to $k$ are bounded.

**Derivative of a function with respect to a probability measure**
(see: course at Institut de France by P.-L. Lions, 2013; notes by Cardaliaguet, 2013; equivalent but more direct approach: Delarue et al. 2015)

A function $h : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is said to be differentiable, if, for all $\mu \in \mathcal{P}_2(\mathbb{R}^d),\ y \in \mathbb{R}^d)$, there exists

+ A measurable function $\frac{\delta}{\delta\mu} f : \mathcal{P}_2(\mathbb{R}^d) \times R^d \to R^d$ s.t.

$$\frac{\delta}{\delta\mu} f(\mu, y) := \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon}\big(f((1-\varepsilon)\mu + \varepsilon\delta_y) - f(\mu)\big),\ (\mu, y) \in \mathcal{P}_2(\mathbb{R}^d) \times R^d,$$

+ A measurable function $\partial_\mu f : \mathcal{P}_2(\mathbb{R}^d) \times R^d \to R^d$ s.t.

$$\partial_\mu f(\mu, y) = \partial_y\big(\frac{\delta}{\delta\mu} f(\mu, y)\big),\ (\mu, y) \in \mathcal{P}_2(\mathbb{R}^d) \times R^d.$$

Remark. If $f$ is differentiable, then, for all $\xi, \eta \in L^2(\Omega, \mathcal{F}, P; R^d)$,

$$f(P_{\xi+\varepsilon\eta}) - f(P_\xi) = E\big[\partial_\mu f(P_\xi, \xi)\eta\big] + o\big(||\eta||_{L^2(P)}\big), \text{ as } o\big(||\eta||_{L^2(P)}\big) \to 0.$$

**Mean-field BDSDEs**: (see, Li, Xing (JMAA, 2022))
Let

$$f : [0,T] \times \Omega \times \mathcal{P}_2(\mathbb{R}^{k+k\times d}) \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \to \mathbb{R}^k,$$
$$g : [0,T] \times \Omega \times \mathcal{P}_2(\mathbb{R}^{k+k\times d}) \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \to \mathbb{R}^{k\times l},$$
$$h : [0,T] \times \mathcal{P}_2(\mathbb{R}^{k+k\times d}) \to \mathbb{R}^{k\times l}$$

be jointly measurable and s.t.:

**(H2.1)** $g(.,.,\delta_0,0,0) \in \mathcal{H}^2(0,T;\mathbb{R}^{k\times l})$, ($\delta_0$ - Dirac measure with $0 \in \mathbb{R}^{k+k\times d}$).

**(H2.2)** $g$ is Lipschitz in $(\mu, y, z)$: $\exists\ C > 0$, $\alpha_1, \alpha_2 > 0$ with $0 < \alpha_1 + \alpha_2 < 1$ s.t., for all $\mu, \mu' \in \mathcal{P}_2(\mathbb{R}^{k+k\times d})$, $y_1, y_2 \in \mathbb{R}^k$, $z_1, z_2 \in \mathbb{R}^{k\times d}$,

$$|g(t,\mu,y_1,z_1) - g(t,\mu',y_2,z_2)|^2 \le C|y_1 - y_2|^2 + \alpha_1|z_1 - z_2|^2 + W_{2,C,\alpha_2}(\mu,\mu')^2.$$

Here we use the weighted Wasserstein distance: for any $\gamma_1, \gamma_2 > 0$,

$$W_{2,\gamma_1,\gamma_2}(\mu,\mu')^2 := \inf\Big\{ E[\gamma_1|\xi - \xi'|^2 + \gamma_2|\eta - \eta'|^2] \Big| (\xi,\eta),(\xi',\eta') \in L^2(\mathcal{F};\mathbb{R}^k \times \mathbb{R}^{k\times d}) : P_{(\xi,\eta)} = \mu, P_{(\xi',\eta')} = \mu' \Big\}.$$

**(H2.3)** $f(t, \omega, \delta_0, 0, 0) \in \mathcal{H}^2(0, T; \mathbb{R}^k)$.

**(H2.4)** $f$ is Lipschitz in $(\mu, y, z)$: There exists a constant $C > 0$ such that, for all $\mu, \mu' \in \mathcal{P}_2(\mathbb{R}^{k+k\times d})$, $y_1, y_2 \in \mathbb{R}^k$, $z_1, z_2 \in \mathbb{R}^{k\times d}$,

$$|f(t, \mu, y_1, z_1) - f(t, \mu', y_2, z_2)| \leq C(W_2(\mu, \mu') + |y_1 - y_2| + |z_1 - z_2|).$$

**(H2.5)** $h(t, \delta_0) \in \mathcal{H}^2(0, T; \mathbb{R}^{k\times l})$.

**(H2.6)** $h$ is Lipschitz in $\mu$: There exists a constant $C > 0$ such that, for all $\mu, \mu' \in \mathcal{P}_2(\mathbb{R}^{k+k\times d})$,

$$|h(t, \mu) - h(t, \mu')|^2 \leq C W_2(\mu, \mu')^2.$$

Given $\xi \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^k)$, we consider the following general mean-field BDSDEs:

$$\begin{aligned} Y_t = \xi &+ \int_t^T f(s, P_{(Y_s,Z_s)}, Y_s, Z_s)ds + \int_t^T g(s, P_{(Y_s,Z_s)}, Y_s, Z_s)d\overleftarrow{B}_s \\ &+ \int_t^T h(s, P_{(Y_s,Z_s)})d\overleftarrow{B}_s - \int_t^T Z_s dW_s,\ 0 \le t \le T, \end{aligned} \tag{2.1}$$

where the integral with respect to $B$ is the Itô backward one, denoted by $d\overleftarrow{B}$.

**Theorem 2.1. (Existence and uniqueness)**

Under the assumptions (H2.1)-(H2.6), the general mean-field BDSDE (3.1) has a unique solution $(Y, Z) \in \mathcal{S}^2_{\mathcal{F}}(0, T; \mathbb{R}^k) \times \mathcal{H}^2_{\mathcal{F}}(0, T; \mathbb{R}^{k\times d})$.

(to gallop)

**Theorem 2.2. (Higher order moment estimates)**

We assume $g$ satisfies (H2.1) and (H2.2), $f$ satisfies (H2.3) and (H2.4), and $h$ satisfies (H2.5) and (H2.6). Moreover, we suppose that, for some $p \geq 2$, $\overline{C}_p(\alpha_1+\alpha_2)^{\frac{p}{2}} < 1$. Here $\overline{C}_p := 2^{p-1}C_p^*((\frac{p}{p-1})^p+1)C_p'$, $C_p^* := 2^{-p-2}3^p p^{3p} + 2^{\frac{p}{2}}$, $C_p' := (\frac{p}{p-1})^p 3^{p-1}\Big(2C^p 5^{p-1} \vee (6p^3)^p 5^{\frac{p}{2}-1}\Big)$, $C$ is the Lipschitz constant in (H2.2), (H2.4) and (H2.6). $(Y,Z)$ is the solution of the mean-field BDSDE (3.1). Then there exists $C_p \in \mathbb{R}_+$ only depending on the Lipschitz constant $C$ of the coefficients and on $p$, such that

$$E[\sup_{s\in[0,T]}|Y_s|^p]+E\Big[\Big(\int_0^T|Z_s|^2ds\Big)^{\frac{p}{2}}\Big] \leq C_pE\Big[|\xi|^p+(\int_0^T|f(s,\delta_0,0,0)|ds)^p + (\int_0^T|g(s,\delta_0,0,0)|^2ds)^{\frac{p}{2}}+(\int_0^T|h(s,\delta_0)|^2ds)^{\frac{p}{2}}\Big]. \tag{2.2}$$

(to gallop)

Now we give a general Itô's formula which will be used later.

### Theorem 2.3. (Itô's formula)

Let $F \in C_b^{1,2,2}([0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d))$. Given $f\in\mathcal{H}^2_{\mathcal{F}}(0,T;\mathbb{R}^d)$, $g\in\mathcal{H}^2_{\mathcal{F}}(0,T;\mathbb{R}^{d\times l})$, $\xi\in L^2(\mathcal{F}_T;\mathbb{R}^d)$ as well as $u\in\mathcal{H}^2_{\mathcal{F}}(0,T;\mathbb{R}^d)$, $v\in\mathcal{H}^2_{\mathcal{F}}(0,T;\mathbb{R}^{d\times l})$, $\eta\in L^2(\mathcal{F}_T;\mathbb{R}^d)$. We consider the solutions $(Y,Z)$, $(U,V)\in\mathcal{S}^2_{\mathcal{F}}(0,T;\mathbb{R}^d)\times\mathcal{H}^2_{\mathcal{F}}(0,T;\mathbb{R}^{d\times d})$ of the following both BDSDEs:

$$Y_t=\xi+\int_t^T f_s ds+\int_t^T g_s d\overleftarrow{B}_s-\int_t^T Z_s dW_s,\ t\in[0,T], \tag{2.3}$$

and

$$U_t=\eta+\int_t^T u_s ds+\int_t^T v_s d\overleftarrow{B}_s-\int_t^T V_s dW_s,\ t\in[0,T]. \tag{2.4}$$

Then, for all $t\in[0,T]$, we have

(to gallop)

**Theorem 2.3. (continued.)**

$$F(t,U_t,P_{Y_t})=F(T,\eta,P_\xi)+\int_t^T\Big\{-(\partial_s F)(s,U_s,P_{Y_s})+\sum_{i=1}^{d}(\partial_{x_i}F)(s,U_s,P_{Y_s})u_s^i$$

$$+\frac{1}{2}\sum_{i,j,k=1}^{d}(\partial^2_{x_ix_j}F)(s,U_s,P_{Y_s})v_s^{ik}v_s^{jk}-\frac{1}{2}\sum_{i,j=1}^{d}\sum_{k=1}^{l}(\partial^2_{x_ix_j}F)(s,U_s,P_{Y_s})V_s^{ik}V_s^{jk}\Big\}ds$$

$$+\int_t^T\widehat{E}\Big[(\partial_\mu F)_i(s,U_s,P_{Y_s},\widehat{Y_s})\widehat{f}_s^i-\frac{1}{2}\sum_{i,j,k=1}^{d}\partial_{y_i}(\partial_\mu F)_j(s,U_s,P_{Y_s},\widehat{Y_s})\widehat{Z_s^{ik}}\widehat{Z_s^{jk}}$$

$$+\frac{1}{2}\sum_{i,j=1}^{d}\sum_{k=1}^{l}\partial_{y_i}(\partial_\mu F)_j(s,U_s,P_{Y_s},\widehat{Y_s})\widehat{g_s^{ik}}\widehat{g_s^{jk}}\Big]ds$$

$$+\int_t^T\sum_{i=1}^{d}\sum_{j=1}^{l}(\partial_{x_i}F)(s,U_s,P_{Y_s})v_s^{ij}d\overleftarrow{B}_s^j-\int_t^T\sum_{i,j=1}^{d}(\partial_{x_i}F)(s,U_s,P_{Y_s})V_s^{ij}dW_s^j.$$

Here $(\widehat{Y}, \widehat{Z}, \widehat{f}, \widehat{g})$ denotes an independent copy of $(Y, Z, f, g)$, defined on another probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{P})$. The expectation $\widehat{E}[\cdot]$ on $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{P})$ concerns only random variables endowed with the superscript "$\widehat{\ }$".

**Remark**: We observe that the Itô formula studied in Buckdahn, Li, Peng, Rainer (2017, AOP) is a special case of Theorem 2.3.

**Mean-field stochastic differential equations**:

From now on let be given deterministic Lipschitz functions $b : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^{d\times d}$ satisfying

**Assumption (H3.1)** $b$ and $\sigma$ are bounded and Lipschitz on $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$.

We consider for the initial data $(t, x) \in [0, T] \times \mathbb{R}^d$ and $\xi \in L^2(\mathcal{G}_t; \mathbb{R}^d)$ the following both stochastic differential equations (SDEs):

$$X_s^{t,\xi} = \xi + \int_t^s b(X_r^{t,\xi}, P_{X_r^{t,\xi}})dr + \int_t^s \sigma(X_r^{t,\xi}, P_{X_r^{t,\xi}})dW_r, \tag{3.1}$$

and

$$X_s^{t,x,\xi} = x + \int_t^s b(X_r^{t,x,\xi}, P_{X_r^{t,\xi}})dr + \int_t^s \sigma(X_r^{t,x,\xi}, P_{X_r^{t,\xi}})dW_r, \;\; s \in [t, T]. \tag{3.2}$$

**Theorem 3.1. (see Buckdahn, Li, Peng, Rainer (2017, AOP))**

Under assumption (H3.1), the equations (3.1) and (3.2) admit unique solutions $X^{t,\xi} = (X_s^{t,\xi})_{s\in[t,T]}$ and $X^{t,x,\xi} = (X_s^{t,x,\xi})_{s\in[t,T]}$ in $\mathcal{S}^2_{\mathcal{G}}(t,T;\mathbb{R}^d)$. The solution $X^{t,x,\xi}$ is independent of $\mathcal{G}_t$.

**Remark**: (i) From the uniqueness of equation (3.1) for $X^{t,\xi}$, we have

$$X_s^{t,\xi} = X_s^{t,x,\xi}\big|_{x=\xi} = X_s^{t,\xi,\xi}, \ s \in [t,T].$$

(ii) The solutions of equations (3.1) and (3.2) satisfy a **Flow Property**:

$$(X_r^{s,X_s^{t,x,\xi},X_s^{t,\xi}}, X_r^{s,X_s^{t,\xi}}) = (X_r^{t,x,\xi}, X_r^{t,\xi}), \ r \in [s,T],$$

for all $0 \le t \le s \le T, \ x \in \mathbb{R}^d, \ \xi \in L^2(\mathcal{G}_t;\mathbb{R}^d)$.

## Proposition 3.1. (Buckdahn, Li, Peng, Rainer (2017, AOP))

Suppose Assumption (H3.1) holds true. Then, for all $p \geq 2$ there is a constant $C_p > 0$ only depending on the Lipschitz constants of $b$ and $\sigma$, such that for all $t \in [0,T]$, $x, \widehat{x} \in \mathbb{R}^d$, $\xi, \widehat{\xi} \in L^2(\mathcal{G}_t;\mathbb{R}^d)$, $P$-a.s.,

$$\begin{aligned}
&\text{(i) } E\Big[\sup_{s\in[t,T]} |X_s^{t,x,\xi} - X_s^{t,\widehat{x},\widehat{\xi}}|^p \Big| \mathcal{G}_t\Big] \leq C_p\Big(|x-\widehat{x}|^p + W_2(P_\xi, P_{\widehat{\xi}})^p\Big),\\
&\text{(ii) } E\Big[\sup_{s\in[t,T]} |X_s^{t,x,\xi}|^p \Big| \mathcal{G}_t\Big] \leq C_p\Big(1+|x|^p\Big),\\
&\text{(iii) } \sup_{s\in[t,T]} W_2(P_{X_s^{t,\xi}}, P_{X_s^{t,\widehat{\xi}}}) \leq C_2 W_2(P_\xi, P_{\widehat{\xi}}),\\
&\text{(iv) } E\Big[\sup_{s\in[t,t+h]} |X_s^{t,x,\xi} - x|^p \Big| \mathcal{G}_t\Big] \leq C_p h^{\frac{p}{2}}.
\end{aligned} \tag{3.3}$$

**Remark**: The processes $X^{t,x,\xi_1}$ and $X^{t,x,\xi_2}$ are indistinguishable, whenever the laws of $\xi_1, \xi_2 \in L^2(\mathcal{G}_t;\mathbb{R}^d)$ are the same. This means that $X^{t,x,\xi}$ depends on $\xi$ only through its law. Hence, we can define

$$X^{t,x,P_\xi} := X^{t,x,\xi},\ (t,x) \in [0,T]\times\mathbb{R}^d,\ \xi \in L^2(\mathcal{G}_t;\mathbb{R}^d). \tag{3.4}$$

**Mean-field BDSDEs**:

Let $\Phi : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$, $f : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d) \to \mathbb{R}$, $g : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d) \to \mathbb{R}^l$, and $h : \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d) \to \mathbb{R}^l$ be deterministic and satisfy:

**Assumption (H3.2)** The functions $f$, $g$, $h$ and $\Phi$ are bounded and Lipschitz, i.e., there exist constants $C > 0$, and $\alpha_1, \alpha_2 > 0$ with $0 < \alpha_1 + \alpha_2 < 1$ such that, for all $x, x' \in \mathbb{R}^d$, $y, y' \in \mathbb{R}$, $z, z' \in \mathbb{R}^d$, $\mu, \mu' \in \mathcal{P}_2(\mathbb{R}^{d+1} \times \mathbb{R}^d)$,

(i) $$|f(x,y,z,\mu) - f(x',y',z',\mu')| + |h(\mu) - h(\mu')| + |\Phi(x,\mu) - \Phi(x',\mu')| \leq C(|x - x'| + |y - y'| + |z - z'| + W_2(\mu,\mu')),$$

(ii) $$|g(x,y,z,\mu) - g(x',y',z',\mu')|^2 \leq C(|x - x'|^2 + |y - y'|^2) + \alpha_1 |z - z'|^2 + W_{2,C,\alpha_2}(\mu,\mu')^2.$$

Given $x \in \mathbb{R}^d$ and $\xi \in L^2(\mathcal{G}_t;\mathbb{R}^d)$ we consider the following split BDSDEs: for $s \in [t,T]$,

$$\begin{cases} dY_s^{t,\xi} = -f(\Pi_s^{t,\xi}, P_{\Pi_s^{t,\xi}})ds - \big(g(\Pi_s^{t,\xi}, P_{\Pi_s^{t,\xi}}) + h(P_{\Pi_s^{t,\xi}})\big)d\overleftarrow{B_s} + Z_s^{t,\xi}dW_s, \\ Y_T^{t,\xi} = \Phi(X_T^{t,\xi}, P_{X_T^{t,\xi}}), \end{cases} \tag{3.5}$$

$$\begin{cases} dY_s^{t,x,\xi} = -f(\Pi_s^{t,x,\xi}, P_{\Pi_s^{t,\xi}})ds - \big(g(\Pi_s^{t,x,\xi}, P_{\Pi_s^{t,\xi}}) + h(P_{\Pi_s^{t,\xi}})\big)d\overleftarrow{B_s} + Z_s^{t,x,\xi}dW_s, \\ Y_T^{t,x,\xi} = \Phi(X_T^{t,x,\xi}, P_{X_T^{t,\xi}}), \end{cases} \tag{3.6}$$

where $\Pi_s^{t,\xi} := (X_s^{t,\xi}, Y_s^{t,\xi}, Z_s^{t,\xi})$, $\Pi_s^{t,x,\xi} := (X_s^{t,x,\xi}, Y_s^{t,x,\xi}, Z_s^{t,x,\xi})$.

## Proposition 3.2.

Under assumptions (H3.1) and (H3.2), the equations (3.5) and (3.6) admit unique solutions $(Y^{t,\xi}, Z^{t,\xi})$ and $(Y^{t,x,\xi}, Z^{t,x,\xi}) \in \mathcal{S}^2_{\mathcal{F}}(t,T;\mathbb{R}) \times \mathcal{H}^2_{\mathcal{F}}(t,T;\mathbb{R}^d)$. The solution $(Y^{t,x,\xi}, Z^{t,x,\xi})$ is independent of $\mathcal{G}_t$.

**Remark**: (i) From the uniqueness of solution we have

$$\Pi^{t,\xi}_s = \Pi^{t,x,\xi}_s\big|_{x=\xi} = \Pi^{t,\xi,\xi}_s, \text{ in } \mathcal{S}^2_{\mathcal{F}}(t,T;\mathbb{R}^d) \times \mathcal{S}^2_{\mathcal{F}}(t,T;\mathbb{R}) \times \mathcal{H}^2_{\mathcal{F}}(t,T;\mathbb{R}^d).$$

(ii) From the flow property and the uniqueness of the solution of (3.5) and (3.6) we have the following properties: For all $0 \le t \le s \le T$, $x \in \mathbb{R}^d$, $\xi \in L^2(\mathcal{G}_t;\mathbb{R}^d)$,

$$\begin{aligned} &\bullet\, (Y_r^{s,X_s^{t,x,\xi},X_s^{t,\xi}}, Y_r^{s,X_s^{t,\xi}}) = (Y_r^{t,x,\xi}, Y_r^{t,\xi}),\ r \in [s,T],\ P\text{-}a.s.; \\ &\bullet\, (Z_r^{s,X_s^{t,x,\xi},X_s^{t,\xi}}, Z_r^{s,X_s^{t,\xi}}) = (Z_r^{t,x,\xi}, Z_r^{t,\xi}),\ drdP\text{-}a.e. \text{ on } [s,T]\times\Omega. \end{aligned} \tag{3.7}$$

**Assumption (H3.3)** For some $p \geq 2$, $\overline{C}_p(\alpha_1+\alpha_2)^{\frac{p}{2}} < 1$. Here $\overline{C}_p := 2^{p-1}C_p^*((\frac{p}{p-1})^p+1)C_p'$, $C_p' := (\frac{p}{p-1})^p 3^{p-1}\big(2C^p 5^{p-1} \vee (6p^3)^p 5^{\frac{p}{2}-1}\big)$, $C_p^* := 2^{-p-2}3^p p^{3p} + 2^{\frac{p}{2}}$, $C$ is the Lipschitz constant in Assumption (H4.1).

## Proposition 3.3.

Suppose the Assumptions (H3.1), (H3.2) and (H3.3) hold true. Then, for all $p \geq 2$, there is a constant $C_p > 0$ only depending on the Lipschitz constants of $b$, $\sigma$, $f$, $g$, $h$ and $\Phi$, such that, for $t \in [0,T]$, $x, \widehat{x} \in \mathbb{R}^d$, $\xi, \widehat{\xi} \in L^2(\mathcal{G}_t;\mathbb{R}^d)$,

(i) $E[\sup\limits_{s\in[t,T]} |Y_s^{t,x,\xi}|^p + (\int_t^T |Z_s^{t,x,\xi}|^2)^{\frac{p}{2}} |\mathcal{G}_t] \leq C_p;$

(ii) $E[\sup\limits_{s\in[t,T]} |Y_s^{t,x,\xi} - Y_s^{t,\widehat{x},\widehat{\xi}}|^p + (\int_t^T |Z_s^{t,x,\xi} - Z_s^{t,\widehat{x},\widehat{\xi}}|^2 ds)^{\frac{p}{2}} |\mathcal{G}_t]$
$\leq C_p\Big(|x-\widehat{x}|^p + W_2(P_\xi, P_{\widehat{\xi}})^p\Big);$

(iii) $\int_t^T W_2(P_{\Pi_s^{t,\xi}}, P_{\Pi_s^{t,\widehat{\xi}}})^2 \leq C W_2(P_\xi, P_{\widehat{\xi}})^2.$

**Remark**: Due to Proposition 3.3 $Y^{t,x,\xi}$ and $Z^{t,x,\xi}$ depend on $\xi$ only through its law, hence we can define

$$Y_s^{t,x,P_\xi} := Y_s^{t,x,\xi},\ Z_s^{t,x,P_\xi} := Z_s^{t,x,\xi}.$$

Now we introduce **the value function**:

$$V(t,x,P_\xi) := Y_t^{t,x,P_\xi},\ (t,x,P_\xi) \in [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d). \tag{3.8}$$

**Remark**: (i) $V(t,x,P_\xi)$ is $\mathcal{F}_{t,T}^B$-measurable, for all $(t,x)$.

(ii) $V(s, X_s^{t,x,P_\xi}, P_{X_s^{t,\xi}}) = Y_s^{s,X_s^{t,x,\xi},P_{X_s^{t,\xi}}} = Y_s^{t,x,P_\xi},\ s \in [t,T].$

Preparation for the study of regularity of $V$:

- Assuming regularity of $\Phi$, $b$, $\sigma$, $f$, $g$
- We study the regularity of $X^{t,x,P_\xi}$, $X^{t,\xi}$, $(Y^{t,x,P_\xi}, Z^{t,x,P_\xi})$, $(Y^{t,\xi}, Z^{t,\xi})$.

**Study of the first order derivatives of $X^{t,x,P_\xi}$** (For details: Buckdahn, Li, Peng, Rainer (2017, AOP))

**Assumption (H4.1)** $(b,\sigma) \in C_b^{1,1}(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d \times \mathbb{R}^{d\times d})$, that is, the components $b_j$, $\sigma_{i,j}$, $1 \le i,j \le d$, have the following properties:

(i) $b_j(x,\cdot)$, $\sigma_{i,j}(x,\cdot) \in C_b^1(\mathcal{P}_2(\mathbb{R}^d))$, $x \in \mathbb{R}^d$;

(ii) $b_j(\cdot,\mu)$, $\sigma_{i,j}(\cdot,\mu) \in C_b^1(\mathbb{R}^d)$, $\mu \in \mathcal{P}_2(\mathbb{R}^d)$;

(iii) $\partial_x b_j$, $\partial_x \sigma_{i,j}$: $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$ , $\partial_\mu b_j$, $\partial_\mu \sigma_{i,j}$: $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}^d$ are bounded and Lipschitz continuous.

(galopp through)

**Theorem 4.1.** (Derivative w.r.t. $x$; results are classical)

Suppose Assumption (4.1) holds true. Then the $L^2$-derivative of $X^{t,x,P_\xi}$ with respect to $x$ exists, it is denoted by $\partial_x X^{t,x,P_\xi} = (\partial_x X^{t,x,P_\xi,j})_{1\le j\le d}$, and it satisfies the following SDE: $s \in [t,T]$, $1 \le i,j \le d$,

$$\begin{aligned}\partial_{x_i} X_s^{t,x,P_\xi,j} &= \delta_{ij} + \sum_{k=1}^d \int_t^s \partial_{x_k} b_j(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}})\partial_{x_i} X_s^{t,x,P_\xi,k} dr \\ &\quad + \sum_{k,l=1}^d \int_t^s \partial_{x_k}\sigma_{j,l}(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}})\partial_{x_i} X_s^{t,x,P_\xi,k} dW_r^l.\end{aligned}$$

Moreover, For all $p \ge 2$, there exists a constant $C_p > 0$ only depending on the Lipschitz constants of $\partial_x b$ and $\partial_x \sigma$, such that, for all $t \in [0,T]$, $x, x' \in \mathbb{R}^d$, $\xi, \xi' \in L^2(\mathcal{G}_t;\mathbb{R}^d)$, $P$-a.s.,

# 4. First and second order derivatives of $X^{t,x,P_\xi}$

## Theorem 4.1. (continued.)

(i) $E\Big[\sup_{s\in[t,T]} |\partial_x X_s^{t,x,P_\xi}|^p \Big| \mathcal{G}_t\Big] \le C_p$;

(ii) $E\Big[\sup_{s\in[t,T]} |\partial_x X_s^{t,x,P_\xi} - \partial_x X_s^{t,x',P_{\xi'}}|^p \Big| \mathcal{G}_t\Big] \le C_p\Big(|x-x'|^p + W_2(P_\xi, P_{\xi'})^p\Big)$;

(iii) $E\Big[\sup_{s\in[t,t+h]} |\partial_x X_s^{t,x,P_\xi} - I_{d\times d}|^p \Big| \mathcal{G}_t\Big] \le C_p h^{\frac{p}{2}},\ 0 \le t \le t+h \le T.$

Here $I_{d\times d}$ denotes the unit matrix in dimension $d$.

Proof. For the proof the reader is referred to Theorem 3.1 in Buckdahn, Li, Peng, Rainer (AOP, 2017), and for the case with jumps also to Theorem 4.1 in Hao, Li (NODEA, 2016).

## Theorem 4.2. (Derivative w.r.t. $P_\xi$; Buckdahn, Li, Peng, Rainer (AOP, 2017))

Let $(b,\sigma)$ satisfy Assumption (H4.1). Then, for all $0 \le t \le s \le T$, $x \in \mathbb{R}^d$, $\mathcal{P}_2(\mathbb{R}^d) \ni \mu \to X_s^{t,x,\mu} \in L^2(\mathcal{G}_s;\mathbb{R}^d)$ is differentiable, and the derivative is given by

$$\partial_\mu X_s^{t,x,\mu}(y) = U_s^{t,x,\mu}(y)$$

where, for $y \in \mathbb{R}^d$, $U^{t,x,P_\xi}(y) = ((U_{s,i,j}^{t,x,P_\xi}(y))_{s\in[t,T]})_{1\le i,j\le d} \in \mathcal{S}_{\mathcal{G}}^2(t,T;\mathbb{R}^{d\times d})$ is the unique solution of the SDE (for shortness: $b=0$): $s \in [t,T]$, $1 \le i,j \le d$,

$$\begin{aligned} U_{s,i,j}^{t,x,P_\xi}(y) = &\sum_{k,l=1}^{d}\int_t^s \partial_{x_k}\sigma_{i,l}(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}})U_{r,k,j}^{t,x,P_\xi}(y)dW_r^l \\ &+ \sum_{k,l=1}^{d}\int_t^s E[(\partial_\mu\sigma_{i,l})(z, P_{X_r^{t,\xi}}, X_r^{t,y,P_\xi})\partial_{x_j}X_s^{t,y,P_\xi,k} \\ &\qquad + (\partial_\mu\sigma_{i,l})(z, P_{X_r^{t,\xi}}, X_r^{t,\xi})U_{r,k,j}^{t,\xi}(y)]\big|_{z=X_r^{t,x,P_\xi}} dW_r^l, \end{aligned}$$

where $U^{t,\xi}(y) = ((U_{s,i,j}^{t,\xi}(y))_{s\in[t,T]})_{1\le i,j\le d} = U^{t,x,P_\xi}(y)\big|_{x=\xi} \in \mathcal{S}_{\mathcal{G}}^2(t,T;\mathbb{R}^{d\times d})$ satisfies this SDE with $x$ replaced by $\xi$.

**Extension to the second order derivatives**:

**Assumption (H4.2)** $(b,\sigma) \in C_b^{2,2}(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d \times \mathbb{R}^{d\times d})$, that is, $(b,\sigma)$ $\in C_b^{1,1}(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d \times \mathbb{R}^{d\times d})$ and:

(i) $\partial_{x_k} b_i(\cdot,\mu)$, $\partial_{x_k}\sigma_{i,j}(\cdot,\mu) \in C_b^1(\mathbb{R}^d)$, for all $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, $1 \le k \le d$;

(ii) $\partial_\mu b_j(x,\mu,\cdot)$, $\partial_\mu \sigma_{i,j}(x,\mu,\cdot) \in C_b^1(\mathbb{R}^d \to \mathbb{R}^d)$, for all $x \in \mathbb{R}^d$, $\mu \in \mathcal{P}_2(\mathbb{R}^d)$;

(iii) All the derivatives of $b_j$, $\sigma_{i,j}$ up to order 2 are bounded and Lipschitz.

**Theorem 4.3. (Buckdahn, Li, Peng, Rainer (AOP, 2017))**

Under Assumption (H4.2) the first order derivatives $x \to \partial_{x_i} X^{t,x,P_\xi}$, $\partial_\mu X^{t,x,P_\xi}(y) \in \mathcal{S}_{\mathcal{G}}^2(t,T;\mathbb{R}^d)$ are differentiable w.r.t. $x$ and $y$, respectively, and for

$$M_{s,i,j}^{t,x,P_\xi}(y) := \left(\partial^2_{x_i x_j} X_s^{t,x,P_\xi}, \partial_{y_i}(\partial_\mu X_s^{t,x,P_\xi}(y))\right),\ 1 \le i,j \le d,$$

we have that, for all $p \ge 2$, $\exists C_p \in \mathbb{R}_+$ such that, for all $t \in [0,T]$, $x,x',y,y' \in \mathbb{R}^d$, $\xi,\xi' \in L^2(\mathcal{G}_t;\mathbb{R}^d)$, $1 \le i,j \le d$,

(i) $E\big[\sup_{s\in[t,T]} |M_{s,i,j}^{t,x,P_\xi}(y) - M_{s,i,j}^{t,x',P_{\xi'}}(y')|^p\big] \le C_p\big(|x-x'|^p + |y-y'|^p + W_2(P_\xi,P_{\xi'})^p\big)$;

(ii) $E\big[\sup_{s\in[t,t+h]} |M_{s,i,j}^{t,x,P_\xi}(y)|^p\big] \le C_p h^{\frac{p}{2}},\ 0 \le t \le t+h \le T.$

**Study of the first order derivatives of** $(Y^{t,x,P_\xi}, Z^{t,x,P_\xi})$:

To simplify, but w.l.o.g: $d=1$, $l=1$ and $f(\Pi_s^{t,x,\xi}, P_{\Pi_s^{t,\xi}}) = f(Z_s^{t,x,\xi}, P_{Z_s^{t,\xi}})$, $g(\Pi_s^{t,x,\xi}, P_{\Pi_s^{t,\xi}}) = g(Z_s^{t,x,\xi}, P_{Z_s^{t,\xi}})$, $h(P_{\Pi_s^{t,\xi}}) = h(P_{Z_s^{t,\xi}})$, $\Phi(X_T^{t,x,P_\xi}, P_{X_T^{t,\xi}}) = \Phi(X_T^{t,x,P_\xi})$.

**Assumption (H5.1)** Let $\Phi \in C_b^1(\mathbb{R})$, $f \in C_b^{1,1}(\mathbb{R} \times \mathcal{P}_2(\mathbb{R}))$, $g \in C_b^{1,1}(\mathbb{R} \times \mathcal{P}_2(\mathbb{R}))$ and $h \in C_b^1(\mathcal{P}_2(\mathbb{R}^2))$. In addition we suppose Assumption (H4.1)-(ii), i.e., there exist constants $\alpha_1, \alpha_2 > 0$ with $0 < \alpha_1 + \alpha_2 < 1$ such that

$$|g(z,\mu) - g(z',\mu')|^2 \le \alpha_1 |z - z'|^2 + \alpha_2 W_2(\mu,\mu')^2.$$

The derivatives for

$$\begin{cases} dY_s^{t,x,\xi} = -f(\Pi_s^{t,x,\xi}, P_{\Pi_s^{t,\xi}})ds - \big(g(\Pi_s^{t,x,\xi}, P_{\Pi_s^{t,\xi}}) + h(P_{\Pi_s^{t,\xi}})\big) d\overleftarrow{B}_s + Z_s^{t,x,\xi} dW_s, \\ Y_T^{t,x,\xi} = \Phi(X_T^{t,x,\xi}, P_{X_T^{t,\xi}}), \ \ s \in [t,T]. \end{cases} \tag{3.6}$$

## Theorem 5.1. (Derivative w.r.t. $x$)

Under the Assumptions (H3.3), (H4.1) and (H5.1), the $L^2$-derivative of the solution of Eq. (3.6) with respect to $x$, $(\partial_x Y^{t,x,P_\xi}, \partial_x Z^{t,x,P_\xi})$, exists and is the unique solution of the following BDSDE: $s \in [t,T]$,

$$\begin{aligned} \partial_x Y_s^{t,x,P_\xi} &= \Phi(X_T^{t,x,P_\xi})\partial_x X_T^{t,x,P_\xi} + \int_s^T \partial_z f(Z_r^{t,x,P_\xi}, P_{Z_r^{t,\xi}})\partial_x Z_r^{t,x,P_\xi} dr \\ &\quad + \int_s^T \partial_z g(Z_r^{t,x,P_\xi}, P_{Z_r^{t,\xi}})\partial_x Z_r^{t,x,P_\xi} d\overleftarrow{B}_r - \int_s^T \partial_x Z_r^{t,x,P_\xi} dW_r. \end{aligned} \tag{5.1}$$

## Proposition 5.1.

For all $p \geq 2$, there exists a constant $C_p > 0$ only depending on the Lipschitz constants of $\partial_x b$ and $\partial_x \sigma$, such that, for all $t \in [0,T]$, $x, x' \in \mathbb{R}$, $\xi, \xi' \in L^2(\mathcal{G}_t;\mathbb{R})$, $P$-a.s.,

(i) $E\Big[\sup_{s\in[t,T]} |\partial_x Y_s^{t,x,P_\xi}|^p + (\int_t^T |\partial_x Z_s^{t,x,P_\xi}|^2 ds)^{\frac{p}{2}} ds \Big| \mathcal{G}_t\Big] \leq C_p;$

(ii) $E\Big[\sup_{s\in[t,T]} |\partial_x Y_s^{t,x,P_\xi} - \partial_x Y_s^{t,x',P_{\xi'}}|^p + (\int_t^T |\partial_x Z_s^{t,x,P_\xi} - \partial_x Z_s^{t,x',P_{\xi'}}|^2 ds)^{\frac{p}{2}} \Big| \mathcal{G}_t\Big]$

$\leq C_p M^p \big(|x-x'|^p + W_2(P_\xi, P_{\xi'})^p\big) + \rho_{M,p}(t,x,P_\xi),$

with $M \geq 1$, $\rho_{M,p}(t,x,P_\xi) \to 0$, as $M \to \infty$, $E[\rho_{M,p}(t,\xi,P_\xi)] \to 0$, as $M \to \infty$.

**Remark**. The term $\rho_{M,p}(t,x,P_\xi)$ in (ii) comes from the estimate of

$$\int_s^T \big(\partial_x g(Z_r^{t,x,P_\xi}, P_{Z_r^{t,\xi}}) - \partial_x g(Z_r^{t,x',P_{\xi'}}, P_{Z_r^{t,\xi'}})\big)\partial_x Z_r^{t,x,P_\xi} d\overleftarrow{B}_r$$ using (i).

Can one make better with Malliavin derivate?

## Proposition 5.2.

Let the Assumptions (H3.3), (H4.1) and (H5.1) hold true. Then for all $(t,x)\in[0,T]\times\mathbb{R}$, $\xi\in L^2(\mathcal{G}_t;\mathbb{R})$, $s\in[t,T]$, $(Y_s^{t,x,P_\xi}, Z_s^{t,x,P_\xi})\in L^2(t,T;(\mathbb{D}^{1,2})^2)$ and a version of $\{D_\theta Y_s^{t,x,P_\xi}, D_\theta Z_s^{t,x,P_\xi} : \theta, s\in[t,T]\}$ is given by:

(i) $D_\theta Y_s^{t,x,P_\xi}=0$, $D_\theta Z_s^{t,x,P_\xi}=0$, $t\le s<\theta\le T$;

(ii) $\{(D_\theta Y^{t,x,P_\xi}, D_\theta Z^{t,x,P_\xi}) : s\in[\theta,T]\}$ is the unique solution of the linear BDSDE: $s\in[t,T]$, $d\theta dP$-a.e., $t\le\theta\le s$,

$$\begin{aligned} D_\theta Y_s^{t,x,P_\xi} &= \partial_x\Phi(X_T^{t,x,P_\xi})D_\theta X_T^{t,x,P_\xi}+\int_s^T \partial_z f(Z_r^{t,x,P_\xi},P_{Z_r^{t,\xi}})D_\theta Z_r^{t,x,P_\xi}dr \\ &+\int_s^T \partial_z g(Z_r^{t,x,P_\xi},P_{Z_r^{t,\xi}})D_\theta Z_r^{t,x,P_\xi}d\overleftarrow{B}_r-\int_s^T D_\theta Z_r^{t,x,P_\xi}dW_r. \end{aligned} \tag{5.2}$$

Moreover,

$$Z_s^{t,x,P_\xi}=P\text{-}\lim_{s<u\downarrow s} D_s Y_u^{t,x,P_\xi},\ dsdP\text{-a.e.} \tag{5.3}$$

## Proposition 5.2. (continued.)

Furthermore, for all $p \geq 2$ there exists a constant $C_p > 0$ such that

(i) $E\big[\sup_{s\in[t,T]} |D_\theta Y_s^{t,x,P_\xi}|^p + (\int_t^T |D_\theta Z_s^{t,x,P_\xi}|^2 ds)^{\frac{p}{2}}\big] \leq C_p;$

(ii) $$E\big[\sup_{s\in[t,T]} |D_\theta Y_s^{t,x,P_\xi} - D_\theta Y_s^{t,x',P_{\xi'}}|^p + (\int_t^T |D_\theta Z_s^{t,x,P_\xi} - D_\theta Z_s^{t,x',P_{\xi'}}|^2 ds)^{\frac{p}{2}}\big]$$
$$\leq C_p M^p \big(|x - x'|^p + W_2(P_\xi, P_{\xi'})^p\big) + \rho_{M,p,\theta}(t, x, P_\xi).$$

In particular,

(i) $E[|Z_s^{t,x,P_\xi}|^p] \leq C_p$;

(ii) $E[|Z_s^{t,x,P_\xi} - Z_s^{t,x',P_{\xi'}}|^p] \leq C_p M^p \big(|x - x'|^p + W_2(P_\xi, P_{\xi'})^p\big) + \rho_{M,p}(t, x, P_\xi)$,

with $M \geq 1$, $\rho_{M,p}(t, x, P_\xi) \to 0$, as $M \to \infty$, $E[\rho_{M,p}(t, \xi, P_\xi)] \to 0$, as $M \to \infty$.

## Theorem 5.2. (Derivative w.r.t. $P_\xi$)

Assume the Assumptions (H3.3), (H4.1) and (H5.1) hold. Then, for all $x \in \mathbb{R}$, $0 \le t \le s \le T$, $\mathcal{P}_2(\mathbb{R}) \ni \mu \to Y_s^{t,x,\mu} \in L^2(\mathcal{F}_s;\mathbb{R})$, and $\mathcal{P}_2(\mathbb{R}) \ni \mu \to Z_s^{t,x,\mu} \in \mathcal{H}^2_{\mathcal{F}}(t,T;\mathbb{R})$ are differentiable, with the derivatives

$$\begin{aligned} &\partial_\mu Y_s^{t,x,\mu}(y) = O_s^{t,x,\mu}(y), \ \ s \in [t,T], \ \ P\text{-}a.s., \\ &\partial_\mu Z_s^{t,x,\mu}(\eta) = Q_s^{t,x,\mu}(y), \ \ dsdP\text{-}a.e., \end{aligned} \tag{5.4}$$

where for all $y \in \mathbb{R}$, $\mu = P_\xi$ $(\xi \in L^2(\mathcal{G}_t,\mathbb{R})$, $(O^{t,x,P_\xi}(y), Q^{t,x,P_\xi}(y)) \in \mathcal{S}^2_{\mathcal{F}}(t,T;\mathbb{R}) \times \mathcal{H}^2_{\mathcal{F}}(t,T;\mathbb{R})$ is the unique solution of the BDSDE:

$$O_s^{t,x,P_\xi}(y) = \Phi(X_T^{t,x,P_\xi})\partial_\mu X_T^{t,x,P_\xi}(y) + \int_s^T (\partial_z f)(Z_r^{t,x,P_\xi}, P_{Z_r^{t,\xi}})Q_r^{t,x,P_\xi}(y)dr$$
$$+ \int_s^T (\partial_z g)(Z_r^{t,x,P_\xi}, P_{Z_r^{t,\xi}})Q_r^{t,x,P_\xi}(y)d\overleftarrow{B}_r +$$

**Theorem 5.2. (continued.)**

$$+\int_s^T E\big[(\partial_\mu f)(z,P_{Z_r^{t,\xi}},Z_r^{t,y,P_\xi})\partial_x Z_r^{t,y,P_\xi}+(\partial_\mu f)(z,P_{Z_r^{t,\xi}},Z_r^{t,\xi})Q_r^{t,\xi}(y)\big]\Big|_{z=Z_r^{t,x,P_\xi}}dr$$
$$+\int_s^T E\big[\big((\partial_\mu g)(z,P_{Z_r^{t,\xi}},Z_r^{t,y,P_\xi})+(\partial_\mu h)(P_{Z_r^{t,\xi}},Z_r^{t,y,P_\xi})\big)\partial_x Z_r^{t,y,P_\xi}\big]\Big|_{z=Z_r^{t,x,P_\xi}}d\overleftarrow{B}_r$$
$$+\int_s^T E\big[\big((\partial_\mu g)(z,P_{Z_r^{t,\xi}},Z_r^{t,\xi})+(\partial_\mu h)(P_{Z_r^{t,\xi}},Z_r^{t,\xi})\big)Q_r^{t,\xi}(y)\big]\Big|_{z=Z_r^{t,x,P_\xi}}d\overleftarrow{B}_r$$
$$-\int_s^T Q_r^{t,x,P_\xi}(y)dW_r,\ \ s\in[t,T],$$

where $(O^{t,\xi},Q^{t,\xi})=(O^{t,\xi,P_\xi},Q^{t,\xi,P_\xi})$ is the unique solution of the above BDSDE with $x$ replaced by $\xi$.

As before, $(O^{t,x,P_\xi}, Q^{t,x,P_\xi})$ is the derivative of $(Y^{t,x,P_\xi}, Z^{t,x,P_\xi})$ w.r.t. the measure $P_\xi$, i.e., $\partial_\mu Y_s^{t,x,P_\xi}(y) := O_s^{t,x,P_\xi}(y)$, $\partial_\mu Z_s^{t,x,P_\xi}(y) := Q_s^{t,x,P_\xi}(y)$.

## Proposition 5.3.

For all $p \geq 2$, $\exists\ C_p > 0$, s.t., for all $t \in [0,T]$, $x, x', y, y' \in \mathbb{R}$, $\xi, \xi' \in L^2(\mathcal{G}_t; \mathbb{R})$,

$$\text{(i) } E\Big[\sup_{s\in[t,T]} |\partial_\mu Y_s^{t,x,P_\xi}(y)|^p + (\int_t^T |\partial_\mu Z_s^{t,x,P_\xi}(y)|^2 ds)^{\frac{p}{2}}\Big] \leq C_p,$$

$$\text{(ii) } E\Big[\sup_{s\in[t,T]} |\partial_\mu Y_s^{t,x,P_\xi}(y) - \partial_\mu Y_s^{t,x',P_{\xi'}}(y')|^p + (\int_t^T |\partial_\mu Z_s^{t,x,P_\xi}(y) - \partial_\mu Z_s^{t,x',P_{\xi'}}(y')|^2 ds)^{\frac{p}{2}}\Big]$$
$$\leq C_p M^p \big(|x-x'|^p + |y-y'|^p + W_2(P_\xi, P_{\xi'})^p\big)$$
$$+\rho_{M,p}(t,x,y,P_\xi) + \rho_{M,p}(t,y,P_\xi) + E[\rho_{M,p}(t,\xi,y,P_\xi)],$$

with $M \geq 1$, $\rho_{M,p}(t,x,y,P_\xi) \to 0$, $\rho_{M,p}(t,y,P_\xi) \to 0$, $E[\rho_{M,p}(t,\xi,y,P_\xi)] \to 0$, as $M \to \infty$.

**Extension to the second order derivatives**:

**Assumption (H5.2)** Let $f \in C_b^{2,2}(\mathbb{R}^{d+1+d} \times \mathcal{P}_2(\mathbb{R}^{d+1+d}))$, $\Phi \in C_b^{2,2}(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$ and $h \in C_b^2(\mathcal{P}_2(\mathbb{R}^{d+1+d}) \to \mathbb{R}^l)$.

**Assumption (H5.3)** The coefficient $g$ is affine in $z$: for all $x \in \mathbb{R}^d$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$, $\mu \in \mathcal{P}_2(\mathbb{R}^{d+1} \times \mathbb{R}^d)$,

$$g(x, y, z, \mu) = g^1(x, y, \mu) + g^2(\mu(\cdot \times \mathbb{R} \times \mathbb{R}^d))z,$$

where $g^1 \in C_b^{2,2}(\mathbb{R}^{d+1} \times \mathcal{P}_2(\mathbb{R}^{d+1+d}) \to \mathbb{R}^l)$ and $g^2 \in C_b^2(\mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^l)$. In addition we suppose $|g^2|^2 \le \alpha_1$, $\sum_{k=1}^d \sum_{i=1}^l |(\partial_\mu g_i^1)_{d+1+k}|^2 \le \alpha_2$, where constants $\alpha_1, \alpha_2 > 0$ with $0 < \alpha_1 + \alpha_2 < 1$.

To simplify, but w.l.o.g: $d = 1$, $l = 1$ and $g(\Pi_s^{t,x,\xi}, P_{\Pi_s^{t,\xi}}) = g^2(P_{X_s^{t,\xi}})Z_s^{t,x,P_\xi}$, $h(P_{\Pi_s^{t,\xi}}) = h(P_{(Y_s^{t,\xi}, Z_s^{t,\xi})})$, $f(\Pi_s^{t,x,\xi}, P_{\Pi_s^{t,\xi}}) = 0$ and $\Phi(X_T^{t,x,P_\xi}, P_{X_T^{t,\xi}}) = \Phi(X_T^{t,x,P_\xi})$.

**Remark**: (i) In order to understand better why we need Assumption (H5.3), consider for $d = l = 1$ and for the functions $\Phi, g \in C_b^2(\mathbb{R})$ with $|\partial_x g|^2 \le \alpha_1 < 1$, the BDSDE

$$Y_s^{t,x,P_\xi} = \Phi(X_T^{t,x,P_\xi}) + \int_s^T g(Z_r^{t,x,P_\xi}) d\overleftarrow{B_r} - \int_s^T Z_r^{t,x,P_\xi} dW_r,\ \ s \in [t,T].$$

Then, as we have seen, $(\partial_x Y_s^{t,x,P_\xi}, \partial_x Z_s^{t,x,P_\xi})$ is the solution of the linear BDSDE

$$\begin{aligned}\partial_x Y_s^{t,x,P_\xi} &= \partial_x \Phi(X_T^{t,x,P_\xi}) \partial_x X_T^{t,x,P_\xi} + \int_s^T \partial_z g(Z_r^{t,x,P_\xi}) \partial_x Z_r^{t,x,P_\xi} d\overleftarrow{B_r} \\ &\quad - \int_s^T \partial_x Z_r^{t,x,P_\xi} dW_r,\ \ s \in [t,T],\end{aligned} \tag{5.5}$$

# 5. First and second order derivatives of $(Y^{t,x,P_\xi}, Z^{t,x,P_\xi})$

and the formal second derivative $(\partial^2_{xx}Y_s^{t,x,P_\xi}, \partial^2_{xx}Z_s^{t,x,P_\xi})$ should solve the BDSDE

$$\partial^2_{xx}Y_s^{t,x,P_\xi} = \partial^2_{xx}\Phi(X_T^{t,x,P_\xi})(\partial_x X_T^{t,x,P_\xi})^2 + \partial_x\Phi(X_T^{t,x,P_\xi})\partial^2_{xx}X_T^{t,x,P_\xi}$$
$$+\int_s^T \left(\partial^2_{zz}g(Z_r^{t,x,P_\xi})(\partial_x Z_r^{t,x,P_\xi})^2 + \partial_z g(Z_r^{t,x,P_\xi})\partial^2_{xx}Z_r^{t,x,P_\xi}\right)d\overleftarrow{B_r} - \int_s^T \partial^2_{xx}Z_r^{t,x,P_\xi}dW_r,$$

$s\in[t,T]$. However, to give sense to $\int_s^T \partial^2_{zz}g(Z_r^{t,x,P_\xi})(\partial_x Z_r^{t,x,P_\xi})^2 d\overleftarrow{B_r}$, $s\in[t,T]$, we need $P(\int_t^T |\partial_x Z_r^{t,x,P_\xi}|^4 dr < \infty) = 1$, while equation (5.5) only allows to conclude that $E[(\int_t^T |\partial_x Z_r^{t,x,P_\xi}|^2 dr)^p] < \infty$, $p \geq 1$.

This is why we suppose (H5.3).

**Remark**:

(ii) For all $p \geq 2$, $\exists\ C_p > 0$, s.t., for all $x, x' \in \mathbb{R}$, $\xi, \xi' \in L^2(\mathcal{G}_t; \mathbb{R})$, $s \in [t, T]$,

$$\begin{aligned} &\bullet\ E[|Z_s^{t,x,P_\xi}|^p] \leq C_p; \\ &\bullet\ E[|Z_s^{t,x,P_\xi} - Z_s^{t,x',P_{\xi'}}|^p] \leq C_p\big(|x - x'|^p + W_2(P_\xi, P_{\xi'})^p\big). \end{aligned} \tag{5.6}$$

(iii) For all $p \geq 2$, $\exists\ C_p > 0$, s.t., for all $x, x' \in \mathbb{R}$, $\xi, \xi' \in L^2(\mathcal{G}_t; \mathbb{R})$, $s \in [t, T]$,

$$\begin{aligned} &\bullet\ E[|\partial_x Z_s^{t,x,P_\xi}|^p] \leq C_p; \\ &\bullet\ E[|\partial_x Z_s^{t,x,P_\xi} - \partial_x Z_s^{t,x',P_{\xi'}}|^p] \leq C_p\big(|x - x'|^p + W_2(P_\xi, P_{\xi'})^p\big). \end{aligned} \tag{5.7}$$

# 5. First and second order derivatives of $(Y^{t,x,P_\xi}, Z^{t,x,P_\xi})$

## Theorem 5.3.

Under Assumptions (H3.3), (H4.2), (H5.2) and (H5.3). For all $t \in [0,T]$, $x \in \mathbb{R}$, $\xi \in L^2(\mathcal{G}_t;\mathbb{R})$, we have

(i) The differentiability (in $L^2$) of the mappings

$$\mathbb{R} \ni x \to (\partial_x Y^{t,x,P_\xi}, \partial_x Z^{t,x,P_\xi}) \in \mathcal{S}^2_{\mathcal{F}}(t,T;\mathbb{R}) \times \mathcal{H}^2_{\mathcal{F}}(t,T;\mathbb{R}),$$

$$\mathbb{R} \ni y \to (\partial_\mu Y^{t,x,P_\xi}(y), \partial_\mu Z^{t,x,P_\xi}(y)) \in \mathcal{S}^2_{\mathcal{F}}(t,T;\mathbb{R}) \times \mathcal{H}^2_{\mathcal{F}}(t,T;\mathbb{R}).$$

(ii) Moreover, for all $p \geq 2$, for $\alpha_1, \alpha_2 > 0$ small enough (depending on $p$) $\exists\, C_p > 0$, s.t. for both $(\zeta_s^{t,x,P_\xi}(y), \delta_s^{t,x,P_\xi}(y)) \in$ $\{(\partial^2_{xx} Y_s^{t,x,P_\xi}, \partial^2_{xx} Z_s^{t,x,P_\xi}), (\partial_y \partial_\mu Y_s^{t,x,P_\xi}(y), \partial_y \partial_\mu Z_s^{t,x,P_\xi}(y))\}$,

(a) $E[\sup_{s\in[t,T]} |\zeta_s^{t,x,P_\xi}(y)|^p + (\int_t^T |\delta_s^{t,x,P_\xi}(y)|^2 ds)^{\frac{p}{2}}] \leq C_p;$

(b) $E[\sup_{s\in[t,T]} |\zeta_s^{t,x,P_\xi}(y) - \zeta_s^{t,x',P_{\xi'}}(y')|^p + (\int_t^T |\delta_s^{t,x,P_\xi}(y) - \delta_s^{t,x',P_{\xi'}}(y')|^2 ds)^{\frac{p}{2}}]$

$\leq C_p\big(|x-x'|^p + W_2(P_\xi, P_{\xi'})^p\big).$

+ We have to study the twofold differentiability of

$$(x, P_\xi) \to V(t, x, P_\xi) = Y_t^{t,x,P_\xi}$$

with Assumptions (H3.3), (H4.2), (H5.2) and (H5.3).

+ Combining the results in Section 5, we know that for all $t \in [0, T]$,

(i) $x \to V(t, x, P_\xi)$ is twice $L^2$-differentiable,

(ii) $V(t, x, \cdot) : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is differentiable,

(iii) $y \to (\partial_\mu V)(t, x, P_\xi, y)$ is $L^2$-differentiable,

(iv) the Lipschitz property in $L^2$ of all these derivatives (with Lipschitz constants independent of $t$).

**Proposition 6.1. (Representation Formulas)**

Under the Assumptions (H3.3), (H4.2), (H5.2) and (H5.3) we have the following representation formulas:

$$\begin{aligned} Y_s^{t,x,P_\xi} &= V(s, X_s^{t,x,P_\xi}, P_{X_s^{t,\xi}}),\ P\text{-}a.s.,\ s\in[t,T]; \\ Z_s^{t,x,P_\xi} &= (\partial_x V)(s, X_s^{t,x,P_\xi}, P_{X_s^{t,\xi}})\sigma(X_s^{t,x,P_\xi}, P_{X_s^{t,\xi}}),\ dsdP\text{-}a.e. \end{aligned} \tag{6.1}$$

Moreover,

$$E\Big[\big|Z_s^{t,x,P_\xi} - (\partial_x V)(s,x,P_\xi)\sigma(x,P_\xi)\big|^2\Big] \le C(s-t),\ 0\le t\le s\le T. \tag{6.2}$$

# 6. Related backward SPDEs of mean-field type

Problem: $V(t,\cdot,\cdot) \in C_b^{2,2}(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$?

$+$ In Pardoux and Peng (PTRF, 1994), Kolmogorov's continuity criterion played a crucial role for the proof;

$+$ $(t,x,P_\xi) \in [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$ runs an infinite dimensional space;

$+$ We cannot apply Kolmogorov's continuity criterion to the value function $V(t,x,P_\xi) = Y_t^{t,x,P_\xi}$.

The consequence is that we have to content with the continuity and differentiability of first and second order of $V(t,\cdot,\cdot)$ in the only $L^2$-sense.

However, we can make the following observation.

## Lemma 6.1

Let $\varphi : \Omega \times \mathbb{R}^d \to \mathbb{R}$ be $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable and $x \to \varphi(\cdot, x)$ $L^2$-differentiable. Then, for all $\eta \in L^\infty(\mathcal{F}; \mathbb{R})$, the deterministic function $\Psi(x) := E[\varphi(\cdot, x) \cdot \eta]$, $x \in \mathbb{R}^d$, is differentiable w.r.t. $x$ on $\mathbb{R}^d$, and $\partial_x \Psi(x) = E[\partial_x \varphi(\cdot, x) \cdot \eta]$, $x \in \mathbb{R}^d$, where $\partial_x \varphi(x)$ denotes the $L^2$-derivative of $\varphi(\cdot, \cdot)$ at $x$.

Proof. For simplicity, let $d = 1$. For all $\eta \in L^2(\mathcal{F})$, $x \in \mathbb{R}$:

$$\frac{1}{q}(\Psi(x+q) - \Psi(x)) = E\Big[\Big(\frac{1}{q}(\varphi(\cdot, x+q) - \varphi(\cdot, x)) - \partial_x \varphi(\cdot, x)\Big) \cdot \eta\Big] + E[\partial_x \varphi(\cdot, x) \cdot \eta]$$

$$\to E[\partial_x \varphi(\cdot, x) \cdot \eta], \text{ as } q \to 0.$$

+ For $\eta \in L^\infty(\mathcal{F};\mathbb{R})$, $(t,x,P_\xi) \in [0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)$, we define

$$\underline{\Psi(t,x,P_\xi) := \Psi_\eta(t,x,P_\xi) := E[V(t,x,P_\xi)\cdot\eta]}.$$

+ It can easily be verified that $\Psi(t,\cdot,\cdot) \in C_b^{2,2}(\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d))$.

+ In order to study the regularity properties w.r.t. $t$ of $\Psi(t,x,P_\xi)$, we make the following additional assumption on $\eta$:

**Assumption (H6.1)** The random variable $\eta \in L^2(\Omega,\mathcal{F}^B_{0,T},P;\mathbb{R})$ is such that, for the $(\mathbb{F}^B = (\mathcal{F}^B_{s,T})_{0\le s\le T})$-adapted process $\theta^\eta \in \mathcal{H}^2_{\mathcal{F}^B_{\cdot,T}}(0,T;\mathbb{R})$ with $\eta = E[\eta] + \int_0^T \theta^\eta_s d\overleftarrow{B}_s$, there exists a constant $C_\eta \in \mathbb{R}_+$, such that $|\theta^\eta_s| \le C_\eta$, $dsdP$-a.e.

## Proposition 6.2

Under assumptions (H3.3), (H4.2), (H5.2) and (H5.3), for all $\eta \in L^\infty(\mathcal{F};\mathbb{R})$ with (H6.2), $\Psi \in C_b^{0,2,2}([0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d))$, and for all $\varphi \in \{\Psi, \partial_x\Psi, \partial_{xx}^2\Psi, \partial_\mu\Psi, \partial_y(\partial_\mu\Psi)\}$ it holds for $0 \le t \le t+q \le T$, $(x, P_\xi, y) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d$,

$$|\varphi(t+q, x, P_\xi, y) - \varphi(t, x, P_\xi, y)| \le C'_\eta \sqrt{q}, \tag{6.3}$$

where $\partial_\mu\Psi(t,x,P_\xi,y) = E[\partial_\mu V(t,x,P_\xi,y)\cdot\eta]$, and the constant $C'_\eta \in \mathbb{R}_+$ depends on $\eta$ and $C_\eta$.

## Definition 6.1

We say that random field $\varphi$ belongs to $\mathfrak{C}^{0,2,2}(\Omega \times [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$, if $\varphi : \Omega \times [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ satisfies:

(i) $\varphi(t,x,\mu)$ is $\mathcal{F}^B_{t,T}$-measurable, $(t,x,\mu) \in [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$;

(ii) $x \to \varphi(t,x,\mu)$ is twice continuously $L^2$-differentiable;

(iii) $\mu \to \varphi(t,x,\mu)$ is differentiable;

(iv) $y \to \partial_\mu \varphi(t,x,\mu,y)$ is continuously $L^2$-differentiable;

(v) The first and second order derivatives are $L^2$-continuous on $[0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d$;

(vi) $\Gamma \in C_b^{0,2,2}([0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$, where $\Gamma(t,x,\mu) := E[\varphi(t,x,\mu) \cdot \eta]$, for all $\eta \in L^\infty(\mathcal{F}^B_{0,T}; \mathbb{R})$ satisfying (H9.1), and all $(t,x,\mu) \in [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$.

$$
\begin{aligned}
V(t,x,P_\xi) = \Phi(x,P_\xi) + \int_t^T \Bigg\{ & \sum_{i=1}^d \partial_{x_i} V(s,x,P_\xi) b_i(x,P_\xi) \\
& + \frac{1}{2} \sum_{i,j,k=1}^d (\partial^2_{x_i x_j} V)(s,x,P_\xi)(\sigma_{i,k}\sigma_{j,k})(x,P_\xi) \\
& + f(x, V(s,x,P_\xi), \sum_{i=1}^d \partial_{x_i} V(s,x,P_\xi)\sigma_i(x,P_\xi), P_{(\xi,\psi(s,\xi,P_\xi))}) \\
& + E\Big[ \sum_{i=1}^d (\partial_\mu V)_i(s,x,P_\xi,\xi) b_i(\xi,P_\xi) \\
& + \frac{1}{2} \sum_{i,j,k=1}^d \partial_{y_i}(\partial_\mu V)_j(s,x,P_\xi,\xi)(\sigma_{i,k}\sigma_{j,k})(\xi,P_\xi) \Big] \Bigg\} ds
\end{aligned} \tag{6.4}
$$

$$+\int_t^T \sum_{j=1}^{l} g_j(x, V(s,x,P_\xi), \sum_{i=1}^{d} \partial_{x_i} V(s,x,P_\xi)\sigma_i(x,P_\xi), P_{(\xi,\psi(s,\xi,P_\xi))}) d\overleftarrow{B}_s^j,$$

$$+\int_t^T \sum_{j=1}^{l} h_j(P_{(\xi,\psi(s,\xi,P_\xi))}) d\overleftarrow{B}_s^j, \ (t,x,\xi,P_\xi) \in [0,T] \times \mathbb{R}^d \times L^2(\mathcal{G}_t;\mathbb{R}^d) \times \mathcal{P}_2(\mathbb{R}^d),$$

where $\psi(s,x,P_\xi) := (V(s,x,P_\xi), \sum_{i=1}^{d} \partial_{x_i} V(s,x,P_\xi)\sigma_i(x,P_\xi))$, and the derivatives $\partial_{x_i} V$, $\partial^2_{x_i x_j} V$ and $\partial_{y_i}(\partial_\mu V)$ are in $L^2$-sense.

# 6. Related backward SPDEs of mean-field type

## Theorem 6.1

Under the Assumptions (H3.3), (H4.2), (H5.2) and (H5.3), $V \in \mathfrak{C}^{0,2,2}(\Omega \times [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$ is a classical solution of backward SPDE (6.4), and it is unique in $\mathfrak{C}^{0,2,2}(\Omega \times [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$.

Sketch of the proof of existence. For simplicity but w.l.o.g., let us restrict to $d=1$, $l=1$ and to $b=0$, $f=0$, $g^1=0$, $h(P_{\Pi_s^{t,\xi}}) = h(P_{X_s^{t,\xi}})$, $\Phi(x,\mu)=\Phi(x)$.

Let $\eta \in L^\infty(\mathcal{F}_{0,T}^B;\mathbb{R})$ be such that (H6.1) holds true, and $s \to \theta_s^\eta$ is continuous. We show that $\Psi(t,x,P_\xi)$ is differentiable w.r.t. $t$, and

$$\partial_t \Psi(t,x,P_\xi) = -\frac{1}{2}(\partial_{xx}^2 \Psi)(t,x,P_\xi)\sigma(x,P_\xi)^2 - \frac{1}{2}\widehat{E}\big[\partial_y(\partial_\mu \Psi)(t,x,P_\xi,\widehat{\xi})\sigma(\widehat{\xi},P_\xi)^2\big]$$
$$-E\big[g^2(P_\xi)(\partial_x V)(t,x,P_\xi)\sigma(x,P_\xi)\cdot\theta_t^\eta\big] - E\big[h(P_\xi)\cdot\theta_t^\eta\big],\ t\in[0,T]. \tag{6.5}$$

Consequently, from the formula for $\partial_t \Psi(t,x,P\xi)$,

$$E[(V(t,x,P_\xi)-\Phi(x,P_\xi))\cdot\eta]=\Psi(t,x,P_\xi)-\Psi(T,x,P_\xi)$$
$$=-\int_t^T \partial_s\Psi(s,x,P_\xi)ds=E[\eta\cdot I(t,x,P_\xi)],$$

where

$$I(t,x,P_\xi):=\int_t^T\Big(\frac{1}{2}(\partial^2_{xx}V)(r,x,P_\xi)\sigma(x,P_\xi)^2+\frac{1}{2}\widehat{E}\big[\partial_y(\partial_\mu V)(r,x,P_\xi,\widehat{\xi})\sigma(\widehat{\xi},P_\xi)^2\big]\Big)dr$$
$$+\int_t^T g^2(P_\xi)(\partial_x V)(r,x,P_\xi)\sigma(x,P_\xi)d\overleftarrow{B_r}+\int_t^T h(P_\xi)d\overleftarrow{B_r}.$$

But, since these $\eta$ satisfying (H9.1) such that $s\to\theta_s^\eta$ is continuous form a dense subset of $L^2(\mathcal{F}^B_{0,T};\mathbb{R})$, we prove that $V(t,x,P_\xi)$ is a solution of (6.4).

Thank you very much
for your attention!