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Mean Field Games with absorption and a model of bank run

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9th Colloquium on BSDEs and
Mean Field Systems

- A model of bank run
- The MFG problem
  - The approximated problem and strategy of the proof;
  - The fixed point procedure for the approximated problem;
  - Tightness and convergence.
- Existence of ε-Nash equilibria
  - ε-Nash equilibria for the approximated problem;
  - Uniform approximation.
- Conclusions

Motivation: A model of bank run

Consider a group of N agents with a deposit in a given bank.

- The initial deposit is $D_0^i > 0$;
- The bank offers an interest rate r so that $D_T^i := e^{rT} D_0^i$ is promised to be paid back at time $T > 0$.
- Any of the depositors has the right to early withdraw the capital (*run*) and to collect the cumulative interests at the run time τ^i:

$$\mathbb{E}\Big[(e^{r\tau^i} D_0^i) \wedge S_{\tau^i}\Big],$$

where S is the market value of the assets of the bank.

We interpret runs as the loss of confidence for the depositor that the bank will be able to pay back the capital at time T.

- X^i is a stochastic process that models the level of trust of agent i. If X^i hits 0, the agent runs.
- We allow the process X^i to depend on the fractions of agents who already left the game:

$$L_t^N = \frac{1}{N} \sum_{i=1}^{N} 1_{[0,\tau^i)}(t), \quad \tau^i := \inf\{t \in [0,T] \ : \ X_t^i \wedge S_t > 0\}.$$

This allows us to capture the self-exciting aspect of runs.

- We allow the process X^i to depend on the performance of the bank S_t and on a private noise:

$$dX_t^i = b^i(t, \alpha_t, X_t^i, S_t, L_t^N)dt + \sigma^i dW_t^i \ .$$

- The value of the bank S_t depends on the common noise and the runs:

$$dS_t = b^0(t, S_t)dt + \sigma^0(t, S_t)dW_t^0 - e^{rt}dL_t^N \ .$$

E. Glaeser, D. Laibson, J. Scheinkman, C. Soutter. *Measuring Trust.* The Quarterly Journal of Economics, 115:811–846, 2000.

The MFG problem

Suppose that a random flow of sub-prob. $(t,\omega)\mapsto \mu_t(\omega)$ is given and $\langle\cdot,\mu_t\rangle$ denotes the integral with respect to μ_t. The state dynamics is:

$$\begin{aligned} dX_t &= b(t, X_t, \langle h, \mu_t\rangle, \alpha_t)dt + \sigma dW_t + \sigma^0 dW_t^0 + \eta(t) d\mathfrak{L}_t \\ X_0 &= \xi \end{aligned}$$

where $\mathfrak{L}_t := \int_0^t k(t-s) L_s ds$ and $L_t = \langle 1, \mu_t\rangle$. The objective is to maximize the function

$$J(\alpha) = \mathbb{E}\left[\int_0^{\tau_{\mathrm{x}}} f(t, X_t, \mu_t, \alpha_t)dt + G(\tau_{\mathrm{x}}, X_{\tau_{\mathrm{x}}})\right],$$

with $\tau_{\mathrm{x}} := \inf\{t\in[0,T] \ : \ X_t \notin \mathcal{O}\}$ and $\mathcal{O}^C$ the absorbing set.

Definition

$(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, \mu)$ with μ a random flow of sub-prob. is a MFG equilibrium if

- there exists an $\mathbb{F}$-progressively measurable α that is optimal for the control problem with input μ;
- $\mu_t(\omega^0, \cdot) = \mathbb{P}\Big(\{X_t^\alpha \in \cdot\} \cap \{\tau_{\mathrm{x}} > t\} \mid \mathcal{F}_t^0\Big)$ where X^α is the state dynamics controlled by α.

Remark

The probability space is part of the solution $\leftrightarrow$ ***weak solutions****.*
The filtration $\mathbb{F}$ *may not be the one generated by* W^0 *only.*

Assumption

The following are standing assumptions:

1. b, f, h, G and η are continuous and bounded on their domains. Moreover, h is Lipschitz continuous.
2. b is affine in the variable a, i.e., $b = b_1(t, x, m) + b_2(t)a$ for some bounded deterministic b_2.
3. σ, σ^0 are constant with σ of full rank.

The MFG problem:
The optimal control question

Key points:

- First consider a process $X_t = X_0 + \sigma W_t + \sigma^0 dW_t^0$. Every control induces a Girsanov transformation so that, after the change of probability, the dynamics is the desired one.
- The optimal one is induced by a maximizer of the Hamiltonian of the system.

The MFG problem:

The fixed point procedure

Some of the challenges of the present framework:

- The desired equilibrium process is a **random** flow of sub-prob., compactness criteria are difficult to handle;
- Some stronger convergence results are needed to take care of $\mathbf{1}_{\tau_{\mathrm{x}}>t}$.

The approach to address them:

- First discretize the input flow and then take the limit;
- Work with compactness in τ-topology.

We define a random environment $\mathfrak{M}^n(\omega^0, dx)$ whose role is to describe the distribution of (X, W) conditional to V^n, which is a sequence of **discretization** of W^0.

$$\mathfrak{M}^n(\omega^0) := \sum_{k=1}^{|\mathcal{V}^n|} \mathfrak{m}_k^n \mathbf{1}_{A_k}(\omega^0), \qquad \omega^0 \in \Omega^0,$$

where $\mathfrak{m}_k^n$ are **deterministic** flows of probabilities. The random environment induces:

$$\mu_t(\omega^0, B) := \int_{\mathcal{C}([0,T];\mathbb{R}^d)} \mathbf{1}_B(x_t) \mathbf{1}_{\{\tau(x)>t\}} \mathfrak{M}^n(\omega^0, dx),$$

with $\tau(x) := \inf\{t \in [0, T] \ : \ x_t \notin \mathcal{O}\}$.

We define

$$\mathcal{E} := \left\{ \mathbb{P}' \in \mathcal{M}_1(\Omega^1) \ : \ \mathbb{P}' \ll \mathbb{P},\ \mathbb{E}\left[\left(d\mathbb{P}'/d\mathbb{P} \right)^2 \right] \leq M \right\}$$

with M depending only on the coefficients of the problem. This is a compact set w.r.t. the **τ-topology**. The iteration is

- Start with $(\mathfrak{m}_1, \dots, \mathfrak{m}_{|\mathcal{V}^n|})$ inducing a discretized input μ;
- Solve the optimal control problem which induces a change of measure $\mathbb{P}^\mu$;
- Compute

$$\hat{\mathfrak{m}}_k(\cdot) := \frac{\mathbb{P}^\mu(A_k \times (X, W)^{-1}(\cdot))}{\mathbb{P}^\mu(A_k \times \Omega^1)}, \quad k = 1, \dots, |\mathcal{V}^n|.$$

Proposition

The map $\Phi : (\mathfrak{m}_1, \dots, \mathfrak{m}_{|\mathcal{V}^n|}) \mapsto (\hat{\mathfrak{m}}_1, \dots, \hat{\mathfrak{m}}_{|\mathcal{V}^n|})$ *is well defined on* $\mathcal{E}^{|\mathcal{V}^n|}$ *and continuous. In particular, since* $\mathcal{E}$ *is* τ*-compact,* Φ *admits a fixed point.*

Key points:

- Suitable stability results for BSDEs;
- Prove convergence in total variation induced by convergence of relative entropy.

The MFG problem:
Tightness and convergence of fixed points

Let $\mathfrak{M}^n$ be the random enivronment induced by the sequence of fixed points of the map Φ.

Proposition

There exists a weak limit of $\mathbb{P}^n \circ (\mathfrak{M}^n)^{-1}$ *denoted by* $\mathbb{P}^\infty \circ (\mathfrak{M}^\infty)^{-1}$. $\mathfrak{M}^\infty$ *is a version of the conditional distribution of* (X, W) *given* $(W^0, \mathfrak{M}^\infty)$ *and the measure* $\mathbb{P}^\infty$ *is induced by an optimal control for the limiting problem.*

Remark

We cannot control the progressive measurability of the limiting point $\mathfrak{M}^\infty$. ***Good news:*** *the extra randomness introduced by* $\mathfrak{M}^\infty$ *does not provide information on the future evolution of the state dynamics. The natural filtration is indeed* ***immersed*** *in the one generated by the common noise.*

Theorem (B., Campi (21))

There exists a weak MFG equilibrium $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, \mathfrak{M}^\infty)$, *namely:*

- *there exists an* $\mathbb{F}$*-progressively measurable* α *that is optimal for the control problem with input* μ^∞ *induced by* $\mathfrak{M}^\infty$*;*
- $\mu_t(\omega^0, \cdot) = \mathbb{P}\Big(\{X_t^\alpha \in \cdot\} \cap \{\tau_{\mathrm{x}} > t\} \mid \mathcal{F}_t^0\Big)$ *where* X^α *is the state dynamics controlled by* α*;*
- $\mathfrak{M}^\infty(\omega^0)(\cdot) = \mathcal{L}(X^\alpha(\omega^0, \cdot), W(\cdot))$ *for every* ω^0 *outside a* $\mathbb{P}$*-null set.*

Existence of ε-Nash equilibria

We aim at constructing ε-Nash equilibria using the solutions to the MFG problem.

- We first construct ε-Nash equilibria for the problem induced by the discretization V^n of the common noise;
- We prove that for $n \in \mathbb{N}$ large enough, the approximation is good enough for the original problem.

Remark

For the second step we need stronger assumptions which are nevertheless satisfied in the bank run model. The advantage is that we do not need to study the Master equation to derive ε*-Nash equilibria.*

$$X_t^i = X_0 + \int_0^t \tilde{b}(s, X_s^i, \rho^{N,n}, \alpha_s)ds + \sigma W_t^i + \sigma^0 W_t^0,$$

where

$$\rho_t^{N,n} = \mathbb{E}^{N,n}\left[\mu_t^N \mid \mathcal{F}_t^{\xi,W,V^n}\right],$$

and

$$\mu_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{X_t^i}(\cdot)1_{[0,\tau_{\mathrm{x}}^i)}(t), \qquad \tau_{\mathrm{x}}^i := \inf\{t \in [0,T] \ : \ X_t^i \notin \mathcal{O}\}$$

The players do not interact with the empirical sub-distribution μ^N but rather on an integrated version with respect to a finite number of events of the common noise.

Proposition (B., Campi (21))

There exist ε-Nash equilibria for the approximated problem.

Key points:

- Extension of strong propagation of chaos results to the case of common noise;
- Combination of large deviation principle and convergence of relative entropy;
- Search for open-loop controls.

Assumption

In this part we further assume that:

1. $b(t, x, m, a)$ *is Lipschitz in the variable* m.
2. *The running cost is of the from* $f = f(t, x)$.

Proposition

$|J^N(\alpha) - J^{N,n}(\alpha)| \to 0$ as $n \to \infty$, uniformly in N and α.

Key points:

- In the presence of common noise, Sanov's Theorem is not enough to guarantee that

$$\mu_t^N := \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^i}(\cdot) 1_{[0,\tau_{\mathrm{x}}^i)}(t) \xrightarrow{N\to\infty} \mu_t,$$

  where μ is the limiting equilibrium flow;
- However, the change of measures measures induced by μ^N and $\rho^{N,n}$ can be made uniformly close in total variation.

Theorem (B., Campi (21))

There exist ε-Nash equilibria for the N-player model.

Conclusions

- We proposed a new model of bank run using mean field games with absorption;
- We studied the existence of equilbria for mean field games with absorption in the presence of common noise;
- We studied the existence of ε-Nash equilibria for the N-player game.

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Thank you for your kind attention!

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