Risk measures and progressive enlargement of filtration: a BSDE approach

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Joint work with Emanuela Rosazza Gianin

Plan of the talk

Question:

How can we make a risk measure react to shocks in financial markets?

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Solution:

Progressive enlargement of filtration and BSDEs with jumps (BSDEJ).

Why BSDEs and progressive enlargement of filtrations?

BSDEs

- BSDEs allow to induce and/or represent dynamic risk measures specifying:
  1. A filtered probability space (i.e., a *probabilistic model*).
  2. A measurable map g, called *driver*.
- Properties of the driver $\longleftrightarrow$ properties of the risk measure.
- The driver can be determined based on investor's preferences, regulatory requirements, etc. . .
- BSDEs flow property $\Longrightarrow$ *time-consistency* (i.e., evaluation of risk is recursive).
- Numerical simulation of BSDEs.

Progressive enlargement of filtration

- *Reference* filtration $\mathbb{F}$: information available prior to a shock (e.g., default).
- *Progressively enlarged* filtration $\mathbb{G}$: information updated after shock (can be generalized to multiple events).

Why BSDEs? Comparison with literature

Nonlinear expectations and g-expectations:

- S. Peng. Backward SDE and related g-expectation. In Backward stochastic differential equations (Paris, 1995–1996), volume 364 of Pitman Res. Notes Math. Ser., pages 141–159. Longman, Harlow, 1997.
- F. Coquet, Y. Hu, J. Mémin, and S. Peng. Filtration-consistent nonlinear expectations and related g-expectations. Probab. Theory Related Fields, 123(1):1–27, 2002.

Representation of risk measures via BSDEs driven by a Wiener process:

- E. Rosazza Gianin. Risk measures via g-expectations. Insurance Math. Econom., 39 (1):19–34, 2006.
- P. Barrieu and N. El Karoui. Pricing, hedging, and designing derivatives with risk measures. In Carmona, R. (ed.) Indifference pricing: theory and applications, pages 77–144. Princeton University Press, Princeton, 2009.

Representation of risk measures via BSDEs driven by a Wiener process and a Poisson random measure:

- M. C. Quenez and A. Sulem. BSDEs with jumps, optimization and applications to dynamic risk measures. Stochastic Process. Appl., 123(8):3328–3357, 2013.

Outline

Outline

Probabilistic setting

We are given the following objects:

- Finite time horizon $T > 0$.
- A complete probability space $(\Omega, \mathcal{A}, \mathbb{P})$.
- A Borel-measurable set $E \subset \mathbb{R}^m$.
- A Wiener process $W = (W_t)_{t \in [0,T]}$.
- A pair of random variables $(\tau, \zeta) \in \mathbb{R}^+ \times E$. Can be generalized to multiple jumps.
- A random counting measure $\mu(\mathrm{d}t\,\mathrm{d}e) := \delta_{(\tau,\zeta)}(\mathrm{d}t\,\mathrm{d}e)$.

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Information flow:

- Reference filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$: completed natural filtration generated by W.
- Progressively enlarged filtration $\mathbb{G} = (\mathcal{G}_t)_{t \in [0,T]}$: completed natural filtration generated by W and μ.
- Initially enlarged filtration $\mathbb{H} = (\mathcal{H}_t)_{t \in [0,T]}$: completed natural filtration generated by W and (τ, ζ).

Notice that: $\mathbb{F} \subset \mathbb{G} \subset \mathbb{H}$.

The fundamental assumption

The following assumption is essential for most of the following results.

Assumption (Density hypothesis[1])

For any $t \geq 0$, the conditional distribution of the pair (τ, ζ) given $\mathcal{F}_t$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^+ \times E$. In particular, there exists a strictly positive $\left(\mathbb{R}^+ \times E\right)$-indexed $\mathbb{F}$-predictable random field γ such that

$$\mathbb{P}\big((\tau, \zeta) \in C \mid \mathcal{F}_t\big) = \int_C \gamma_t(\vartheta, e)\,\mathrm{d}\vartheta\,\mathrm{d}e, \quad C \in \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}(E),\, t \geq 0.$$

[1] It is related to Condition (A') in: J. Jacod. Grossissement initial, hypothèse (H') et théorème de Girsanov. In Lect. Notes. Math., volume 1118, pages 15–35. Springer-Verlag, 1985.

The fundamental decompositions

Lemma (Callegaro et al., 2013, ESAIM PS; Pham, 2010, SPA)

1. *For any $t \geq 0$, a random variable ξ is $\mathcal{G}_t$-measurable if and only if it is of the form*

$$\xi(\omega) = \xi^0(\omega)\mathbb{1}_{t<\tau(\omega)} + \xi^1(\omega, \tau(\omega), \zeta(\omega))\mathbb{1}_{t\geq\tau(\omega)},$$

   for some $\mathcal{F}_t$-measurable random variable ξ^0 and a $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}(E)$-measurable function ξ^1.

2. *A process $Y = (Y_t)_{t\geq0}$ is $\mathbb{G}$-predictable if and only if it is of the form*

$$Y_t = Y_t^0 \mathbb{1}_{t\leq\tau} + Y_t^1(\tau, \zeta)\mathbb{1}_{t>\tau}, \quad t \geq 0,$$

   where Y^0 is an $\mathbb{F}$-predictable process and Y^1 is a $\left(\mathbb{R}^+ \times E\right)$-indexed $\mathbb{F}$-predictable random field.

Lemma (Pham, 2010, SPA; Song, 2014, ESAIM PS)

Under the Density hypothesis, any $\mathbb{G}$-optional process $Y = (Y_t)_{t\geq0}$ can be decomposed as

$$Y_t = Y_t^0 \mathbb{1}_{t<\tau} + Y_t^1(\tau, \zeta)\mathbb{1}_{t\geq\tau}, \quad t \geq 0,$$

where Y^0 is an $\mathbb{F}$-optional process and Y^1 is a $\left(\mathbb{R}^+ \times E\right)$-indexed $\mathbb{F}$-optional random field.

Outline

Backward Stochastic Differential Equations with Jumps (BSDEJ)

Let us define the following sets:

- $\mathcal{S}^{\infty}_{\mathbb{G}}[a,b]$, real-valued $\mathbb{G}$-progressive processes Y such that:

$$\|Y\|_{\mathcal{S}^{\infty}[a,b]} \coloneqq \operatorname*{ess\,sup}_{t \in [a,b]} |Y_t| < \infty.$$

- $\mathrm{L}^2_{\mathbb{G}}[a,b]$, $\mathbb{R}^d$-valued $\mathbb{G}$-predictable processes Z such that:

$$\|Z\|_{\mathrm{L}^2[a,b]} \coloneqq \left(\mathbb{E}\left[\int_a^b |Z_t|^2 \, \mathrm{d}t\right]\right)^{\frac{1}{2}} < \infty.$$

- $\mathrm{L}^2(\mu)$, real-valued E-indexed $\mathbb{G}$-predictable processes U such that:

$$\|U\|_{\mathrm{L}^2(\mu)} \coloneqq \left(\mathbb{E}\left[\int_0^T \int_E |U_s(e)|^2 \, \mu(\mathrm{d}s \, \mathrm{d}e)\right]\right)^{\frac{1}{2}} < \infty.$$

A triple $(Y, Z, U) \in S^{\infty}_{\mathbb{G}}[0,T] \times \mathrm{L}^2_{\mathbb{G}}[0,T] \times \mathrm{L}^2(\mu)$ is a solution to the BSDEJ if it satisfies:

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s, U_s(\cdot)) \, \mathrm{d}s - \int_t^T Z_s \, \mathrm{d}W_s - \int_t^T \int_E U_s(e) \, \mu(\mathrm{d}s \, \mathrm{d}e), \quad t \in [0,T],$$

where:

- ξ is a $\mathcal{G}_T$-measurable r.v., the terminal condition.
- $g \colon \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \times \mathrm{B}(E) \to \mathbb{R}$ is a measurable map, the driver.

The decomposition of the BSDEJ

We assume that for any $(y, z, u) \in \mathbb{R} \times \mathbb{R}^d \times \mathrm{B}(E)^2$ the map $(\omega, t) \mapsto g(\omega, t, y, z, u)$ is $\mathbb{G}$-predictable. Hence the driver can be written (fundamental decomposition):

$$g(t, y, z, u) = g^0(t, y, z, u)\mathbb{1}_{t \leq \tau} + g^1(t, y, z, u, \tau, \zeta)\mathbb{1}_{t > \tau}.$$

The terminal condition can be always decomposed as:

$$\xi = \xi^0 \mathbb{1}_{T < \tau} + \xi^1(\tau, \zeta)\mathbb{1}_{T \geq \tau}.$$

Idea: solve the BSDEJ through a system of indexed Brownian BSDEs before and after the jump time τ.

$$\underbrace{\begin{cases} (Y^0, Z^0) \\ \big(Y^1(\vartheta, e), Z^1(\vartheta, e)\big) \end{cases}}_{\text{solutions to Brownian BSDEs}} \Longrightarrow \underbrace{\begin{cases} Y_t = Y_t^0 \, \mathbb{1}_{t < \tau} + Y_t^1(\tau, \zeta) \, \mathbb{1}_{t \geq \tau}, \\ Z_t = Z_t^0 \, \mathbb{1}_{t \leq \tau} + Z_t^1(\tau, \zeta) \, \mathbb{1}_{t > \tau}, \\ U_t(\cdot) = U_t^0(\cdot) \, \mathbb{1}_{t \leq \tau} = \Big[Y_t^1(t, \cdot) - Y_t^0 \Big] \, \mathbb{1}_{t \leq \tau}. \end{cases}}_{\text{solution to the BSDEJ}}$$

[2] We define $\mathrm{B}(E) \coloneqq \{f \colon E \to \mathbb{R}, \text{ Borel-measurable}\}$ equipped with the pointwise convergence topology.

Existence and uniqueness

Immersion hypothesis: Any $\mathbb{F}$-martingale remains a $\mathbb{G}$-martingale.

Theorem (Kharroubi, Lim, 2014, J. Theoret. Prob.)

Under the Density hypothesis and the immersion hypothesis (plus other technical hypotheses), the BSDEJ admits a unique solution (Y, Z, U) *on* $[0, T]$*, where*

$$\begin{cases} Y_t = Y_t^0 \, \mathbb{1}_{t<\tau} + Y_t^1(\tau, \zeta) \, \mathbb{1}_{t \geq \tau}, \\ Z_t = Z_t^0 \, \mathbb{1}_{t \leq \tau} + Z_t^1(\tau, \zeta) \, \mathbb{1}_{t > \tau}, \\ U_t(\cdot) = U_t^0(\cdot) \, \mathbb{1}_{t \leq \tau} = \Big[Y_t^1(t, \cdot) - Y_t^0 \Big] \, \mathbb{1}_{t \leq \tau}. \end{cases}$$

and (Y^0, Z^0), (Y^1, Z^1) *are the unique solutions to the BSDEs*

$$\begin{aligned} Y_t^1(\vartheta, e) = \xi^1(\vartheta, e) + \int_t^T g^1(s, Y_s^1(\vartheta, e), Z_s^1(\vartheta, e), 0, \vartheta, e) \, \mathrm{d}s \\ - \int_t^T Z_s^1(\vartheta, e) \, \mathrm{d}W_s, \quad \vartheta \wedge T \leq t \leq T, \quad (\vartheta, e) \in \mathbb{R}^+ \times E, \end{aligned}$$

$$Y_t^0 = \xi^0 + \int_t^T g^0(s, Y_s^0, Z_s^0, Y_s^1(s, \cdot) - Y_s^0) \, \mathrm{d}s - \int_t^T Z_s^0 \, \mathrm{d}W_s, \quad 0 \leq t \leq T.$$

Outline

Dynamic risk measures

Let $\mathbb{D} \in \{\mathbb{F}, \mathbb{G}, \mathbb{H}\}$.

Definition (Dynamic risk measure)

A $\mathbb{D}$-*dynamic risk measure* is a family $\rho := (\rho_t)_{t \in [0,T]}$ of $\mathbb{D}$-*conditional risk measures* ρ_t.

Definition (Conditional risk measure)

A $\mathbb{D}$-*conditional risk measure* is a map ρ_t such that:

1. $\rho_t \colon \mathrm{L}^\infty(\mathcal{D}_T) \to \mathrm{L}^\infty(\mathcal{D}_t)$, for all $t \in [0, T]$;
2. ρ_0 is a *static risk measure*, i.e., a functional $\rho_0 \colon \mathrm{L}^\infty(\mathcal{D}_T) \to \mathbb{R}$;
3. $\rho_T(\xi) = -\xi$, for all $\xi \in \mathrm{L}^\infty(\mathcal{D}_T)$.

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3. $\rho_T(\xi) = -\xi$, for all $\xi \in \mathrm{L}^\infty(\mathcal{D}_T)$.

Example

$\rho_t := \mathbb{E}[-\xi \mid \mathcal{F}_t]$, $\xi \in \mathrm{L}^\infty(\mathcal{F}_T)$, $t \in [0, T]$, is a $\mathbb{F}$-conditional risk measure.

The induced dynamic risk measure

$\mathrm{L}^\infty(\mathcal{G}_T) \ni \xi \rightsquigarrow (Y^\xi, Z^\xi, U^\xi)$, unique solution of the BSDEJ. Define

$$\rho_t(\xi) := Y_t^{-\xi}, \quad t \in [0,T].$$

It is easy to show that $\rho = (\rho_t)_{t \in [0,T]}$ is a $\mathbb{G}$-dynamic risk measure.

A note

If $g(t, y, 0, 0) = 0$ for any $t \in [0,T]$ and any $y \in \mathbb{R}$, then

$$\rho_t(\xi) = \mathcal{E}_g(-\xi \mid \mathcal{G}_t), \quad t \in [0,T],$$

where $\mathcal{E}_g(\cdot)$ denotes the $\mathbb{G}$*-conditional* g*-expectation* associated to the BSDEJ.

The induced dynamic risk measure

The $\mathbb{G}$-dynamic risk measure induced by the BSDEJ can be decomposed as follows.

Proposition (C., Rosazza Gianin, 2020)

Under the assumptions of the existence and uniqueness theorem, there exist a $\mathbb{F}$-dynamic risk measure $\rho^0 := (\rho^0_t)_{t \in [0,T]}$ and a $\mathbb{H}$-dynamic risk measure $\rho^1 := (\rho^1_t)_{t \in [0,T]}$ such that:

$$\rho_t(\xi) = \rho^0_t(\xi^0)\, \mathbb{1}_{t<\tau} + \rho^1_t\big(\xi^1(\tau,\zeta)\big)\, \mathbb{1}_{t \geq \tau}, \quad t \in [0,T],\, \xi \in \mathrm{L}^\infty(\mathcal{G}_T).$$

Outline

Properties of dynamic risk measures

Dynamic risk measures, as previously defined, may not be sufficient to assess riskiness of financial positions in a meaningful way.

$$\Downarrow$$

We can impose on dynamic risk measures some mathematical requirement to reflect financial motivations.

Properties of dynamic risk measures

1. *Zero-one law:* For all $t \in [0,T]$ and all $A \in \mathcal{G}_t$:
$$\rho_t(\xi \mathbb{1}_A) = \mathbb{1}_A \, \rho_t(\xi), \quad \xi \in \mathrm{L}^\infty(\mathcal{G}_T).$$
2. *Translation invariance:* For all $t \in [0,T]$ and all $\eta \in \mathrm{L}^\infty(\mathcal{G}_t)$:
$$\rho_t(\xi + \eta) = \rho_t(\xi) - \eta, \quad \xi \in \mathrm{L}^\infty(\mathcal{G}_T).$$
3. *Positive homogeneity:* For all $t \in [0,T]$ and all $\eta \in \mathrm{L}^\infty(\mathcal{G}_t)$, $\eta \geq 0$:
$$\rho_t(\xi\eta) = \eta \, \rho_t(\xi), \quad \xi \in \mathrm{L}^\infty(\mathcal{G}_T).$$
4. *Monotonicity:* For all $\xi, \eta \in \mathrm{L}^\infty(\mathcal{G}_T)$, with $\xi \leq \eta$:
$$\rho_t(\xi) \geq \rho_t(\eta), \quad t \in [0,T].$$
5. *Convexity:* For all $\xi, \eta \in \mathrm{L}^\infty(\mathcal{G}_T)$ and all $\alpha \in [0,1]$:
$$\rho_t\big(\alpha\xi + (1-\alpha)\eta\big) \leq \alpha\rho_t(\xi) + (1-\alpha)\rho_t(\eta), \quad t \in [0,T].$$
6. *Fatou property:* For any sequence $(\xi_n)_{n\in\mathbb{N}} \subset \mathrm{L}^\infty(\mathcal{G}_T)$ and $\xi \in \mathrm{L}^\infty(\mathcal{G}_T)$ such that $\xi_n \to \xi$:
$$\rho_t(\xi) \leq \liminf_{n\to\infty} \rho_t(\xi_n), \quad t \in [0,T].$$
7. *Time-consistency:* For any $\mathbb{G}$-stopping time $\sigma \leq T$, and $\xi \in \mathrm{L}^\infty(\mathcal{G}_T)$:
$$\rho_t(\xi) = \rho_t\big(-\rho_\sigma(\xi)\big), \quad t \leq \sigma.$$

Properties of the induced risk measure – Part I

Proposition (C., Rosazza Gianin, 2020)

Under the assumptions of the existence and uniqueness theorem, the dynamic risk measure ρ satisfies the following properties:

(a) *Zero-one law* *if either $g(t, 0, 0, 0) = 0$, $\mathbb{P}$-a.s., for all $t \in [0, T]$, or both ρ^0 and ρ^1 satisfy this property.*

(b) *Translation invariance* *if either g does not depend on y or both ρ^0 and ρ^1 satisfy this property.*

(c) *Positive homogeneity* *if either g is positively homogeneous with respect to (y, z, u), $\mathbb{P}$-a.s., for all $t \in [0, T]$, or both ρ^0 and ρ^1 satisfy this property.*

(d) *Monotonicity.*

(e) *Convexity* *if either g is convex with respect to (y, z, u), $\mathbb{P}$-a.s., for all $t \in [0, T]$, or both ρ^0 and ρ^1 satisfy this property.*

(f) *Strong time-consistency.*

Properties of the induced risk measure – Part II

Proposition (C., Rosazza Gianin, 2020)

Let the assumptions of the existence and uniqueness theorem hold. Let $\bar{\xi}, \hat{\xi} \in \mathrm{L}^\infty(\mathcal{G}_T)$ *and denote by* $(\bar{Y}, \bar{Z}, \bar{U})$ *(resp.* $(\hat{Y}, \hat{Z}, \hat{U})$*) the solution to the BSDEJ with driver* g *and terminal condition* $\bar{\xi}$ *(resp.* $\hat{\xi}$*).*
Suppose, moreover, that for each $(\vartheta, e) \in \mathbb{R}^+ \times E$*:*

$$\|\bar{Y}^0 - \hat{Y}^0\|_{\mathcal{S}^\infty[0,T]} \leq K^0 \|\bar{\xi}^0 - \hat{\xi}^0\|_{\mathrm{L}^\infty},$$
$$\|\bar{Y}^1(\vartheta, e) - \hat{Y}^1(\vartheta, e)\|_{\mathcal{S}^\infty[\vartheta,T]} \leq K^1(\vartheta, e) \|\bar{\xi}^1(\vartheta, e) - \hat{\xi}^1(\vartheta, e)\|_{\mathrm{L}^\infty},$$

for some finite constants $K^0,\ K^1(\vartheta, e) > 0$*, and that*

$$\sup_{(\vartheta,e)\in\mathbb{R}^+\times E} K^1(\vartheta, e)\|\bar{\xi}^1(\vartheta, e) - \hat{\xi}^1(\vartheta, e)\|_{\mathrm{L}^\infty} < +\infty.$$

Then there exists a finite constant $M > 0$ *such that*

$$\|\bar{Y} - \hat{Y}\|_{\mathcal{S}^\infty[0,T]} \leq 2M.$$

Proposition (C., Rosazza Gianin, 2020)

Under the assumptions of the existence and uniqueness theorem and the above Proposition, the dynamic risk measure ρ *satisfies the Fatou property.*

Outline

Example: dynamic entropic risk measure

Suppose that an agent wants to assess the riskiness of a financial position ξ (that we assume bounded) and that she/he evaluates her/his preferences based on the utility function $u(x) := -\gamma e^{-\frac{x}{\gamma}}$, where $\gamma > 0$ is the *risk tolerance* parameter.

$$\Downarrow$$

Dynamic entropic risk measure:

$$\rho_t(\xi) := \gamma \log\bigl(\mathbb{E}[e^{-\frac{\xi}{\gamma}} \mid \mathcal{G}_t]\bigr), \quad t \in [0,T].$$

How can we make this risk measure react to shocks in the financial market? Is the agent allowed to change her/his preferences based on this event? Can we make γ depend on it in a time-consistent way?

Example: dynamic entropic risk measure

We use the parameter γ to introduce a dependence of the risk aversion of the agent (the inverse of γ) on the possible default times and values.

After default the agent becomes more risk averse.

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After default the agent becomes more risk averse.

Suppose that the driver g of the BSDEJ is $g(\omega, t, z) := \frac{1}{2}\|z\|^2 f(t, \tau(\omega), \zeta(\omega))$, where

$$f(t, \vartheta, e) = \begin{cases} 1, & \text{if } t \leq \vartheta, \\ \dfrac{1}{\gamma(\vartheta, e)}, & \text{if } t > \vartheta, \end{cases}$$

and $\gamma \colon \mathbb{R}^+ \times E \to (0, 1)$ is a measurable function. The driver can be decomposed as

$$g(t, z) = g^0(z)\mathbb{1}_{t \leq \tau} + g^1(z, \tau, \zeta)\mathbb{1}_{t > \tau},$$
$$g^0(z) = \frac{1}{2}\|z\|^2, \quad g^1(z, \vartheta, e) = \frac{1}{2\gamma(\vartheta, e)}\|z\|^2.$$

Example: dynamic entropic risk measure

Define the $\mathbb{G}$-dynamic risk measure $\rho_t(\xi) \coloneqq Y_t^{-\xi}$, $t \in [0,T]$, $\xi \in \mathrm{L}^\infty(\mathcal{G}_T)$. Then

$$\rho_t(\xi) = \rho_t^0(\xi^0)\,\mathbb{1}_{t<\tau} + \rho_t^1\big(\xi^1(\tau,\zeta)\big)\,\mathbb{1}_{t\geq\tau},$$

where $\rho_t^0(\xi^0) = Y^0$, with

$$Y_t^0 = -\xi^0 + \int_t^T g^0(Z_s^0)\,\mathrm{d}s - \int_t^T Z_s^0\,\mathrm{d}W_s, \quad 0 \leq t \leq T$$

and, on the event $\{t \geq \tau\}$, $\rho_t^1(\xi^1(\tau,\zeta)) = Y^1(\tau,\zeta)$, with

$$Y_t^1(\vartheta,e) = -\xi^1(\vartheta,e) + \int_t^T g^1(Z_s^1(\vartheta,e),\vartheta,e)\,\mathrm{d}s - \int_t^T Z_s^1(\vartheta,e)\,\mathrm{d}W_s, \quad \vartheta\wedge T \leq t \leq T.$$

More explicitly

$$\rho_t^0(\xi^0) = \log\mathbb{E}[\mathrm{e}^{-\xi^0} \mid \mathcal{F}_t], \quad t\in[0,T], \quad \text{Reference risk measure,}$$

$$\rho_t^1(\xi^1(\tau,\zeta)) = \gamma(\tau,\zeta)\log\mathbb{E}[\mathrm{e}^{-\frac{\xi^1(\tau,\zeta)}{\gamma(\tau,\zeta)}} \mid \mathcal{H}_t], \quad \text{on } \{t\geq\tau\}, \quad \text{Updated risk measure.}$$

Outline

The dual representation

Suppose that the $\mathbb{G}$-dynamic risk measure ρ induced by the BSDEJ satisfies the zero-one law, translation invariance, convexity and Fatou properties (monotonicity is granted by the comparison theorem).

Under these assumptions, ρ admits the dual (or robust) representation

$$\rho_t(\xi) = \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathcal{Q}} \bigl\{\mathbb{E}_{\mathbb{Q}}[-\xi \,|\, \mathcal{G}_t] - \alpha_t(\mathbb{Q})\bigr\}, \quad \xi \in \mathrm{L}^\infty(\mathcal{G}_T),\, t \in [0,T],$$

where $\mathcal{Q} \coloneqq \{\mathbb{Q},$ probability measures on $(\Omega, \mathcal{G}_T),$ such that $\mathbb{Q} \sim \mathbb{P}_{|\mathcal{G}_T}\}$. The map α_t is the $\mathbb{G}$-penalty term:

$$\alpha_t(\mathbb{Q}) = \operatorname*{ess\,sup}_{\xi\in\mathrm{L}^\infty(\mathcal{G}_T)} \bigl\{\mathbb{E}_{\mathbb{Q}}[-\xi \,|\, \mathcal{G}_t] - \rho_t(\xi)\bigr\} = \operatorname*{ess\,sup}_{\xi\in\mathrm{L}^\infty(\mathcal{G}_T),\, \rho_t(\xi)\le 0} \bigl\{\mathbb{E}_{\mathbb{Q}}[-\xi \,|\, \mathcal{G}_t]\bigr\}, \quad \mathbb{Q} \in \mathcal{Q}.$$

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Question

Can we decompose the penalty term as we did with the dynamic risk measure ρ?

Decomposition of the penalty term

The *candidate* penalty terms to provide a decomposition of the $\mathbb{G}$-penalty α are those associated to the $\mathbb{F}$-risk measure ρ^0 and the $\mathbb{H}$-risk measure ρ^1, i.e.:

$$\alpha_t^0(\mathbb{Q}^0) = \operatorname*{ess\,sup}_{\xi^0 \in \mathrm{L}^\infty(\mathcal{F}_T)} \big\{\mathbb{E}_{\mathbb{Q}^0}[-\xi^0 \mid \mathcal{F}_t] - \rho_t^0(\xi^0)\big\}, \quad \mathbb{Q}^0 \in \mathcal{Q}^0,$$
$$\alpha_t^1(\mathbb{Q}^1) = \operatorname*{ess\,sup}_{\xi^1 \in \mathrm{L}^\infty(\mathcal{H}_T)} \big\{\mathbb{E}_{\mathbb{Q}^1}[-\xi^1 \mid \mathcal{H}_t] - \rho_t^1(\xi^1)\big\}, \quad \mathbb{Q}^1 \in \mathcal{Q}^1,$$

where

$$\mathcal{Q}^0 \coloneqq \{\mathbb{Q}^0, \text{ probability measures on } (\Omega, \mathcal{F}_T), \text{ such that } \mathbb{Q}^0 \sim \mathbb{P}_{|\mathcal{F}_T}\},$$
$$\mathcal{Q}^1 \coloneqq \{\mathbb{Q}^1, \text{ probability measures on } (\Omega, \mathcal{H}_T), \text{ such that } \mathbb{Q}^1 \sim \mathbb{P}_{|\mathcal{H}_T}\}.$$

Decomposition of the penalty term

Recall that $\mathcal{Q} \coloneqq \{\mathbb{Q},$ probability measures on $(\Omega, \mathcal{G}_T),$ such that $\mathbb{Q} \sim \mathbb{P}_{|\mathcal{G}_T}\}$. Define:

$$\mathcal{Q}^{\hookrightarrow} \coloneqq \{\mathbb{Q} \in \mathcal{Q} \colon \text{ any } \mathbb{F}\text{-martingale is a } \mathbb{G}\text{-martingale under } \mathbb{Q}\}.$$

Proposition (C., Rosazza Gianin, 2020)

For any $t \in [0, T]$ and any $\mathbb{Q} \in \mathcal{Q}^{\hookrightarrow}$ the following holds for the $\mathbb{G}$-penalty α

$$\alpha_t(\mathbb{Q}) \geq k_t(\mathbb{Q})\alpha_t^0(\mathbb{Q}^0), \quad \textit{on } \{t < \tau\}, \qquad \alpha_t(\mathbb{Q}) = \alpha_t^1(\mathbb{Q}^1), \quad \textit{on } \{t \geq \tau\},$$

where $\mathbb{Q}^0$ and $\mathbb{Q}^1$ are probability measures on $(\Omega, \mathcal{F}_T)$ and $(\Omega, \mathcal{H}_T)$, respectively, such that

$$\mathrm{d}\mathbb{Q}^0 = \mathbb{E}[L \mid \mathcal{F}_T]\, \mathrm{d}\mathbb{P}_{|\mathcal{F}_T}, \quad \mathrm{d}\mathbb{Q}^1 = L \mathrm{d}\mathbb{P}_{|\mathcal{H}_T}, \quad L \coloneqq \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}_{|\mathcal{G}_T}},$$

and $k_t(\mathbb{Q})$ is a $\mathcal{F}_t$-measurable random variable satisfying $k_t(\mathbb{Q}) \geq 1$ $\mathbb{P}$-a.s.

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The stochastic factor $k_t(\mathbb{Q})$ is linked to the ratio $\dfrac{\mathbb{Q}(\tau > t \mid \mathcal{F}_t)}{\mathbb{P}(\tau > t \mid \mathcal{F}_t)}$, $t \in [0, T]$. Financially, it represents an added penalization due to lack of information prior to the default event.

Conclusion and future developments

Summary:

- Definition of dynamic risk measure induced by a BSDEJ in a progressive enlargement of filtration setting.
- Link between properties of the dynamic risk measure and properties of the driver and/or the decomposed dynamic risk measures.
- Partial results for the decomposition of the penalty term appearing in the dual representation of the dynamic risk measure.
- Updating preferences or risk-aversion feature.

To do:

- Find a BSDEJ representation for a given dynamic risk measure (more generally, for a non-linear expectation) in a progressive enlargement of filtration setting.
- Unspecified (possibily infinite) number of jumps.

Thank you for your attention!

Talk based on: A. Calvia, E. Rosazza Gianin, *Risk measures and progressive enlargement of filtration: a BSDE approach*, SIAM J. Financial Math., 11 (2020), pp. 815-848.