Convergence rate for the optimal control of McKean-Vlasov dynamics

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Based on joint works in progress with S. Daudin (Dauphine), Joe Jackson (Austin) and P.E. Souganidis (Chicago).

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- Large population stochastic wireless power control problems (Huang and al. ('03), ...)
- Swarm robotic systems (Lerman and al. ('04), ...)
- Smart charging of PEVs (Le Floch and al. ('15), Sheppard and al ('17),...)
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- ... and mean field games.

Optimal control of large particle systems (2)

We consider an optimal control of large particle systems of the form

$$\min_{(\alpha^{N,i})_{i=1,\dots,N}} \mathbb{E}\left[\int_{t_0}^{T} \left(\frac{1}{N}\sum_{i=1}^{N} L(X_t^{N,i},\alpha_t^{N,i}) + \mathcal{F}(m^N_{\mathbf{X}_t^N})\right)dt + \mathcal{G}(m^N_{\mathbf{X}_T^N})\right],$$

where, for $i = 1,\dots,N$,

$$X_t^{N,i} = x_0^{N,i} + \int_{t_0}^{T} \alpha_t^{N,i} dt + \sqrt{2}(B_t^i - B_{t_0}^i) + \sqrt{2a_0}(B_t^0 - B_{t_0}^0), \qquad m^N_{\mathbf{X}_t^N} = \frac{1}{N}\sum_{i=1}^{N}\delta_{X_t^{N,i}}$$

and

- N is the (large) number of particles,
- $X_t^{N,i} \in \mathbb{R}^d$ is the position of a particle at time t,
- $\alpha_t^{N,i} \in \mathbb{R}^d$ is the control for particle $i \in \{1,\dots,N\}$ at time t,
- $(B^i)_{i\in\mathbb{N}}$ is a family of $d-$dimension independent Brownian motions
- $T > 0$ is the terminal time horizon,
- $(t_0, \mathbf{x}_0^N) = (t_0, (x_0^{N,i})_{i=1,\dots,N}) \in [0,T] \times (\mathbb{R}^d)^N$ is the initial position of the particles,
- $L : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a kinetic cost,
- $\mathcal{F}, \mathcal{G} : \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}$ are interaction costs,

Let $\mathcal{V}^N$ be the value function of the problem:

$$\mathcal{V}^N(t_0, \mathbf{x}_0^N) := \min_{(\alpha^{N,i})_{i=1,\dots,N}} \mathbb{E}\left[\int_{t_0}^T \left(\frac{1}{N}\sum_{i=1}^N L(X_t^{N,i}, \alpha_t^{N,i}) + \mathcal{F}(m^N_{\mathbf{X}_t^N})\right)dt + \mathcal{G}(m^N_{\mathbf{X}_T^N})\right],$$

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To understand:

- The behavior of $\mathcal{V}^N$ as $N \to +\infty$,
- and the behavior of the optimal trajectories.

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The limit optimal control problem

Following Lacker ('17) and Djete et al. ('22) the limit problem as $N \to +\infty$ is expected to be an optimal control problem of a McKean-Vlasov equation (here in a strong form)

$$\mathcal{U}(t_0, m_0) = \inf_{\alpha} \mathbb{E}\Big[\int_{t_0}^{T} \big(L(X_t, \alpha_t) + \mathcal{F}(\mathcal{L}(X_t|\mathcal{F}_t^{B^0}))\big) + \mathcal{G}(\mathcal{L}(X_T|\mathcal{F}_T^{B^0}))\Big]$$

where $\mathbb{F}^{B^0} = (\mathcal{F}_t^{B^0})_{0 \le t \le T}$ denotes the filtration generated by B^0, and

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Here B is another Brownian motion, $\bar{X}_{t_0}$ is a random initial condition with law m_0 and B^0, B and $\bar{X}_{t_0}$ are independent.

Main results (in more general context) **of Lacker ('17) and Djete, Possamaï and Tan ('22)**

- Convergence of $\mathcal{V}^N$ to $\mathcal{U}$,
- Convergence of optimal solutions for $\mathcal{V}^N$ to relaxed optimal solutions for $\mathcal{U}$.

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Aim of the talk

- Let $\mathbf{X}^{N,*} = (X^{N,1,*}, \dots, X^{N,N,*})$ be optimal in for $\mathcal{V}^N$. Has $m^N_{\mathbf{X}^{N,*}}$ a limit adapted to (B^0)?
- Main issue: Lack of smoothness in the limit problem (multiplicity of solutions)
- Solving this issue requires a careful analysis of $\mathcal{U}$ and a quantitative approach. Namely
  - **Quantify** the difference between $\mathcal{V}^N$ and $\mathcal{U}$,
  - **Understand** the structure of the MFC problem, i.e., regularity of $\mathcal{U}$
  - Use this regularity to obtain **propagation of chaos properties**.

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A few references

- Early references: Huang-Caines-Malhamé ('03), Lasry-Lions ('07), Andersson-Djehiche ('10) for max. principle, Carmona-Delarue-Lachapelle ('13) for comparison MFG/MFC, Laurière-Pironneau ('14) for dyn. program.,...
- Analysis of mean field control (MFC) problems:
  - Deterministic setting: Fornasier-Solombrino ('14), Fornasier-Lisini-Orrieri-Savaré ('17), Cesaroni-Cirant ('21) for pbs with density constraints, Cavagnari-Lisini-Orrieri-Savaré ('22) with $\Gamma-$convergence techniques, ...
  - Stochastic setting: Buckdahn-Li-Ma ('17) for pbs with partial observations, Lacker ('17), Barrasso-Touzi ('22) for exit-time pbs, Djete-Possamaï-Tan ('22) for dyn. prog. with common noise,...
- Analysis of the mean field limit: Kolokoltsov ('12) in finite state, Lacker ('17), Cecchin ('21) in finite state, Gangbo-Mayorga-Swiech ('21) for pbs without idyo. noise, Germain-Pham-Warin ('21) for rate in the smooth case, Talbi-Touzi-Zhang ('21) for exit-time pbs, Djete-Possamaï-Tan ('22) with common noise, Djete ('22) extended MFC,
- Analysis of the HJ eq.: C.-Quincampoix ('08) for pbs arising in diff. games, Feng-Katsoulakis ('09) for controlled gradient flows, Lasry-Lions ('08) for first order pbs, Ambrosio-Feng ('14) for first order pbs, Burzoni-Ignazio-Reppen-Soner ('20) under a structure condition, Gangbo-Mayorga-Swiech ('21) for viscosity sols without idyo. noise, Wu-Zhang ('20) for viscosity sols, Conforti-Kraaij-Tonon ('21), Cosso-Gozzi-Kharroubi-Pham-Rosestolato ('21) for an intrinsic approach, Cecchin-Delarue ('22) for semiconcave sols,...

Outline

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Heuristic arguments (when $a_0 = 0$)

- The value function $\mathcal{V}^N$ of the $N-$particle system is a classical solution to

$$\left\{\begin{array}{l} -\partial_t \mathcal{V}^N(t,\mathbf{x}) - \sum_{j=1}^N \Delta_{x^j}\mathcal{V}^N(t,\mathbf{x}) + \frac{1}{N}\sum_{j=1}^N H(x^j, ND_{x^j}\mathcal{V}^N(t,\mathbf{x})) = \mathcal{F}(m^N_{\mathbf{x}}) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } (0,T)\times(\mathbb{R}^d)^N \\ \mathcal{V}^N(T,\mathbf{x}) = \mathcal{G}(m^N_{\mathbf{x}}) \qquad \text{in } (\mathbb{R}^d)^N \end{array}\right.$$

  where $H(x,p) = \sup_{a\in\mathbb{R}^d} -p.a - L(x,a)$.

- The value function $\mathcal{U}$ of the limit problem is expected to satisfy

$$\left\{\begin{array}{l} -\partial_t \mathcal{U}(t,m) - \int_{\mathbb{R}^d} \mathrm{div}(D_m\mathcal{U}(t,m,y))m(dy) + \int_{\mathbb{R}^d} H(y, D_m\mathcal{U}(t,m,y))m(dy) = \mathcal{F}(m) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } (0,T)\times\mathcal{P}_1(\mathbb{R}^d) \\ \mathcal{U}(T,m) = \mathcal{G}(m) \qquad \text{in } \mathcal{P}_1(\mathbb{R}^d) \end{array}\right.$$

  However $\mathcal{U}$ is not smooth in general and the equation has just to be understood in a weak sense (see Cosso and al. (preprint '21) or Cecchin-Delarue ('22) when for $a_0 = 0$).

- **Assume $\mathcal{U}$ is smooth** (as in Germain and al. ('21)). Then setting $\mathcal{U}^N(t,\mathbf{x}) := \mathcal{U}(t, m^N_{\mathbf{x}})$, we have

$$D_{x_i}\mathcal{U}^N(t,\mathbf{x}) = \frac{1}{N} D_m\mathcal{U}(t, m^N_{\mathbf{x}}, x_i), \qquad etc...$$

and therefore $\mathcal{U}^N$ satisfies

$$\begin{cases} -\partial_t\mathcal{U}^N(t,\mathbf{x}) - \sum_{j=1}^N \Delta_{x^j}\mathcal{U}^N(t,\mathbf{x}) + \frac{1}{N}\sum_{j=1}^N H(x^j, ND_{x^j}\mathcal{U}^N(t,\mathbf{x})) \\ \qquad = \mathcal{F}(m^N_{\mathbf{x}}) + E_N(t,\mathbf{x}) \qquad \text{in } (0,T)\times(\mathbb{R}^d)^N \\ \mathcal{U}^N(T,\mathbf{x}) = \mathcal{G}(m^N_{\mathbf{x}}) \qquad \text{in } (\mathbb{R}^d)^N \end{cases}$$

where $E_N(t,\mathbf{x}) = -\frac{1}{N^2}\sum_{j=1}^N \mathrm{tr}(D_{mm}\mathcal{U}(t, m^N_{\mathbf{x}}, x_i, x_i)) = O(1/N)$.

- By comparison **we can then conclude** the convergence rate

$$|\mathcal{U}^N - \mathcal{V}^N| \leq C/N$$

and (following C.-Delarue-Lasry-Lions) a quantified propagation of chaos.

- Unfortunately, **argument not correct in general** when $\mathcal{U}$ is not smooth.

Outline

1. Heuristic arguments
2. The convergence rate
3. Propagation of chaos

- **$\mathcal{V}^N$ is the value function for the $N-$particle system:**

$$\mathcal{V}^N(t_0, \mathbf{x}_0^N) := \min_{(\alpha^{N,i})_{i=1,\dots,N}} \mathbb{E}\left[\int_{t_0}^T (\frac{1}{N}\sum_{i=1}^N L(X_t^{N,i}, \alpha_t^{N,i}) + \mathcal{F}(m^N_{\mathbf{X}_t^N}))dt + \mathcal{G}(m^N_{\mathbf{X}_T^N})\right],$$

where, for $i = 1, \dots, N$,

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- **Definition of the value function $\mathcal{U}$ for the limit system:** Given $(t_0, m_0) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$, we define a control rule $\mathcal{R} \in \mathcal{A}(t_0, m_0)$ to be a tuple $\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, B^0, m, \alpha)$, where
  1. $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ is a filtered probability space supporting the d-dimensional Brownian motion B^0
  2. $\alpha = (\alpha_t)_{t_0\le t\le T}$ is a $\mathbb{F}$-progressively measurable taking values in $L^\infty(\mathbb{R}^d; \mathbb{R}^d)$ and such that α is uniformly bounded,
  3. m satisfies the stochastic McKean-Vlasov equation

$$dm_t(x) = [(1 + a_0)\Delta m_t(x) - \mathrm{div}(m_t\alpha_t(x))]\, dt + \sqrt{2a^0} Dm_t(x) \cdot dB_t^0, \qquad m_{t_0} = m_0.$$

We define

$$\mathcal{U}(t_0, m_0) := \inf_{\mathcal{R}\in\mathcal{A}(t_0,m_0)} \mathbb{E}^{\mathbb{P}}\Big[\int_{t_0}^T (\int_{\mathbb{R}^d} L(x, \alpha_t(x))m_t(dx) + \mathcal{F}(m_t))dt + \mathcal{G}(m_T)\Big].$$

Standing assumptions

The maps $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, $\mathcal{F} : \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}$ and $\mathcal{G} : \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}$ satisfy

- H is of class C^2 and strictly convex. In addition we assume that there exists a constant $C > 0$ such that

$$C^{-1}|p|^2 - C \leq H(x,p) \leq C(|p|^2+1) \qquad \forall (x,p) \in \mathbb{R}^d \times \mathbb{R}^d,$$

$$|D_x H(x,p)| \leq C(|p|+1) \qquad \forall (x,p) \in \mathbb{R}^d \times \mathbb{R}^d$$

  and that, for any $R > 0$, there exists $C_R > 0$ such that

$$|D^2_{xx}H(x,p)| + |D^2_{xp}H(x,p)| \leq C_R \qquad \forall (x,p) \in \mathbb{R}^d \times \mathbb{R}^d, \ |p| \leq R.$$

- The map $\mathcal{F} : \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}$ is of class C^2 with $\mathcal{F}$, $D_m\mathcal{F}$, $D^2_{ym}\mathcal{F}$ and $D^2_{mm}\mathcal{F}$ uniformly bounded. The map $\mathcal{G} : \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}$ is of class C^4 with all derivatives (in m and then in the additional variables) up to order 4 uniformly bounded.

$\longrightarrow$ **Note that** $\mathcal{F}$ and $\mathcal{G}$ are not assumed to be convex and thus $\mathcal{U}$ is not smooth in general. (cf. Briani-C. ('18), Bardi-Fischer ('19))

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Main result on the convergence rate

Theorem (C.-Daudin-Jackson-Souganidis)

Under our standing assumptions, there exists $\beta \in (0, 1]$ (depending only on d) and $C > 0$ (depending on the data) such that, for any $(t, \mathbf{x}) \in [0, T] \times (\mathbb{R}^d)^N$,

$$\left|\mathcal{V}^N(t, \mathbf{x}) - \mathcal{U}(t, m^N_{\mathbf{x}})\right| \leq CN^{-\beta}(1 + M_2(m^N_{\mathbf{x}})).$$

The proof relies on

- (uniform in N) regularity estimates for $\mathcal{V}^N$
- and concentration inequalities

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Idea of proof (1): regularity estimates

Lemma

Under our standing assumptions, there exists a constant $C > 0$ such that,

- for any $N \geq 1$,
$$\|\mathcal{V}^N\|_\infty + N \sup_j \|D_{x^j} \mathcal{V}^N\|_\infty + \|\partial_t \mathcal{V}^N\|_\infty \leq C.$$
- (Semiconcavity) for any $\xi = (\xi^i) \in (\mathbb{R}^d)^N$ and $\xi^0 \in \mathbb{R}$,

$$\sum_{i,j=1}^N D^2_{x^i x^j} \mathcal{V}^N(t, \mathbf{x}) \xi^i \cdot \xi^j + 2 \sum_{i=1}^N D^2_{x^i t} \mathcal{V}^N(t, \mathbf{x}) \cdot \xi^i \xi^0 + D^2_{tt} \mathcal{V}^N(t, \mathbf{x})(\xi^0)^2 \leq \frac{C}{N} \sum_{i=1}^N |\xi^i|^2 + C(\xi^0)^2.$$

Remark: As a consequence, the limit value function $\mathcal{U}$ is Lipschitz continuous in $[0, T] \times \mathcal{P}_1(\mathbb{R}^d)$ and (displacement) semiconcave.

Idea of proof (2): The easy inequality

Let

$$\hat{\mathcal{V}}^N(t,m) := \int_{(\mathbb{R}^d)^N} \mathcal{V}^N(t,\mathbf{x}) \prod_{j=1}^N m(dx^j) \qquad \forall (t,m) \in [0,T] \times \mathcal{P}_1(\mathbb{R}^d).$$

Lemma

The map $\hat{\mathcal{V}}^N$ is smooth and satisfies the inequality

$$\begin{cases} -\partial_t \hat{\mathcal{V}}^N(t,m) - \int_{\mathbb{R}^d} \mathrm{div}(D_m\hat{\mathcal{V}}^N(t,m,y))m(dy) + \int_{\mathbb{R}^d} H(y, D_m\hat{\mathcal{V}}^N(t,m,y))m(dy) \leq \hat{\mathcal{F}}(m) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } (0,T) \times \mathcal{P}_1(\mathbb{R}^d) \\ \hat{\mathcal{V}}^N(T,m) = \hat{\mathcal{G}}(m) \qquad \text{in } \mathcal{P}_1(\mathbb{R}^d) \end{cases}$$

where $\hat{\mathcal{F}}^N(m) := \int_{(\mathbb{R}^d)^N} \mathcal{F}(m^N_{\mathbf{x}}) \prod_{j=1}^N m(dx^j)$ and $\hat{\mathcal{G}}^N(m) := \int_{(\mathbb{R}^d)^N} \mathcal{G}(m^N_{\mathbf{x}}) \prod_{j=1}^N m(dx^j)$.
Hence, the exists constants $C, \beta > 0$ such that, for any $(t,\mathbf{x}_0) \in [0,T] \times (\mathbb{R}^d)^N$,

$$\mathcal{V}^N(t, m^N_{\mathbf{x}_0}) \leq \mathcal{U}(t, m^N_{\mathbf{x}_0}) + C(1 + M_2^{1/2}(m^N_{\mathbf{x}_0}))N^{-\beta},$$

Proposition

There exists a constant $\beta \in (0, 1]$ (depending on dimension only) and a constant $C > 0$ (depending on the data) such that, for any $N \geq 1$ and any $(t, \mathbf{x}) \in [0, T] \times (\mathbb{R}^d)^N$, it holds:

$$\mathcal{U}(t, m^N_{\mathbf{x}}) - \mathcal{V}^N(t, \mathbf{x}) \leq CN^{-\beta}(1 + \frac{1}{N}\sum_{i=1}^N |x^i|^2).$$

Proof by penalization: we consider, for $\theta, \lambda \in (0, 1)$,

$$M^N := \max_{(t,\mathbf{x}),(s,\mathbf{y}) \in [0,T]\times(\mathbb{R}^d)^N} e^s(\mathcal{U}(s, m^N_{\mathbf{y}}) - \mathcal{V}^N(t, \mathbf{x})) - \frac{1}{2\theta N}\sum_{i=1}^N |x^i - y^i|^2 - \frac{1}{2\theta}|s-t|^2 - \frac{\lambda}{2N}\sum_{i=1}^N |y^i|^2.$$

By combining Lipschitz and semiconcavity estimates and concentration inequalities we show that, for a suitable choice of θ, λ,

$$M^N \leq CN^{-\beta}.$$

Outline

Our aim is to study the behavior of optimal trajectories of $\mathcal{V}^N$ and prove a (quantitative) propagation of chaos property.

For this **we assume from now on that there is no common noise**: $a_0 = 0$. Then the value function of the limit problem is given by

$$\mathcal{U}(t_0, m_0) := \inf \left\{ \int_{t_0}^{T} \left(\int_{\mathbb{R}^d} L(x, \alpha(t,x)) m(t,dx) + \mathcal{F}(m(t)) \right) dt + \mathcal{G}(m(T)) \right\}$$

where the infimum is taken over the pairs $(m, \alpha) \in C^0([t_0, T], \mathcal{P}_1(\mathbb{R}^d)) \times L^0([t_0, T] \times \mathbb{R}^d; \mathbb{R}^d)$ such that $\int_{t_0}^{T} \int_{\mathbb{R}^d} |\alpha(t,x)|^2 m(t,dx) dt < +\infty$ and (m, α) satisfies in the sense of distributions

$$\partial_t m - \Delta m + \mathrm{div}(m\alpha) = 0 \text{ in } (t_0, T) \times \mathbb{R}^d, \qquad m(0) = m_0 \text{ in } \mathbb{R}^d.$$

The analysis is split into two parts:

- Regularity properties of the function $\mathcal{U}$,
- Propagation of chaos.

Regularity of $\mathcal{U}$

Theorem (C.-Souganidis)

The map $\mathcal{U}$ is globally Lipschitz continuous on $[0, T] \times \mathcal{P}_1(\mathbb{R}^d)$ and there exists an open and dense subset $\mathcal{O}$ of $[0, T) \times \mathcal{P}_2(\mathbb{R}^d)$ on which $\mathcal{U}$ is of class C^1. Moreover $\mathcal{U}$ satisfies in a classical sense in $\mathcal{O}$ the Hamilton-Jacobi equation:

$$-\partial_t \mathcal{U}(t, m) - \int_{\mathbb{R}^d} \mathrm{div}(D_m \mathcal{U}(t, m, y)) m(dy) + \int_{\mathbb{R}^d} H(y, D_m \mathcal{U}(t, m, y)) m(dy) = \mathcal{F}(m).$$

(Compare with Cosso and al. ('21) and Cecchin-Delarue ('22))

The set $\mathcal{O}$ is defined as follows:

$$\mathcal{O} := \left\{ (t_0, m_0) \in [0, T) \times \mathcal{P}_2(\mathbb{R}^d), \begin{array}{l} \text{there exists a unique minimizer for } \mathcal{U}(t_0, m_0) \\ \text{and this minimizer is stable} \end{array} \right\}.$$

Proof (1): Stability of a minimizer

Proposition (Lasry-Lions)

Let (m, α) be a minimizer for $\mathcal{U}(t_0, m_0)$. There exists a unique multiplier $u : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}$ of class $C^{1,2}$ such that $\alpha = -D_p H(x, Du)$ and the pair (u, m) satisfies

$$\left\{\begin{array}{ll} -\partial_t u - \Delta u + H(x, Du) = F(x, m(t)) & \text{in } (t_0, T) \times \mathbb{R}^d \\ \partial_t m - \Delta m - \mathrm{div}(H_p(x, Du)m) = 0 & \text{in } (t_0, T) \times \mathbb{R}^d \\ m(t_0) = m_0, \ u(T, x) = G(x, m(T)) & \text{in } \mathbb{R}^d \end{array}\right.$$

where $F(x, m) = \dfrac{\delta \mathcal{F}}{\delta m}(m, x), \qquad G(x, m) = \dfrac{\delta \mathcal{G}}{\delta m}(m, x).$

We say that (m, α) is stable if $(z, \mu) = (0, 0)$ is the only solution to the linearized system

$$\left\{\begin{array}{ll} -\partial_t z - \Delta z + H_p(x, Du) \cdot Dz = \dfrac{\delta F}{\delta m}(x, m(t))(\mu(t)) & \text{in } (t_0, T) \times \mathbb{R}^d \\ \partial_t \mu - \Delta \mu - \mathrm{div}(H_p(x, Du)\mu) - \mathrm{div}(H_{pp}(x, Du) Dz m) = 0 & \text{in } (t_0, T) \times \mathbb{R}^d \\ \mu(t_0) = 0, \ z(T, x) = \dfrac{\delta G}{\delta m}(x, m(T))(\mu(T)) & \text{in } \mathbb{R}^d \end{array}\right.$$

Proof (2): Key property of stable solutions

Proposition

1. Assume that there is a unique minimizer (m, α) for $\mathcal{U}(t_0, m_0)$ and that this minimizer is stable. Then there exists a neighborhood $\mathcal{O}'$ of $\{(t, m(t)),\ t \in [t_0, T]\}$ such that, for any $(t_1, m_1) \in \mathcal{O}'$, there is a unique minimizer for $\mathcal{U}(t_1, m_1)$ and this minimizer is stable.
2. If (m, α) is a minimizer for $\mathcal{U}(t_0, m_0)$, then for any $t_1 \in (t_0, T)$ there is a unique minimizer for $\mathcal{U}(t_1, m(t_1))$ and this minimizer is stable.

- Reminiscent of similar results in finite dimension.
- The proof uses on a Lions-Malgrange ('60) type argument, generalized by Cannarsa-Tessitore ('94) to forward-backward systems.
- Similar result obtained by Briani-C. ('18) in the torus.

Proof (3): Regularity of $\mathcal{U}$

Proposition

The map $\mathcal{U}$ is of class C^1 in $\mathcal{O}$ with $D_m\mathcal{U}(t_0, m_0, \cdot) = Du(t_0, \cdot)$ for any $(t_0, m_0) \in \mathcal{O}$, where u is the multiplier associated to the unique minimizer (m, α) for $\mathcal{U}(t_0, m_0)$.

- Relies on constructions developed in C.-Delarue-Lasry-Lions ('19) for mean field games.
- In contrast with this paper, stability replaces the monotonicity condition.

Main result on the propagation of chaos

Theorem (C.-Souganidis)

Fix $(t_0, m_0) \in \mathcal{O}$. There exists a constant $\gamma \in (0,1)$ (depending on dimension only) and $C > 0$ (depending on (t_0, m_0)) such that, if (Z^k) is a sequence of independent r.v. with law m_0 and $\mathbf{Y}^N = (Y^{N,k})$ is **the optimal trajectories** for $\mathcal{V}^N(t_0, (Z^k)_{k=1,\dots,N})$:

$$Y_t^{N,k} = Z^k - \int_{t_0}^{t} H_p(Y_s^k, D_{x^k}\mathcal{V}^N(s, \mathbf{Y}_s^N))ds + \sqrt{2}(B_t^k - B_{t_0}^k),$$

then

$$\mathbb{E}\left[\sup_{t\in[t_0,T]} \mathbf{d_1}(m^N_{\mathbf{Y}_t^N}, m(t))\right] \leq CN^{-\gamma},$$

where (m, α) is optimal for $\mathcal{U}(t_0, m_0)$.

Following Sznitman, this implies the propagation of chaos for the $(Y^{N,k})$.

Idea of proof

- For $\delta > 0$, let $\mathcal{W}_\delta :=$ $\delta-$neighborhood of $\{(t, m(t)),\ t \in [t_0, T]\}$ contained in $\mathcal{O}$ and $\mathcal{W}_\delta^N := \{\mathbf{x} \in (\mathbb{R}^d)^N,\ m_{\mathbf{x}}^N \in \mathcal{W}_\delta\}$.
- Let $\mathbf{X}^N = (X_t^{N,i})$ be the solution to

$$dX_t^{N,j} = Z^j - \int_{t_0}^t H_p(X_s^{N,j}, D_m\mathcal{U}(s, m^N_{\mathbf{X_s^N}}, X_s^{N,j}))ds + \sqrt{2}(B_s^j - B_{t_0}^j),$$

on the time interval $[t_0, \tau^N]$, where $\tau^N = \inf\left\{t \in [t_0, T],\ (t, \mathbf{X}_t^N) \notin \mathcal{V}^N_{\delta/2}\right\}$.

- Following Horowitz-Karandikar ('94) and standard argument on the propagation of chaos,

$$\mathbb{E}\left[\sup_{t\in[t_0,\tau^N]} \mathbf{d}_1(m^N_{\mathbf{X}_t^N}, m(t))\right] \leq CN^{-1/(d+8)} \text{ and } \mathbb{P}\left[\tau^N < T\right] \leq CN^{-1/(d+8)}.$$

- By the strict convexity of H and the estimate $\|\mathcal{U}^N - \mathcal{V}^N\|_\infty \leq CN^{-\beta}$, we get

$$\mathbb{E}\Big[\int_{t_0}^{\tau^N} N^{-1}\sum_j |H_p(Y_t^{N,j}, ND_{x^j}\mathcal{U}^N) - H_p(Y_t^{N,j}, ND_{x^j}\mathcal{V}^N)|^2 dt\Big] \leq CN^{-2\beta},$$

- ...from which we infer that $\mathbb{E}\left[\sup_{s\in[t_0,t\wedge\tau^N]} N^{-1}\sum_j |X_s^{N,j} - Y_s^{N,j}|\right] \leq CN^{-\beta}$.

Conclusion and open problems

Conclusion: in these works we have obtained

- a converge rate for the value function,
- the smoothness of the limit value function in an open and dense set,
- and the propagation of chaos for initial data in this set.

Open problems

- sharper convergence rate
- generalization of the propagation of chaos to problems with a common noise
- propagation of chaos for general initial conditions
- application to potential mean field game problems.

Thank you!

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