Mean field games master equations: from discrete to continuous state space

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Introduction

Mean field games were introduced by [Huang-Malhamé-Caines '06] and [Lasry-Lions '06] as limit models for symmetric non-zero sum dynamic games, when the number N of players tends to infinity.

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- We consider here games in continuous time and finite horizon.
- Players are small and symmetric, interaction is mean field, control their dynamics in order to minimize a cost.
  Notion of optimality: Nash equilibrium. N-player game typically untractable because of curse of dimensionality.
  Letting $N = \infty$ may restore some tractability of the model.
- Mean field games have seen a wide variety of applications, including models of oil production, volatility formation, economic growth, energy production, bitcoin mining...
- Importance of numerical methods:
  We present here a space discretization.

Outline

1. Continuous state mean field game
   - Diffusion-based model
   - Mean field game system and master equation
2. Space discretization
   - Finite state mean field game
   - Controlled Markov chain
3. Results: Convergence of master equation and MFG system as number of states grows, with convergence rate
   - with classical solution to limit master equation
   - without such solution.

   Here without common noise.

Mean field dynamics

Dynamics on one-dimensional torus $\mathbb{T}$, finite horizon T.

One reference player X chooses its control $\alpha : [0, T] \times \mathbb{T} \to \mathbb{R}$ (in feedback form)

$$dX_t = \alpha(t, X_t)dt + \sqrt{2}dB_t$$

in order to minimize

$$J(\alpha, \mu) = \mathbb{E}\left[\int_0^T \frac{1}{2}|\alpha(t, X_t)|^2 + f(X_t, \mu_t)dt + g(X_T, \mu_T)\right]$$

for fixed flow $\mu : [0, T] \to \mathcal{P}(\mathbb{T})$ deterministic.

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Definition

A solution of the mean field game is a couple (α, μ) such that

1. *Optimality*: $J(\alpha, \mu) \leq J(\beta, \mu)$ for every β;
2. *Mean field condition*: $\mathrm{Law}(X_t^\alpha) = \mu_t$ for any $t \in [0, T]$.

Mean field game system

Fixed point: $\mu \to \alpha^*_\mu \to \mathrm{Flow}(X^{\alpha^*_\mu}) = \mu$.

- Given a flow of measures μ find the optimal control via the HJB equation: $\alpha^*_\mu(t,x) = -\partial_x u(t,x)$, u value function.
- Hamiltonian $H(x,p) = \sup_a \left\{ -ap - \frac{1}{2}|a|^2 \right\} = \frac{1}{2}|p|^2$
  Unique maximizer $a^*(x,p) = -p$.
- Then put α^*_μ into the KFP equation for $\mathrm{Law}(X^\alpha_t)$
- Solution if $\mu_t = \mathrm{Law}(X^\alpha_t)$, fixed point.

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A solution of the mean field game system is a couple (u,μ) solving the forward-backward system of PDEs

$$\begin{cases} -\partial_t u - \partial^2_x u + \frac{1}{2}|\partial_x u|^2 = f(x,\mu_t) \\ \partial_t \mu - \partial^2_x \mu - \partial_x(\mu \partial_x u) = 0 \\ u(T,x) = g(x,\mu_T) \qquad \mu_0 = m_0. \end{cases} \tag{MFG}$$

Monotonicity

Existence: if f, g are W_1-Lipschitz in m and
$\sup_{m\in\mathcal{P}(\mathbb{T})} ||f(\cdot,m)||_\gamma < \infty$, $\sup_{m\in\mathcal{P}(\mathbb{T})} ||g(\cdot,m)||_{2+\gamma} < \infty$
then $\exists$ sol. $u \in C^{1+\frac{\gamma}{2},2+\gamma}([0,T]\times\mathbb{T})$, $\mu \in C^{1+\frac{\gamma}{2},2+\gamma}((0,T]\times\mathbb{T})$

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Uniqueness either for small T or under Lasry-Lions monotonicity condition on f and g:

$$\begin{aligned}
&\int_{\mathbb{T}}(f(x,m)-f(x,\tilde{m}))(m-\tilde{m})(dx) \geq 0 \qquad \forall m,\tilde{m}\in\mathcal{P}(\mathbb{T})\\
&\int_{\mathbb{T}}(g(x,m)-g(x,\tilde{m}))(m-\tilde{m})(dx) \geq 0 \qquad \forall m,\tilde{m}\in\mathcal{P}(\mathbb{T})
\end{aligned} \tag{1}$$

- Example: $f(x,m) = x\mathrm{Mean}(m) = x\int_{\mathbb{T}} y m(dy)$. Monotonicity means that players prefer to spread, instead of aggregate.

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- Monotonicity implies also stability of the system.

Master equation

MFG completely understood by means of the master equation
$U : [0, T] \times \mathbb{T} \times \mathcal{P}(\mathbb{T}) \to \mathbb{R}$

- U is decoupling field of forward-backward system:
  $u(t, x) = U(t, x, \mu_t)$
- MFG system is the system of characteristics of (M):
  $U(t_0, x, m_0) := u(t_0, x)$ defines a solution, where (u, μ) solves the MFG system with $\mu_{t_0} = m_0$.

$$\begin{cases} -\partial_t U + \frac{1}{2}|\partial_x U|^2 + \int_{\mathbb{T}} \partial_x U(t, y, m)\partial_y \frac{\delta U}{\delta m}(t,x,m; y)m(dy) \\ \qquad -\partial_x^2 U - \int_{\mathbb{T}} \partial_y^2 \frac{\delta U}{\delta m} U(t, x, m; y)m(dy) = f(x, m) \\ U(T, x, m) = g(x, m). \end{cases} \tag{M}$$

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Requires chain rule for flat derivative on $\mathcal{P}(\mathbb{T})$:
for a function $U : \mathcal{P}(\mathbb{T}) \to \mathbb{R}$, $\frac{\delta U}{\delta m}(m;y)$ is defined by

$$\lim_{h\to 0^+} \frac{U(m+h(m'-m)) - U(m)}{h} = \int_{\mathbb{T}} \frac{\delta U}{\delta m}(m;y)(m-m')(dy)$$

Classical solution

U is a classical solution if all derivatives
$\partial_t U, \partial_x U, \partial_x^2 U, \partial_y \frac{\delta U}{\delta m}(t,x,m;y), \partial_y^2 \frac{\delta U}{\delta m} U(t,x,m;y)$ exist continuous.

Theorem [Cardaliaguet-Delarue-Lasry-Lions '19]: There exists a classical solution if f, g monotone and smooth in the measure argument.

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- existence of classical solutions implies uniqueness of MFG system.
- Assumption on f, g smooth is typically too strong, they might be just W_1-Lipschitz in m.
- Notions of *weak solutions*, assuming monotonicity and thus uniqueness of MFG system, considered in [Bertucci '20, 21], [Mou-Zhang '20], [Gangbo-Meszaros '20], ...

Numerical methods

- - [Achdou, Capuzzo-Dolcetta '10], [Achdou, Capuzzo-Dolcetta, Camilli '12]: finite difference scheme for MFG sysyem.
- - [Benamou, Carlier '15]: augmented Lagrangian methods.
- - [Chassagneux, Crisan, Delarue '19]: McKean-Vlasov forward-Backward SDEs.
- - [Laurière '21 (survey)]: machine learning based methods.
- - [Hadikhanloo, Silva '19]: probabilistic method for deterministic MFG, based of Kushner's Markov chain approximation method for stochastic control.

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We consider a space discretization (for diffusions) such that discretized model is a continuous time finite state MFG, and corresponding MFG system is finite difference scheme.
Study convergence of the master equations, the main result being to provide a convergence rate.

Space discretization

For any n, consider n states $S^n = \{x_1^n, \dots, x_n^n\} = \{1/n, \dots, 1\}$ with mutual distance $1/n$, with the convention $x_0^n = x_n^n$, $x_{n+1}^n = x_1^n$.

X^n Markov chain in $[0, T]$, continuous time.
Control the jump rate on the right and on the left by 2 feedback functions $\alpha_+^n, \alpha_-^n : [0, T] \times S^n \to [0, +\infty)$:

$$\mathbb{P}(X_{t+\Delta t}^n = x_{i+1}^n | X_t^n = x_i^n) = \left(\frac{\alpha_+^n(t, x_i^n)}{1/n} + \frac{1}{1/n^2} \right) \Delta t + o(\Delta t),$$
$$\mathbb{P}(X_{t+\Delta t}^n = x_{i-1}^n | X_t^n = x_i^n) = \left(\frac{\alpha_-^n(t, x_i^n)}{1/n} + \frac{1}{1/n^2} \right) \Delta t + o(\Delta t), \tag{2}$$

The cost is given by

$$J^n(\alpha_\pm^n, \mu^n) = \mathbb{E}\left[\int_0^T \frac{1}{2} |\alpha_+^n(t, X_t^n)|^2 + \frac{1}{2} |\alpha_-^n(t, X_t^n)|^2 + f(X_t^n, \mu_t^n) dt + g(X_T^n, \mu_T^n) \right]$$

with $\mu^n : [0, T] \to \mathcal{P}(S^n)$ fixed and deterministic.

Discrete mean field game

$\mathcal{P}(S^n) \approxeq$ simplex of probability measures on $\mathbb{R}^n$,
elements $m^n = \sum_{j=1}^n m_j^n \delta_{x_j^n}$.

MFG solution $(\alpha_\pm^n, \mu^n)$, with $\alpha_\pm^n$ optimal for μ^n and $\mu_t^n = \mathrm{Law}(X_t^n)$

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For $u : \mathbb{T} \to \mathbb{R}$, denote the right and left first order finite difference and the second order finite difference by

$$\Delta_+^n u(x) = \frac{u(x+1/n) - u(x)}{1/n}, \qquad \Delta_-^n u(x) = \frac{u(x-1/n) - u(x)}{1/n},$$

$$\Delta_2^n u(x) = \frac{u(x+1/n) - 2u(x) + u(x-1/n)}{1/n^2}$$

Obtain unique optimal controls, given by

$$\alpha_+^n(t,x) = (\Delta_+^n u(t,x))_-, \qquad \alpha_-^n(t,x) = (\Delta_-^n u(t,x))_-, \qquad x \in S^n.$$

where $u(t,x)$ is the value function and r_- is the negative part of r.

Discrete mean field game system

The discrete MFG system is a system of ODEs, indexed by $x \in S^n$, backward HJB equation and forward KFP equation.

$$\begin{cases} -\frac{d}{dt}u^n + \frac{1}{2}(\Delta_+^n u^n(x))_-^2 + \frac{1}{2}(\Delta_-^n u^n(x))_-^2 - \Delta_2^n u^n(x) = f(x, \mu_t^n), \\ \frac{d}{dt}\mu^n(t,x) - \Delta_2^n \mu^n(t,x) - \Delta_-^n[(\Delta_+^n u^n(x))_- \mu^n(t,x)] \\ -\Delta_+^n[(\Delta_-^n u^n(x))_- \mu^n(t,x)] = 0, \\ u^n(t,x) = g(x, \mu_t^n), \qquad \mu_0^n = m_0^n \end{cases} \tag{MFG:n}$$

Existence and uniqueness under monotonicity assumptions

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Existence and uniqueness under monotonicity assumptions

Formally, we should have $\lim_{n\to\infty} \Delta_\pm^n u(x) = \pm\partial_x u(x)$ and $\lim_{n\to\infty} \Delta_2^n u(x) = \partial_x^2 u(x)$.

Thus, heuristically, we see that $u^n \to u$ and $\mu^n \to \mu$.

Convergence of trajectories

Convergence of optimal trajectories X^n to X in distribution, (formally) by means of convergence of the generators:

The generator of X^n is given by

$$\begin{aligned}
\mathcal{L}_t^n \phi(x) = & \left(\frac{(\Delta_+^n u(x))_-}{1/n} + \frac{1}{1/n^2} \right) [\phi(x+1/n) - \phi(x)] \\
& + \left(\frac{(\Delta_-^n u(x))_-}{1/n} + \frac{1}{1/n^2} \right) [\phi(x-1/n) - \phi(x)] \\
& = (\Delta_+^n u(x))_- \Delta_+^n \phi(x) + (\Delta_-^n u(x))_- \Delta_-^n \phi(x) + \Delta_2^n \phi(x). \\
& \approx (\partial_x u(t,x))_- \partial_x \phi(x) - (\partial_x u(t,x))_+ \partial_x \phi(x) + \partial_x^2 \phi(x) \\
& = -\partial_x u(t,x) \partial_x \phi(x) + \partial_x^2 \phi(x),
\end{aligned}$$

which is the generator of X: $dX_t = -\partial_x u(t,x)dt + \sqrt{2}dB_t$.

Discrete master equation

Decoupling field $U^n : [0,T] \times S^n \times \mathcal{P}(S^n)$, such that $u^n(t,x) = U^n(t,x,\mu^n_t)$ solves the first order PDE on the simplex:

$$
\begin{aligned}
&-\partial_t U^n(x,m) + \frac{1}{2}(\Delta^n_+ U^n(x,m))^2_- + \frac{1}{2}(\Delta^n_- U^n(x,m))^2_- - \Delta^n_2 U^n(x,m) - f(x,m) \\
&- \sum_{y\in S^n} m_y \left(\frac{(\Delta^n_+ U^n(y,m))_-}{1/n} + \frac{1}{1/n^2} \right) \left(\partial_{m_{y+1/n}} U^n(x,m) - \partial_{m_y} U^n(x,m) \right) \\
&- \sum_{y\in S^n} m_y \left(\frac{(\Delta^n_- U^n(y,m))_-}{1/n} + \frac{1}{1/n^2} \right) \left(\partial_{m_{y-1/n}} U^n(x,m) - \partial_{m_y} U^n(x,m) \right) = 0 \\
&U^n(T,x,m) = g(x,m)
\end{aligned}
\tag{M:n}
$$

For U defined on $\mathcal{P}(S^n)$, we denote by $\partial_{m_j} U$ its derivative along direction e_j; and equivalently $\partial_{m_j} U = \partial_{m_{x_j}} U$, because we view $m \in \mathcal{P}(S^n)$ as $m = \sum_{j=1}^n m_j \delta_{x_j}$.

Heuristic limit

Assume that, formally, $U^n(t,x,m^n) \approx U(t,x,\sum_{j=1}^n m_j^n \delta_{x_j^n})$.
By definition, if U is C^1 on $\mathcal{P}(\mathbb{T})$, we have
$\partial_{m_i} U(\sum_{j=1}^n m_j^n \delta_{x_j^n}) = \frac{\delta U}{\delta m}(\sum_{j=1}^n m_j^n \delta_{x_j^n}; x_i)$

Thus in the master equation we get

$$\int_{\mathbb{T}} m(dy) \frac{(\Delta_+^n U^n(y,m))_-}{1/n} \left(\partial_{m_{y+1/n}} U^n(x,m) - \partial_{m_y} U^n(x,m)\right)$$

$$\int_{\mathbb{T}} m(dy) \frac{(\Delta_-^n U^n(y,m))_-}{1/n} \left(\partial_{m_{y-1/n}} U^n(x,m) - \partial_{m_y} U^n(x,m)\right)$$

$$\int_{\mathbb{T}} m(dy) \frac{1}{1/n^2} \left(\partial_{m_{y+1/n}} U^n(x,m) - 2\partial_{m_y} U^n(x,m) + \partial_{m_{y-1/n}} U^n(x,m)\right)$$

$$\approx \int_{\mathbb{T}} m(dy) \left[(\partial_y U(y,m))_- \partial_y \frac{\delta U}{\delta m}(x,m;y) - (\partial_y U(y,m))_+ \partial_y \frac{\delta U}{\delta m}(x,m;y)\right]$$

$$+ \int_{\mathbb{T}} m(dy) \partial_y^2 \frac{\delta U}{\delta m}(x,m;y)$$

which provide the terms in (M), thus U "solves" limit master equation.

Concergence of classical solutions

U^n, U master equation, X^n, X optimal trajectories.

Theorem

Assume that (M:n) *and* (M) *has a classical solution* U*, limit* U *with Lipschitz derivatives. Then*

$$\sup_{t\in[0,T],x\in S^n,m\in\mathcal{P}(S_n)} |U^n(t,x,m) - U(t,x,m)| \leq \frac{C}{n}$$

$$\mathbb{E}\int_0^T |\Delta^n_{\pm}(U^n - U)(t, X^n_t, \mathrm{Law}(X^n_t))|^2 dt \leq \frac{C}{n^2}$$

We obtain also (assume $W_1(m^n_0, m_0) \leq \frac{1}{n}$):

- Y^n Markov chain (2) with rates given by limit master equation: $\mathbb{E}\left[\sup_{t\in[0,T]} |X^n_t - Y^n_t|\right] \leq \frac{C}{n}$
- $\lim_n X^n = X$ in law in $D([0,T],\mathbb{T})$

$$\sup_{t\in[0,T]} W_1(\mathrm{Law}(X^n_t), \mathrm{Law}(X_t)) \leq \frac{C}{n^{1/3}}$$

Idea of the proof

- $U^n(m^n) := U(\sum_{i=1}^n m_i^n \delta_{x_i^n})$ almost solves (M:n), with a reminder of order $\mathcal{O}(1/n)$
- Argument of stability of forward-backward systems, if exists classical decoupling field ([Ma-Protter-Yong '94]): expand $d|U^n(t, X_t^n, \mathrm{Law}(X_t^n)) - U(t, X_t^n, \mathrm{Law}(X_t^n))|^2$. Method employed also in [Cardaliaguet, Delarue, Lasry, Lions '19] to prove convergence of N-player game
- Laplacian (non-degeneracy) required to get estimate on the gradients

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To obtain estimate on trajectories in W_1, use the following:
Proposition. Let $dY_t = \alpha(t, Y_t)dt + \sqrt{2}dB_t$, with α regular (Hölder), and Y^n the Markov chain (2) with rates α_+, α_- positive and negative part of α. Then

$$\sup_{t\in[0,T]} W_1(\mathrm{Law}(Y_t^n), \mathrm{Law}(Y_t)) \leq \frac{C}{n^{1/3}}$$

Convergence via MFG systems

If there are no classical solutions (coefficients not smooth), convergence via MFG system

Theorem

Assume f, g W_1-Lipschitz and monotone, f Lipschitz in x, g $(2+\gamma)$-Hölder, $\gamma \geq 1/3$. (u^n, μ^n) and (u, μ) solutions of MFG systems. Then

$$\sup_{0\leq t\leq T} \sup_{x\in S^n} |u^n(t,x) - u(t,x)| + \sup_{0\leq t\leq T} W_1(\mu^n_t, \mu_t) \leq \frac{C}{n^{1/6}}.$$

$$\sup_{t\in[0,T], x\in S^n, m\in\mathcal{P}(S_n)} |U^n(t,x,m) - U(t,x,m)| \leq \frac{C}{n^{1/6}}$$

- Worse convergence rate

Idea of the proof

- show that (u,μ) almost solves the discrete MFG system (MFG:n)

From previous proposition,

$$\sup_{t\in[0,T]} W_1(\mathrm{Law}(Y_t^n),\mathrm{Law}(X_t)) \leq \frac{C}{n^{1/3}},$$

with X limit optimal trajectory $dX_t = -\partial_u(t,X_t)dt + \sqrt{2}dB_t$, and Y^n the Markov chain (2) with rates $\alpha_+^n(t,x) = (\Delta_+^n u(t,x))_-$, $\alpha_-^n(t,x) = (\Delta_-^n u(t,x))_-$, $\widetilde{\mu}_t^n = \mathrm{Law}(Y_t^n)$, then

$$-\frac{d}{dt}u+\frac{1}{2}(\Delta_+^n u(x))_-^2+\frac{1}{2}(\Delta_-^n u(x))_-^2-\Delta_2^n u(x) = f(x,\widetilde{\mu}_t^n)+\mathcal{O}\Big(\frac{1}{n^{1/3}}\Big)$$

Idea of the proof

- show that (u, μ) almost solves the discrete MFG system (MFG:n)

From previous proposition,

$$\sup_{t\in[0,T]} W_1(\mathrm{Law}(Y_t^n), \mathrm{Law}(X_t)) \leq \frac{C}{n^{1/3}},$$

with X limit optimal trajectory $dX_t = -\partial_u(t, X_t)dt + \sqrt{2}dB_t$, and Y^n the Markov chain (2) with rates $\alpha_+^n(t,x) = (\Delta_+^n u(t,x))_-$, $\alpha_-^n(t,x) = (\Delta_-^n u(t,x))_-$, $\widetilde{\mu}_t^n = \mathrm{Law}(Y_t^n)$, then

$$-\frac{d}{dt}u + \frac{1}{2}(\Delta_+^n u(x))_-^2 + \frac{1}{2}(\Delta_-^n u(x))_-^2 - \Delta_2^n u(x) = f(x, \widetilde{\mu}_t^n) + \mathcal{O}\left(\frac{1}{n^{1/3}}\right)$$

- Rely on variant of the the stability argument for MFG system under monotonicity (plus uniform convexity of Lagrangian).

Conclusions

Without common noise, we show convergence of the discretized master equation to the continuous one, with a convergence rate, in case there is a classical solution and in case there is not (with a worse rate).

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We also deal with a type of common noise made of common jumps of the whole population (introduced in [Bertucci-Lasry-Lions '18])

- Use notion of monotone solution to the master equation introduced by [Bertucci '20, '21]
- Convergence via compactness arguments, PDE method.

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THANK YOU FOR YOUR ATTENTION