Propagation of monotonicity for mean field games

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Outline

1. Master equation with common noise
2. Known results
3. Road map
4. Propagation of monotonicity for MFG
   - Lasry-Lions monotonicity
   - Displacement monotonicity
   - Anti-monotonicity
5. Propagation of monotonicity for MFGC
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The master equation with common noise

\[\begin{align*}
\partial_t V(t, x, \mu) &+ \frac{1+\beta^2}{2} \Delta V(t, x, \mu) - H(x, \mu, \partial_x V(t, x, \mu)) \\
+ \mathcal{N}V & = 0, \quad \text{in } (0, T) \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d), \\
V(T, x, \mu) & = G(x, \mu),
\end{align*}\]

where \( H : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R} \rightarrow \mathbb{R}, \ G : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R} \) and

\[\mathcal{N}V := \text{tr}\left( \mathbb{E}\left[ \frac{1+\beta^2}{2} \partial_{x\mu} V(t, x, \mu, \xi) + \beta^2 \partial_{x\mu} V(t, x, \mu, \xi) \\
- \langle \partial_{\mu} V(t, x, \mu, \xi), \partial_p H(\xi, \mu, \partial_x V(t, \xi, \mu)) \rangle + \frac{\beta^2}{2} \tilde{\mathbb{E}}[\partial_{\mu\mu} V(t, x, \mu, \tilde{\xi}, \xi)] \right]\right).\]
Mean field game

- Let r.v. $\xi$ be such that $\mathcal{L}_\xi = \mu$ and let $X^{\xi,\alpha'}$, $X^{\xi,\alpha}$ be

  $$
  X^{\xi,\alpha'}_t = \xi + \int_0^t \alpha'_s ds + B_t + \beta B^0_t,
  $$

  $$
  X^{\xi,\alpha}_t = \xi + \int_0^t \alpha_s ds + B_t + \beta B^0_t.
  $$

- Let $(Y^{\xi;\alpha',\alpha}, Z^{\xi;\alpha',\alpha}, Z^{0,\xi;\alpha',\alpha})$ solve

  $$
  Y^{\xi;\alpha',\alpha}_t = G(X^{\xi,\alpha'}_T, \mathcal{L} X^{\xi,\alpha}_T|B^0) + \int_t^T L(X^{\xi,\alpha'}, \alpha'_s, \mathcal{L} X^{\xi,\alpha}_s|B^0)
  $$

  $$
  - \int_t^T Z^{\xi;\alpha',\alpha}_s dB_s - \int_t^T Z^{0,\xi;\alpha',\alpha}_s dB^0_s.
  $$
Mean field game

- The cost functional

\[ J(\mu; \alpha', \alpha) = \mathbb{E}[Y_0^{\xi, \alpha'}, \alpha]. \]

- The minimization problem

\[ V(\mu; \alpha) = \inf_{\alpha'} J(\mu; \alpha', \alpha). \]

**Definition (Nash equilibrium)**

We say that \((\alpha^*, \mu^*)\) is a Nash equilibrium for the above mean field game if

\[ V(\mu; \alpha^*) = J(\mu; \alpha^*, \alpha^*) \quad \text{and} \quad \mu_t^* = \mathcal{L}_{X_t^{\xi, \alpha^*}|B_0}. \]
Characterization of Nash equilibrium

By the Girsanov Theorem and comparison principle for BSDEs, we have

\[
X_t^{\alpha^*} = \xi + B_t^{\alpha^*} + \beta B_t^0,
\]

\[
Y_t^{\alpha^*} = G(X_T^{\alpha^*}, \mu_T^*) - \int_t^T H(X_s^{\alpha^*}, \mu_s^*, Z_s^{\alpha^*}) ds
- \int_t^T Z_s^{\alpha^*} dB_s^{\alpha^*} - \int_t^T Z_s^0, \alpha^* dB_s^0,
\]

where \( \alpha_t^* = -\partial_p H(X_t^{\alpha^*}, \mu_t^*, Z_t^{\alpha^*}) \) and \( dB_t^{\alpha^*} = \alpha_t^* dt + dB_t. \)
At the Nash equilibrium, we have the following FBSDE system:

\[ X_t^\xi = \xi - \int_0^t \partial_p H(X_s^\xi, \mathcal{L}_{X_s^\xi|B^0}, Z_s^\xi) + B_t + \beta B_0^t, \]

\[ Y_t^\xi = G(X_T^\xi, \mathcal{L}_{X_T^\xi|B^0}) + \int_t^T L(X_s^\xi, \mathcal{L}_{X_s^\xi|B^0}, -\partial_p H(X_s^\xi, \mathcal{L}_{X_s^\xi|B^0}, Z_s^\xi)) \, ds \]

\[- \int_t^T Z_s^\xi \, dB_s - \int_t^T Z_0^\xi \, dB_s^0. \]

\[ Y_0^\xi = V(0, \xi, \mu). \]
The master equation (1) is equivalent to the following forward-backward McKean-Vlasov SDEs

\begin{align*}
X_t^\xi &= \xi - \int_0^t \partial_p H(X_s^\xi, \mathcal{L}_{X_s^\xi|B_0}, Z_s^\xi) + B_t + \beta B_0^t, \\
Y_t^\xi &= G(X_T^\xi, \mathcal{L}_{X_T^\xi|B_0}) + \int_t^T L(X_s^\xi, \mathcal{L}_{X_s^\xi|B_0}, -\partial_p H(X_s^\xi, \mathcal{L}_{X_s^\xi|B_0}, Z_s^\xi)) ds \\
&\quad - \int_t^T Z_s^\xi dB_s - \int_t^T Z_s^{0,\xi} dB_s^0, \\
X_t^{x,\xi} &= x - \int_0^t \partial_p H(X_s^{x,\xi}, \mathcal{L}_{X_s^{x,\xi|B_0}}, Z_s^{x,\xi}) + B_t + \beta B_0^t, \\
Y_t^{x,\xi} &= G(X_T^{x,\xi}, \mathcal{L}_{X_T^{x,\xi|B_0}}) + \int_t^T L(X_s^{x,\xi}, \mathcal{L}_{X_s^{x,\xi|B_0}}, -\partial_p H(X_s^{x,\xi}, \mathcal{L}_{X_s^{x,\xi|B_0}, Z_s^{x,\xi}})) ds \\
&\quad - \int_t^T Z_s^{x,\xi} dB_s - \int_t^T Z_s^{0,x,\xi} dB_s^0,
\end{align*}
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Literature for global wellposedness

- Buckdahn-Li-Peng-Rainer (2017)
  - Linear equation, not MFG, so monotonicity is not required

  - Separable $H$ and Lasry-Lions monotonicity

- M.-Zhang (2019), Bertucci (2021), Cardaliaguet-Souganidis (2021)
  - same as above, weak solutions

  - Potential MFG with displacement monotonicity

- Cecchin-Delarue (2022)
  - Potential MFG without monotonicity, weak solutions
Literature for global wellposedness

  - Finite state MFG with Lasry-Lions monotonicity

  - Finite state MFG without monotonicity

  - MFGC with Lasry-Lions monotonicity
  - MFG system, not master equation
Our works

- Gangbo-Meszaros-M.-Zhang (2021)
  - MFG master equation with non-separable $H$ and displacement monotonicity

- M.-Zhang (2022a)
  - MFG master equation with anti-monotonicity

- M.-Zhang (2022b)
  - MFGC master equation with Lasry-Lions monotonicity, displacement monotonicity, anti-monotonicity
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Road map

- Step 1: Assume that $H$ and $G$ satisfy certain monotonicity condition. Show that the solution $V$ of the master equation propagates the monotonicity condition.
- Step 2: Using the monotonicity condition of $V$ (not the data $H$ and $G$), show that $V$ is Lipschitz continuous in $\mu$ with respect to the metric $W_2/W_1$.
- Step 2': $W_2$-Lipschitz continuity implies $W_1$-Lipschitz continuity.
- Step 3: Use $W_1$-Lipschitz continuity of $V$ to patch local solutions to obtain a global one.
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Lasry-Lions monotonicity

**Definition (Lasry-Lions monotonicity)**

We say that $G : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ is Lasry-Lions monotone if $\forall \xi_1, \xi_2$

$$\mathbb{E}[G(\xi_1, \mathcal{L}\xi_1) + G(\xi_2, \mathcal{L}\xi_2) - G(\xi_1, \mathcal{L}\xi_2) - G(\xi_2, \mathcal{L}\xi_1)] \geq 0.$$  

- If $G$ is smooth, then the Lasry-Lions monotonicity is equivalent to $\forall \xi, \eta$

$$\mathbb{E}[\langle \tilde{\mathbb{E}}[\partial_{x\mu} G(\xi, \mathcal{L}\xi, \tilde{\xi})\tilde{\eta}], \eta \rangle] \geq 0.$$
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Definition (Displacement monotonicity)

We say that \( G : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \rightarrow \mathbb{R} \) is displacement monotone if \( \forall \xi_1, \xi_2 \)

\[
\mathbb{E}\left[ \langle \partial_x G(\xi_1, \mathcal{L}_{\xi_1}) - \partial_x G(\xi_2, \mathcal{L}_{\xi_2}), \xi_1 - \xi_2 \rangle \right] \geq 0.
\]

- If \( G \) is smooth, then the displacement monotonicity is equivalent to \( \forall \xi, \eta \)

\[
\mathbb{E}\left[ \langle \partial_{xx} G(\xi, \mathcal{L}_{\xi}) \eta, \eta \rangle + \langle \tilde{\mathbb{E}}[\partial_{x\mu} G(\xi, \mathcal{L}_{\xi}, \tilde{\xi})\tilde{\eta}], \eta \rangle \right] \geq 0.
\]
We find the displacement monotonicity assumption for non-separable $H$ to guarantee the uniqueness of MFG system and thus we are able to derive the global well-posedness of the master equation.

**Definition (Displacement monotonicity on $H$)**

We say that $H : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$ is displacement monotone if $\forall \xi, \eta$ and $\forall \varphi \in C^1(\mathbb{R}^d; \mathbb{R}^d)$

$$\operatorname{displ}_\xi H(\eta, \eta) := \mathbb{E} \left[ \langle \partial_{xx} H(\xi, L_\xi, \varphi(\xi)) \eta, \eta \rangle + \langle \tilde{\mathbb{E}}[\partial_{\mu x} H(\xi, L_\xi, \tilde{\xi}, \varphi(\xi))\tilde{\eta}], \eta \rangle \right] + \frac{1}{4} \left| (\partial_{pp} H(\xi, \mu, \varphi(\xi))) - \frac{1}{2} \tilde{\mathbb{E}}[\partial_{\mu \mu} H(\xi, \mu, \tilde{\xi}, \varphi(\xi))\tilde{\eta}] \right|^2 \leq 0.$$
Displacement monotonicity

- It remains a challenge to extend the Lasry-Lions monotonicity assumption for non-separable Hamiltonian $H$ to guarantee the uniqueness of the mean field game.
Displacement monotonicity

Propagation of monotonicity

- Consider

\[
X_t = \xi - \int_0^T \partial_p H(X_s, \mathcal{L}X_s, \partial_x V(s, X_s, \mathcal{L}X_s)) ds + B_t + \beta B_0^t;
\]

\[
\delta X_t = \eta - \int_0^t \partial_{px} H(X_s) + \tilde{\mathbb{E}} \mathcal{F}_t [\partial_{p\mu} H(X_s, \tilde{X}_s) \delta \tilde{X}_t]
+ \partial_{pp} H(X_s) [\tilde{\mathbb{E}} \mathcal{F}_s [\partial_{x\mu} V(X_s, \tilde{X}_s) \delta \tilde{X}_s] + \partial_{xx} V(X_s) \delta X_s] ds.
\]

- Define

\[
D_t := J_t^1 + J_t^2 := \mathbb{E} [\langle l_t, \delta X_t \rangle] + \langle \tilde{l}_t, \delta X_t \rangle
\]

where

\[
l_t = \tilde{\mathbb{E}} \mathcal{F}_t [\partial_{x\mu} V(X_t, \tilde{X}_t) \delta \tilde{X}_t] \quad \text{and} \quad \tilde{l}_t := \partial_{xx} V(X_t) \delta X_t.
\]
Displacement monotonicity

Propagation of monotonicity

\[ \dot{J}_t = \mathbb{E} \left[ -\langle \partial_{pp} H(X_t) I_t, I_t \rangle - \langle \tilde{E}_F t [\partial_{p \mu} H(X_t, \tilde{X}_t) \delta \tilde{X}_t], \bar{I}_t - I_t \rangle 
+ \langle \tilde{E}_F t [\partial_{x \mu} H(X_t, \tilde{X}_t) \delta \tilde{X}_t], \delta X_t \rangle \right]. \]

and

\[ \dot{D}_t = \mathbb{E} \left[ -|\partial_{pp} H(X_t)|^{\frac{1}{2}} [I_t + \bar{I}_t]|^2 - \langle \tilde{E}_F t [\partial_{p \mu} H(X_t, \tilde{X}_t) \delta \tilde{X}_t], I_t + \bar{I}_t \rangle 
+ \langle \tilde{E}_F t [\partial_{x \mu} H(X_t, \tilde{X}_t) \delta \tilde{X}_t] + \partial_{xx} H(X_t) \delta X_t, \delta X_t \rangle \right] 
= \mathbb{E} \left[ -|\partial_{pp} H(X_t)|^{\frac{1}{2}} [I_t + \bar{I}_t] + \frac{1}{2} \partial_{pp} H(X_t) - \frac{1}{2} \tilde{E}_F t [\partial_{p \mu} H(X_t, \tilde{X}_t) \delta \tilde{X}_t]|^2 
+ \text{displ}_{X_t} V(t, \cdot, \rho_t)(\delta X_t, \delta X_t) \right]. \]
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Definition (Anti-monotonicity)

We say that $G : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ is anti-monotone if, for some appropriate constants $c_1, c_2 > 0$,

$$
\mathbb{E}\left[\langle \partial_{xx} G(\xi, L\xi)\eta, \eta \rangle + \langle \tilde{\mathbb{E}}[\partial_{x\mu} G(\xi, L\xi, \tilde{\xi})\tilde{\eta}], \eta \rangle \right] \\
\leq \; -\mathbb{E}\left[ c_1 \| \partial_{xx} G(\xi, L\xi)\eta \|^2 - c_2 \| \tilde{\mathbb{E}}[\partial_{x\mu} G(\xi, L\xi, \tilde{\xi})\tilde{\eta}] \|^2 \right] \; \forall \xi, \eta.
$$

- See a more general condition in the paper.
Denote

\[ A_t := \mathbb{E}\left[ c_1 \| \partial_{xx} G(\xi, \mathcal{L}_\xi) \eta \|^2 + c_2 \| \tilde{E}_F^T [\partial_x \mu G(\xi, \mathcal{L}_\xi, \tilde{\xi}) \tilde{\eta}] \|^2 \right]. \]

Assume that \( G \) is anti-monotone, i.e. \( D_T + A_T \leq 0 \). Then, under certain condition on \( H \), we are able to show that

\[ \dot{D}_t + \dot{A}_t \geq 0, \quad \forall t \]

which implies that

\[ D_t + A_t \leq 0, \quad \forall t. \]
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MFGC

The dynamic at the Nash equilibrium:

\[ X_t = \xi + \int_0^t \alpha_s^* ds + B_t + \beta B^0_t; \quad \nu_t := \mathcal{L}(X_t, \alpha_t^*)|_{B^0}; \quad \mu_t := \mathcal{L}X_t|_{B^0}. \]

The control at the Nash equilibrium:

\[ \alpha_t^* = -\partial_p H(X_t, \nu_t, Z_t), \quad Z_t := \partial_x V(t, X_t, \mu_t) \]

The fixed point:

\[ \nu_t = \mathcal{L}(X_t, \alpha_t^*) = \mathcal{L}(X_t, -\partial_p H(X_t, \nu_t, Z_t)) \Rightarrow \nu_t = \psi(\mathcal{L}(X_t, Z_t)). \]
The MFGC master equation

\[
\begin{aligned}
\partial_t V(t, x, \mu) + \frac{1+\beta^2}{2} \Delta V(t, x, \mu) - \hat{H}(x, \mathcal{L}(\xi, \partial_x V(t, \xi, \mu)), \partial_x V(t, x, \mu)) + \hat{N} V &= 0, \text{ in } (0, T) \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d), \\
V(T, x, \mu) &= G(x, \mu),
\end{aligned}
\]

where \( \hat{H}(x, \mathcal{L}(\xi, \eta), p) := H(x, \psi(\mathcal{L}(\xi, \eta)), p) \) and

\[
\hat{N} V := \text{tr} \left( \mathbb{E} \left[ \frac{1+\beta^2}{2} \partial_{\tilde{x}\mu} V(t, x, \mu, \xi) + \beta^2 \partial_{x\mu} V(t, x, \mu, \xi) \\
- \langle \partial_\mu V(t, x, \mu, \xi), \partial_p \hat{H}(\xi, \mathcal{L}(\xi, \partial_x V(t, \xi, \mu)), \partial_x V(t, \xi, \mu)) \rangle \\
+ \frac{\beta^2}{2} \mathbb{E} [\partial_{\mu\mu} V(t, x, \mu, \tilde{\xi}, \xi)] \right] \right).
\]
The condition for $\hat{H}$ to propagate the displacement monotonicity:

$$
\mathbb{E} \left[ \langle \partial_{xx} \hat{H}(\xi) \eta, \eta \rangle \right] + \mathbb{E} \left[ \langle \partial_{x\nu_1} \hat{H}(\xi, \tilde{\xi}) \tilde{\eta} \rangle, \eta \right] \\
+ \frac{1}{4} \left| \partial_{pp} \hat{H}(\xi) - \| \partial_{x\nu_2} \hat{H}(\xi, \cdot) \|_{\infty} \right|^{-\frac{1}{2}} \mathbb{E} \left[ \left( \partial_{p\nu_1} \hat{H}(\xi, \tilde{\xi}) + \partial_{p\nu_2} \hat{H}(\xi, \tilde{\xi}) \right) \tilde{\eta} \right]^2 \leq 0.
$$

Assume above and $G$ is displacement monotone, then $V(t, \cdot, \cdot)$ is displacement monotone for all $t$.

Similarly we can derive conditions for Lasry-Lions monotonicity and anti-monotonicity.
Thank you for your attention!