Propagation of monotoncity for mean field games

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The 9th colloquium on Backward Stochastic Differential Equations and Mean Field Systems

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Outline

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Master equation with common noise

- The master equation with common noise

$$\begin{cases} \partial_t V(t,x,\mu) + \frac{1+\beta^2}{2}\Delta V(t,x,\mu) - H(x,\mu,\partial_x V(t,x,\mu)) \\ +\mathcal{N}V = 0, \quad \text{in } (0,T)\times\mathbb{R}^d\times\mathcal{P}(\mathbb{R}^d), \\ V(T,x,\mu) = G(x,\mu), \end{cases} \tag{1}$$

where $H : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$, $G : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ and

$$\mathcal{N}V := \mathrm{tr}\Big(\mathbb{E}\Big[\frac{1+\beta^2}{2}\partial_{\tilde{x}\mu}V(t,x,\mu,\xi) + \beta^2\partial_{x\mu}V(t,x,\mu,\xi)$$

$$-\langle\partial_\mu V(t,x,\mu,\xi), \partial_p H(\xi,\mu,\partial_x V(t,\xi,\mu))\rangle + \frac{\beta^2}{2}\tilde{\mathbb{E}}[\partial_{\mu\mu}V(t,x,\mu,\tilde{\xi},\xi)]\Big]\Big).$$

Mean field game

- Let r.v. ξ be such that $\mathcal{L}_\xi = \mu$ and let $X^{\xi,\alpha'}$, $X^{\xi,\alpha}$ be

$$X_t^{\xi,\alpha'} = \xi + \int_0^t \alpha'_s ds + B_t + \beta B_t^0,$$

$$X_t^{\xi,\alpha} = \xi + \int_0^t \alpha_s ds + B_t + \beta B_t^0.$$

- Let $(Y^{\xi;\alpha',\alpha}, Z^{\xi;\alpha',\alpha}, Z^{0,\xi;\alpha',\alpha})$ solve

$$\begin{aligned} Y_t^{\xi;\alpha',\alpha} &= G(X_T^{\xi,\alpha'}, \mathcal{L}_{X_T^{\xi,\alpha}|B^0}) + \int_t^T L(X_s^{\xi,\alpha'}, \alpha'_s, \mathcal{L}_{X_s^{\xi,\alpha}|B^0}) \\ &\quad - \int_t^T Z_s^{\xi;\alpha',\alpha} dB_s - \int_t^T Z_s^{0,\xi;\alpha',\alpha} dB_s^0. \end{aligned}$$

Mean field game

- The cost functional

$$J(\mu; \alpha', \alpha) = \mathbb{E}[Y_0^{\xi;\alpha',\alpha}].$$

- The minimization problem

$$V(\mu; \alpha) = \inf_{\alpha'} J(\mu; \alpha', \alpha).$$

Definition (Nash equilibrium)

We say that (α^*, μ^*) is a Nash equilibrium for the above mean field game if

$$V(\mu; \alpha^*) = J(\mu; \alpha^*, \alpha^*) \quad \text{and} \quad \mu_t^* = \mathcal{L}_{X_t^{\xi,\alpha^*}|B^0}.$$

Characterization of Nash eqilibrium

- By the Girsanov Theorem and comparison principle for BSDEs, we have

$$\begin{aligned} X_t^{\alpha^*} &= \xi + B_t^{\alpha^*} + \beta B_t^0, \\ Y_t^{\alpha^*} &= G(X_T^{\alpha^*}, \mu_T^*) - \int_t^T H(X_s^{\alpha^*}, \mu_s^*, Z_s^{\alpha^*}) ds \\ &\quad - \int_t^T Z_s^{\alpha^*} dB_s^{\alpha^*} - \int_t^T Z_s^{0,\alpha^*} dB_s^0, \end{aligned} \tag{2}$$

where $\alpha_t^* = -\partial_p H(X_t^{\alpha^*}, \mu_t^*, Z_t^{\alpha^*})$ and $dB_t^{\alpha^*} = \alpha_t^* dt + dB_t$.

Characterization of the Nash eqilibrium

- At the Nash equilibrium, we have the following FBSDE system

$$
\begin{aligned}
X_t^\xi &= \xi - \int_0^t \partial_p H(X_s^\xi, \mathcal{L}_{X_s^\xi|B^0}, Z_s^\xi) + B_t + \beta B_t^0, \\
Y_t^\xi &= G(X_T^\xi, \mathcal{L}_{X_T^\xi|B^0}) + \int_t^T L(X_s^\xi, \mathcal{L}_{X_s^\xi|B^0}, -\partial_p H(X_s^\xi, \mathcal{L}_{X_s^\xi|B^0}, Z_s^\xi))ds \\
&- \int_t^T Z_s^\xi dB_s - \int_t^T Z_s^{0,\xi} dB_s^0.
\end{aligned}
$$

- $Y_0^\xi = V(0, \xi, \mu)$.

FBSDEs

- The master equation (1) is equivalent to the following forward-backward McKean-Vlasov SDEs

$$
\begin{aligned}
X_t^{\xi} &= \xi - \int_0^t \partial_p H(X_s^{\xi}, \mathcal{L}_{X_s^{\xi}|B^0}, Z_s^{\xi}) + B_t + \beta B_t^0, \\
Y_t^{\xi} &= G(X_T^{\xi}, \mathcal{L}_{X_T^{\xi}|B^0}) + \int_t^T L(X_s^{\xi}, \mathcal{L}_{X_s^{\xi}|B^0}, -\partial_p H(X_s^{\xi}, \mathcal{L}_{X_s^{\xi}|B^0}, Z_s^{\xi}))ds \\
&- \int_t^T Z_s^{\xi} dB_s - \int_t^T Z_s^{0,\xi} dB_s^0, \\
X_t^{x,\xi} &= x - \int_0^t \partial_p H(X_s^{x,\xi}, \mathcal{L}_{X_s^{\xi}|B^0}, Z_s^{x,\xi}) + B_t + \beta B_t^0, \\
Y_t^{x,\xi} &= G(X_T^{x,\xi}, \mathcal{L}_{X_T^{\xi}|B^0}) + \int_t^T L(X_s^{x,\xi}, \mathcal{L}_{X_s^{\xi}|B^0}, -\partial_p H(X_s^{x,\xi}, \mathcal{L}_{X_s^{\xi}|B^0}, Z_s^{x,\xi}))ds \\
&- \int_t^T Z_s^{x,\xi} dB_s - \int_t^T Z_s^{0,x,\xi} dB_s^0,
\end{aligned}
$$

Outline

Literature for global wellposedness

- Buckdahn-Li-Peng-Rainer (2017)
  - ◇ Linear equation, not MFG, so monotonicity is not required
- Chassagneux-Crisan-Delarue (2014), Carmona-Delarue (2018), Cardaliaguet-Delarue-Lasry-Lions (2019)
  - ◇ Separable H and Lasry-Lions monotonicity
- M.-Zhang (2019), Bertucci (2021), Cardaliaguet-Souganidis (2021)
  - ◇ same as above, weak solutions
- Gangbo-Meszaros (2020), Bensoussan-Graber-Yam (2020)
  - ◇ Potential MFG with displacement monotonicity
- Cecchin-Delarue (2022)
  - ◇ Potential MFG without monotonicity, weak solutions

Literature for global wellposedness

- Bayraktar-Cohen (2018), Bertucci-Lasry-Lions (2019), Bertucci (2020), Bertucci-Cecchin (2022)
  - ◇ Finite state MFG with Lasry-Lions monotonicity
- Bayraktar-Cecchin-Cohen-Delarue (2019), Cecchin-Delarue (2020)
  - ◇ Finite state MFG without monotonicity
- Gomes-Voskanyan (2013), Carmona-Lacker (2015), Carmona-Delarue (2018), Cardaliaguet-Lehalle (2018), Kobeissi (2020)
  - ◇ MFGC with Lasry-Lions monotonicity
  - ◇ MFG system, not master equation

Our works

- Gangbo-Meszaros-M.-Zhang (2021)
  ⋄ MFG master equation with non-separable H and displacement monotonicity
- M.-Zhang (2022a)
  ⋄ MFG master equation with anti-monotonicity
- M.-Zhang (2022b)
  ⋄ MFGC master equation with Lasry-Lions monotonicity, displacement monotonicity, anti-monotonicity

Outline

1. Master equation with common noise
2. Known results
3. Road map
4. Propagation of monotonicity for MFG
   - Lasry-Lions monotonicity
   - Displacement monotonicity
   - Anti-monotonicity
5. Propagation of monotonicity for MFGC

Road map

- Step 1: Assume that H and G satisfy certain monotonicity condition. Show that the solution V of the master equation propagates the monotonicity condition.
- Step 2: Using the monotonicity condition of V (not the data H and G), show that V is Lipschitz continuous in μ with respect to the metric W_2/W_1.
- Step 2': W_2-Lipschitz continuity implies W_1-Lipschitz continuity.
- Step 3: Use W_1-Lipschitz continuity of V to patch local solutions to obtain a global one.

Outline

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Outline

Lasry-Lions monotonicity

Definition (Lasry-Lions monotonicity)

We say that $G : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ is Lasry-Lions monotone if $\forall \xi_1, \xi_2$

$$\mathbb{E}[G(\xi_1, \mathcal{L}_{\xi_1}) + G(\xi_2, \mathcal{L}_{\xi_2}) - G(\xi_1, \mathcal{L}_{\xi_2}) - G(\xi_2, \mathcal{L}_{\xi_1})] \geq 0.$$

- If G is smooth, then the Lasry-Lions monotonicity is equivalent to $\forall \xi, \eta$

$$\mathbb{E}\big[\langle \tilde{\mathbb{E}}[\partial_{x\mu} G(\xi, \mathcal{L}_\xi, \tilde{\xi})\tilde{\eta}], \eta\rangle\big] \geq 0.$$

Outline

Displacement monotonicity

Definition (Displacement monotonicity)

We say that $G : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ is displacement monotone if $\forall \xi_1, \xi_2$

$$\mathbb{E}[\langle \partial_x G(\xi_1, \mathcal{L}_{\xi_1}) - \partial_x G(\xi_2, \mathcal{L}_{\xi_2}), \xi_1 - \xi_2 \rangle] \geq 0.$$

- If G is smooth, then the displacement monotonicity is equivalent to $\forall \xi, \eta$

$$\mathbb{E}\big[\langle \partial_{xx} G(\xi, \mathcal{L}_\xi)\eta, \eta\rangle + \langle \tilde{\mathbb{E}}[\partial_{x\mu} G(\xi, \mathcal{L}_\xi, \tilde{\xi})\tilde{\eta}], \eta\rangle\big] \geq 0.$$

Displacement monotonicity

- We find the displacement monotonicity assumption for non-separable H to guarantee the uniqueness of MFG system and thus we are able to derive the global well-posedness of the master equation.

Definition (Displacement monotonicity on H)

We say that $H : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$ is displacement monotone if $\forall \xi, \eta$ and $\forall \varphi \in C^1(\mathbb{R}^d; \mathbb{R}^d)$

$$\begin{aligned}\mathrm{displ}_\xi^\varphi H(\eta,\eta) := \mathbb{E}\Big[&\langle \partial_{xx} H(\xi, \mathcal{L}_\xi, \varphi(\xi))\eta, \eta\rangle] + \langle \tilde{\mathbb{E}}[\partial_{x\mu} H(\xi, \mathcal{L}_\xi, \tilde{\xi}, \varphi(\xi))\tilde{\eta}], \eta\rangle \\ &+ \frac{1}{4}\big|(\partial_{pp} H(\xi, \mu, \varphi(\xi)))^{-\frac{1}{2}} \tilde{\mathbb{E}}[\partial_{p\mu} H(\xi, \mu, \tilde{\xi}, \varphi(\xi))\tilde{\eta}]\big|^2\Big] \le 0.\end{aligned}$$

Displacement monotonicity

- It remains a challenge to extend the Lasry-Lions monotonicity assumption for non-separable Hamiltonian H to guarantee the uniqueness of the mean field game.

Propagation of monotonicity

- Consider

$$
\begin{aligned}
X_t &= \xi - \int_0^T \partial_p H(X_s, \mathcal{L}_{X_s}, \partial_x V(s, X_s, \mathcal{L}_{X_s}))ds + B_t + \beta B_t^0; \\
\delta X_t &= \eta - \int_0^t \partial_{px} H(X_s) + \tilde{\mathbb{E}}_{\mathcal{F}_t}[\partial_{p\mu} H(X_s, \tilde{X}_s)\delta \tilde{X}_t] \\
&\quad + \partial_{pp} H(X_s)[\tilde{\mathbb{E}}_{\mathcal{F}_s}[\partial_{x\mu} V(X_s, \tilde{X}_s)\delta \tilde{X}_s] + \partial_{xx} V(X_s)\delta X_s] ds.
\end{aligned}
$$

- Define

$$
D_t := J_t^1 + J_t^2 := \mathbb{E}[\langle I_t, \delta X_t \rangle] + \langle \bar{I}_t, \delta X_t \rangle]
$$

where

$$
I_t = \tilde{\mathbb{E}}_{\mathcal{F}_t}[\partial_{x\mu} V(X_t, \tilde{X}_t)\delta \tilde{X}_t] \quad \text{and} \quad \bar{I}_t := \partial_{xx} V(X_t)\delta X_t.
$$

Propagation of monotonicity

- $$
\begin{aligned}
\dot{J}_t^1 &= \mathbb{E}\Big[-\langle \partial_{pp} H(X_t) I_t, I_t \rangle - \langle \tilde{\mathbb{E}}_{\mathcal{F}_t}[\partial_{p\mu} H(X_t, \tilde{X}_t)\delta\tilde{X}_t], \bar{I}_t - I_t \rangle \\
&\quad + \langle \tilde{\mathbb{E}}_{\mathcal{F}_t}[\partial_{x\mu} H(X_t, \tilde{X}_t)\delta\tilde{X}_t], \delta X_t \rangle \Big].
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{D}_t &= \mathbb{E}\Big[-\big|\partial_{pp} H(X_t)^{\frac{1}{2}}[I_t + \bar{I}_t]\big|^2 - \langle \tilde{\mathbb{E}}_{\mathcal{F}_t}[\partial_{p\mu} H(X_t, \tilde{X}_t)\delta\tilde{X}_t], I_t + \bar{I}_t \rangle \\
&\quad + \langle \tilde{\mathbb{E}}_{\mathcal{F}_t}[\partial_{x\mu} H(X_t, \tilde{X}_t)\delta\tilde{X}_t] + \partial_{xx} H(X_t)\delta X_t, \delta X_t \rangle \Big] \\
&= \mathbb{E}\Big[-\big|\partial_{pp} H(X_t)^{\frac{1}{2}}[I_t + \bar{I}_t] + \frac{1}{2}\partial_{pp} H(X_t)^{-\frac{1}{2}}\tilde{\mathbb{E}}_{\mathcal{F}_t}[\partial_{p\mu} H(X_t, \tilde{X}_t)\delta\tilde{X}_t]\big|^2 \\
&\quad + \mathrm{displ}_{X_t}^{\partial_x V(t,\cdot,\rho_t)}(\delta X_t, \delta X_t) \Big].
\end{aligned}
$$

Outline

Anti-monotonicity

Definition (Anti-monotonicity)

We say that $G : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ is anti-monotone if, for some appropriate constants $c_1, c_2 > 0$,

$$\begin{aligned} & \mathbb{E}\Big[\langle \partial_{xx} G(\xi, \mathcal{L}_\xi)\eta, \eta\rangle + \langle \tilde{\mathbb{E}}[\partial_{x\mu} G(\xi, \mathcal{L}_\xi, \tilde{\xi})\tilde{\eta}], \eta\rangle\Big] \\ \leq \; & -\mathbb{E}\Big[c_1 \|\partial_{xx} G(\xi, \mathcal{L}_\xi)\eta\|^2 - c_2 \|\tilde{\mathbb{E}}[\partial_{x\mu} G(\xi, \mathcal{L}_\xi, \tilde{\xi})\tilde{\eta}]\|^2\Big] \quad \forall \xi, \eta. \end{aligned}$$

- See a more general condition in the paper.

Propagation of anti-monotonicity

- Denote

$$A_t := \mathbb{E}\Big[c_1\|\partial_{xx}G(\xi,\mathcal{L}_\xi)\eta\|^2 + c_2\|\tilde{\mathbb{E}}_{\mathcal{F}_T}[\partial_{x\mu}G(\xi,\mathcal{L}_\xi,\tilde{\xi})\tilde{\eta}]\|^2\Big].$$

- Assume that G is anti-monotone, i.e. $D_T + A_T \le 0$. Then, under certain condition on H, we are able to show that

$$\dot{D}_t + \dot{A}_t \ge 0, \quad \forall t$$

which implies that

$$D_t + A_t \le 0, \quad \forall t.$$

Outline

MFGC

- The dynamic at the Nash equilibrium:

$$X_t = \xi + \int_0^t \alpha_s^* ds + B_t + \beta B_t^0; \quad \nu_t := \mathcal{L}_{(X_t, \alpha_t^*)|B^0}; \quad \mu_t := \mathcal{L}_{X_t|B^0}.$$

- The control at the Nash equilibrium:

$$\alpha_t^* = -\partial_p H(X_t, \nu_t, Z_t), \quad Z_t := \partial_x V(t, X_t, \mu_t)$$

- The fixed point :

$$\nu_t = \mathcal{L}_{(X_t, \alpha_t^*)} = \mathcal{L}_{(X_t, -\partial_p H(X_t, \nu_t, Z_t))} \Rightarrow \nu_t = \psi(\mathcal{L}_{(X_t, Z_t)}).$$

MFGC Master equations

- The MFGC master equation

$$\begin{cases} \partial_t V(t,x,\mu) + \frac{1+\beta^2}{2}\Delta V(t,x,\mu) - \hat{H}(x, \mathcal{L}_{(\xi,\partial_x V(t,\xi,\mu))}, \partial_x V(t,x,\mu)) \\ +\hat{\mathcal{N}} V = 0, \quad \text{in } (0,T)\times\mathbb{R}^d\times\mathcal{P}(\mathbb{R}^d), \\ V(T,x,\mu) = G(x,\mu), \end{cases} \tag{5}$$

where $\hat{H}(x, \mathcal{L}_{(\xi,\eta)}, p) := H(x, \psi(\mathcal{L}_{(\xi,\eta)}), p)$ and

$$\begin{aligned} \hat{\mathcal{N}} V := \mathrm{tr}\Big(\mathbb{E}\Big[\frac{1+\beta^2}{2}\partial_{\tilde{x}\mu} V(t,x,\mu,\xi) + \beta^2 \partial_{x\mu} V(t,x,\mu,\xi) \\ -\langle \partial_\mu V(t,x,\mu,\xi), \partial_p \hat{H}(\xi, \mathcal{L}_{(\xi,\partial_x V(t,\xi,\mu))}, \partial_x V(t,\xi,\mu))\rangle \\ +\frac{\beta^2}{2}\tilde{\mathbb{E}}[\partial_{\mu\mu} V(t,x,\mu,\tilde{\xi},\xi)]\Big]\Big). \end{aligned}$$

Propagation of monotonicity

- The condition for $\hat{H}$ to propagate the displacement monotonicity:

$$\mathbb{E}\Big[\langle \partial_{xx}\hat{H}(\xi)\eta,\eta\rangle] + \langle \tilde{\mathbb{E}}[\partial_{x\nu_1}\hat{H}(\xi,\tilde{\xi})\tilde{\eta}],\eta\rangle$$

$$+\frac{1}{4}\Big|(\partial_{pp}\hat{H}(\xi) - \|\partial_{x\nu_2}\hat{H}(\xi,\cdot)\|_{\infty})^{-\frac{1}{2}}\tilde{\mathbb{E}}[(\partial_{p\nu_1}\hat{H}(\xi,\tilde{\xi}) + \partial_{p\nu_2}\hat{H}(\xi,\tilde{\xi}))\tilde{\eta}]\Big|^2\Big] \le 0.$$

- Assume above and G is displacement monotone, then $V(t,\cdot,\cdot)$ is displacement monotone for all t.
- Similarly we can derive conditions for Lasry-Lions monotonicity and anti-monotoncity.

Thank you for your attention!