JYVÄSKYLÄN YLIOPISTO
UNIVERSITY OF JYVÄSKYLÄ

Convergence Rates of Random Walk Approximations of Forward-Backward SDEs

Christel Geiss (University of Jyväskylä)

joint work with
Philippe Briand, Céline Labart (Université Savoie Mont-Blanc)
Stefan Geiss (University of Jyväskylä)

BSDE 2022

Annecy June 27 - July 1

Forward Backward Stochastic Differential Equations (FBSDEs)

$$X_t = x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s, \quad 0 \le t \le T$$

$$Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dB_s$$

Replacing $(B_t)_{t\in[0,T]}$ by a random walk $(B_t^n)_{t\in[0,T]}$ what kind of convergence one can expect:

$$((Y_t^n)_{t\in[0,T]}, (Z_t^n)_{t\in[0,T]}) \to ((Y_t)_{t\in[0,T]}, (Z_t)_{t\in[0,T]})?$$

Forward Backward Stochastic Differential Equations (FBSDEs)

$$X_t = x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s, \quad 0 \le t \le T$$

$$Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dB_s$$

Replacing $(B_t)_{t\in[0,T]}$ by a random walk $(B_t^n)_{t\in[0,T]}$ what kind of convergence one can expect:

$$((Y_t^n)_{t\in[0,T]}, (Z_t^n)_{t\in[0,T]}) \to ((Y_t)_{t\in[0,T]}, (Z_t)_{t\in[0,T]})?$$

Briand, Delyon and Mémin (2001)

If b, σ, f and g are Lipschitz and $(B_t^n)_{t\in[0,T]}$ such that

$\sup_{0\le t\le T} |B_t^n - B_t| \to 0, \quad n \to \infty,$ in probability, then

$$\sup_{0\le t\le T} |Y_t^n - Y_t|^2 + \int_0^T |Z_s^n - Z_s|^2 ds \to 0 \text{ when } n \to \infty \text{ in probability.}$$

Random walk approximation of the Brownian motion

Let $t_k := kh$, $k = 0, ..., n$ be a regular grid of $[0, T]$, where $h = \frac{T}{n}$ and define

$$B_t^n := \sqrt{h} \sum_{j=1}^{[t/h]} \varepsilon_j, \quad (\varepsilon_j)_{j=1}^n \text{ i.i.d. Rademacher r.v.: } \mathbb{P}(\varepsilon_j = \pm 1) = \frac{1}{2}$$

$$[B^n]_t = h \sum_{j=1}^{n} j \mathbf{1}_{(t_{j-1}, t_j]}(t) \qquad \text{quadratic variation.}$$

Random walk approximation of the Brownian motion

Let $t_k := kh$, $k = 0, ..., n$ be a regular grid of $[0, T]$, where $h = \frac{T}{n}$ and define

$$B_t^n := \sqrt{h} \sum_{j=1}^{[t/h]} \varepsilon_j, \quad (\varepsilon_j)_{j=1}^n \text{ i.i.d. Rademacher r.v.: } \mathbb{P}(\varepsilon_j = \pm 1) = \frac{1}{2}$$

$$[B^n]_t = h \sum_{j=1}^{n} j \mathbf{1}_{(t_{j-1}, t_j]}(t) \qquad \text{quadratic variation.}$$

- Donsker's Theorem:
  Convergence of the processes in the Skorokhod space $D[0, T]$:

$$(B_t^n)_{t \in [0,T]} \to (B_t)_{t \in [0,T]} \quad \text{in distribution.}$$

Random walk approximation of the Brownian motion

Let $t_k := kh$, $k = 0, ..., n$ be a regular grid of $[0, T]$, where $h = \frac{T}{n}$ and define

$$B_t^n := \sqrt{h} \sum_{j=1}^{[t/h]} \varepsilon_j, \quad (\varepsilon_j)_{j=1}^n \text{ i.i.d. Rademacher r.v.: } \mathbb{P}(\varepsilon_j = \pm 1) = \frac{1}{2}$$

$$[B^n]_t = h \sum_{j=1}^{n} j \mathbf{1}_{(t_{j-1}, t_j]}(t) \qquad \text{quadratic variation.}$$

- Donsker's Theorem:
  Convergence of the processes in the Skorokhod space $D[0, T]$:

$$(B_t^n)_{t \in [0,T]} \to (B_t)_{t \in [0,T]} \quad \text{in distribution.}$$

- We get the FBSΔE (Forward-backward stochastic difference equation)

$$X_t^n = x + \int_{(0,t]} b(s, X_{s-}^n) d[B^n]_s + \int_{(0,t]} \sigma(s, X_{s-}^n) dB_s^n,$$

$$Y_t^n = g(X_T^n) + \int_{(t,T]} f(s, X_{s-}^n Y_{s-}^n, Z_s^n) d[B^n]_s - \int_{(t,T]} Z_s^n dB_s^n, \quad 0 \le t \le T.$$

BSDEs and BSΔEs

$$Y_t = g(B_{s_1}, .., B_{s_K}) + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s$$

B d-dimensional $d \geq 2$, $\implies$ (B_t^n) does not possess the representation property.
(M_t^n) is a martingale orthogonal to (B_t^n) :

$$Y_t^n = g(B_{s_1}^n, .., B_{s_K}^n) + \int_{(t,T]} f(s, Y_{s^-}^n, Z_s^n)d[B^n]_s - \int_{(t,T]} Z_s^n dB_s^n - (M_T^n - M_t^n)$$

Cheridito and Stadje (2013): BSDEs and BSΔEs

f sub-quadratic growth in z, Lipschitz in y, g bounded and Lipschitz and $(B_t^n)_{t\in[0,T]}$ such that

$\mathbb{E}[\sup_{0\leq t\leq T} |B_t^n - B_t|^2] \to 0, \quad n \to \infty$, then

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left(|Y_t^n - Y_t| + \left|\int_0^t Z_s^n dB_s^n - \int_0^t Z_s dB_s\right| + |M_t^n|\right)^2\right] \to 0 \text{ when } n \to \infty.$$

Other results

Random walk schemes: **convergence in probability or weak convergence**:

- **Nakayama** (2002) (multidimensional), **Toldo** (2005) (with random terminal time), Numerical schemes: **Ma, Protter, San Martín** and **Torres** (2002) (path-dependent terminal condition), **Peng, Xu** (2008) (Implicit and explicit schemes for BSDEs) **Mémin, Peng** and **Xu** (2008), **Martinez, San Martín** and **Torres** (2011) (reflected BSDEs), **Jańczak** (2008, 2009) (generalized reflected BSDEs with random terminal time),...

Other results

Random walk schemes: **convergence in probability or weak convergence**:

- **Nakayama** (2002) (multidimensional), **Toldo** (2005) (with random terminal time), Numerical schemes: **Ma, Protter, San Martín** and **Torres** (2002) (path-dependent terminal condition), **Peng, Xu** (2008) (Implicit and explicit schemes for BSDEs) **Mémin, Peng** and **Xu** (2008), **Martinez, San Martín** and **Torres** (2011) (reflected BSDEs), **Jańczak** (2008, 2009) (generalized reflected BSDEs with random terminal time),...

Time discretization schemes **with L_2 or L_p-rate ($p \geq 2$)**

- **Zhang** (2004) **Bouchard & Touzi** (2004) ,..., **Richou** (2011), **Lionnet & dos Reis & Szpruch** (2016), **S. Geiss & Ylinen** (2018) (regularity of Y), **Han & Jentzen** (2017), **Chassagneux, Richou** (2019), **Sun et al.** (2022),...

Other results

Random walk schemes: **convergence in probability or weak convergence**:

- **Nakayama** (2002) (multidimensional), **Toldo** (2005) (with random terminal time), Numerical schemes: **Ma, Protter, San Martín** and **Torres** (2002) (path-dependent terminal condition), **Peng, Xu** (2008) (Implicit and explicit schemes for BSDEs) **Mémin, Peng** and **Xu** (2008), **Martinez, San Martín** and **Torres** (2011) (reflected BSDEs), **Jańczak** (2008, 2009) (generalized reflected BSDEs with random terminal time),...

Time discretization schemes **with L_2 or L_p-rate ($p \geq 2$)**

- **Zhang** (2004) **Bouchard & Touzi** (2004) ,..., **Richou** (2011), **Lionnet & dos Reis & Szpruch** (2016), **S. Geiss & Ylinen** (2018) (regularity of Y), **Han & Jentzen** (2017), **Chassagneux, Richou** (2019), **Sun et al.** (2022),...

Random walk schemes: **with L_2-rate**

- **C.G., Labart, Luoto** (2020, 2021)
  f Lipschitz, g ε-Hölder continuous:

$$\sup_{0\leq t<T} (\mathbb{E}|Y_t - Y_t^n|^2)^{\frac{1}{2}} \leq Cn^{-\frac{\varepsilon}{4}}, \left(\mathbb{E}\int_0^T |Z_t - Z_t^n|^2 dt\right)^{\frac{1}{2}} \leq C_2 n^{-\beta} \text{ for } \beta \in (0, \tfrac{\varepsilon}{4})$$

Why only $n^{-\frac{\varepsilon}{4}}$?

- **C.G., Labart, Luoto** (2020, 2021)
  f Lipschitz, g ε-Hölder continuous:

$$\sup_{0\le t<T}(\mathbb{E}|Y_t-Y_t^n|^2)^{\frac{1}{2}} \le Cn^{-\frac{\varepsilon}{4}}, \left(\mathbb{E}\int_0^T |Z_t-Z_t^n|^2 dt\right)^{\frac{1}{2}} \le C_2 n^{-\beta} \text{ for } \beta \in (0, \tfrac{\varepsilon}{4})$$

Why only $n^{-\frac{\varepsilon}{4}}$?

- **C.G., Labart, Luoto** (2020, 2021)
  f Lipschitz, g ε-Hölder continuous:

$$\sup_{0\le t<T}(\mathbb{E}|Y_t-Y_t^n|^2)^{\frac{1}{2}} \le Cn^{-\frac{\varepsilon}{4}}, \left(\mathbb{E}\int_0^T |Z_t-Z_t^n|^2 dt\right)^{\frac{1}{2}} \le C_2 n^{-\beta} \text{ for } \beta \in (0, \tfrac{\varepsilon}{4})$$

- B^n is constructed from B by Skorohod embedding:

$$(\mathbb{E}|B_T^n - B_T|^2)^{\frac{1}{2}} \le Cn^{-\frac{1}{4}}$$

$\Longrightarrow$ the rate for $(Y^n, Z^n) \to (Y, Z)$ can not be expected to be better.

Why only $n^{-\frac{\varepsilon}{4}}$?

- **C.G., Labart, Luoto** (2020, 2021)
  f Lipschitz, g ε-Hölder continuous:

$$\sup_{0\le t<T}(\mathbb{E}|Y_t-Y_t^n|^2)^{\frac{1}{2}} \le Cn^{-\frac{\varepsilon}{4}}, \left(\mathbb{E}\int_0^T |Z_t-Z_t^n|^2 dt\right)^{\frac{1}{2}} \le C_2 n^{-\beta} \text{ for } \beta \in (0, \tfrac{\varepsilon}{4})$$

- B^n is constructed from B by Skorohod embedding:

$$(\mathbb{E}|B_T^n - B_T|^2)^{\frac{1}{2}} \le Cn^{-\frac{1}{4}}$$

  $\Longrightarrow$ the rate for $(Y^n, Z^n) \to (Y, Z)$ can not be expected to be better.

- Changing the metric to improve the rate?

Wasserstein distances

For $r > 0$ put $\psi_r(x) := e^{|x|^r} - 1$. For a real random variable X we define the Orlicz norm

$$\|X\|_{\psi_r} := \inf\{a > 0 : \mathbb{E}\left[\psi_r(X/a)\right] \leq 1\}, \qquad \inf \emptyset := +\infty.$$

Then for any $p > 0$,

$$\|X\|_{\mathrm{L}^p} \leq \left(\sup_{x>0}\left\{\frac{x^{p\vee r}}{\psi_r(x)}\right\}\right)^{1/(p\vee r)} \|X\|_{\psi_r}.$$

For X, X' random variables with $\mathsf{law}(X) = \mu, \mathsf{law}(X') = \nu$ and $r \geq 1$,

$$W_{\psi_r}(\mu,\nu) = W_{\psi_r}(X,X') := \inf\left\{\|Y - Y'\|_{\psi_r} : \mathsf{law}(Y) = \mu,\, \mathsf{law}(Y') = \nu\right\}.$$

is a metric, the **Wasserstein distance** associated to ψ_r.

Wasserstein convergence rates for $B^n \to B$

Theorem 1 (Rio (2009))

$(X_k)_{k\geq 1}$ *i.i.d. with*
$\mathbb{E}[X_1] = 0$,
$\mathbb{E}\left[X_1^2\right] = 1$, *and*
$\mathbb{E}\left[e^{c|X_1|}\right] < +\infty$ *for some* $c > 0$.
Let $\mathcal{G} \sim N(0,1)$.

Then $\exists C > 0$ *such that, for* $n \geq 1$,

$$W_{\psi_1}\left(\frac{X_1 + \ldots + X_n}{n^{1/2}}, \mathcal{G}\right) \leq C\, n^{-1/2}.$$

For $x \in \mathbb{R}$ and $0 \leq t \leq s \leq T$ we put

$$B_s^{t,x} := x + B_s - B_t \quad B_s^{n,t,x} := x + B_s^n - B_t^n.$$

Lemma 2 (Briand, C.G., S.Geiss, Labart (2021))

1. $\exists C > 0$ *such that* $\forall x \in \mathbb{R}$ *and* $0 \leq t \leq s \leq T$,

$$W_{\psi_1}\left(B_s^{n,t,x}, B_s^{t,x}\right) \leq C\sqrt{T}n^{-1/2}.$$

2. *If* $g : \mathbb{R} \longrightarrow \mathbb{R}$ *is* ε*-Hölder continuous (* $0 < \varepsilon \leq 1$*), then* $\forall x \in \mathbb{R}$ *and* $0 \leq t \leq s \leq T$,

$$\left|\mathbb{E}\left[g\left(B_s^{n,t,x}\right)\right] - \mathbb{E}\left[g\left(B_s^{t,x}\right)\right]\right| \leq C\,\|g\|_\varepsilon\, n^{-\varepsilon/2},$$

for some $C = C(T)$.

$$Y_t = g(B_T) + \int_t^T f(s, B_s, Y_s, Z_s)ds - \int_t^T Z_s dB_s$$

Assumption 1

$\exists \varepsilon, \alpha \in (0,1]$:

1. $g : \mathbb{R} \longrightarrow \mathbb{R}$ *is* ε*-Hölder continuous:* $\forall x, x' \in \mathbb{R}$

$$|g(x) - g(x')| \leq \|g\|_\varepsilon \, |x - x'|^\varepsilon .$$

2. $f : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ $\forall (t,x,y,z)$ *and* (t',x',y',z')

$$\begin{aligned} &|f(t,x,y,z) - f(t',x',y',z')| \\ &\leq \|f_t\|_\alpha \, |t - t'|^\alpha + \|f_x\|_\varepsilon \, |x - x'|^\varepsilon + \|f\|_{\mathrm{Lip}} \left(|y - y'| + |z - z'|\right). \end{aligned}$$

Theorem 3 (Briand, C.G., S.Geiss, Labart (2021))

Assume that Assumption 1 holds. Then there exists a constant $C_\varepsilon > 0$, depending at most on $(T, \alpha, \varepsilon, f, g)$ such that for all $x \in \mathbb{R}$,

(i) $W_{\psi_{1/\varepsilon}}(Y_s^{n,t,x}, Y_s^{t,x}) \leq C_\varepsilon (1+|x|)^\varepsilon n^{-(\alpha \wedge \frac{\varepsilon}{2})}$ *for all* $0 \leq t \leq s \leq T$,

(ii) $W_{\psi_{1/\varepsilon}}(Z_s^{n,t,x}, Z_s^{t,x}) \leq C_\varepsilon \frac{(1+|x|)^\varepsilon}{\sqrt{T-s}} n^{-(\alpha \wedge \frac{\varepsilon}{2})}$ *for all* $s \in [t, T[$.

In particular, for any $p \in [1, \infty[$, there exists a constant $C_p > 0$, depending at most on $(T, \alpha, \varepsilon, f, g, p)$ such that for all $x \in \mathbb{R}$,

(i) $W_p(Y_s^{n,t,x}, Y_s^{t,x}) \leq C_p (1+|x|)^\varepsilon n^{-(\alpha \wedge \frac{\varepsilon}{2})}$ *for all* $0 \leq t \leq s \leq T$,

(ii) $W_p(Z_s^{n,t,x}, Z_s^{t,x}) \leq C_p \frac{(1+|x|)^\varepsilon}{\sqrt{T-s}} n^{-(\alpha \wedge \frac{\varepsilon}{2})}$ *for all* $s \in [t, T[$.

The rate is optimal: If $g(x) = x$ and $f = 0$ we have $Y_T^{n,0,0} = B_T^n$ and $Y_T^{0,0} = B_T$, and it holds

$$\lim_{n \to \infty} n^{\frac{1}{2}} W_1\big(Y_T^{n,0,0}, Y_T^{0,0}\big) \geq \frac{1}{2} T^{\frac{1}{2}}.$$

Idea of the proof: using the connection

$$\text{FBSDE} \iff \text{semilinear heat equation}$$

$$\text{FBS}\Delta\text{E} \iff \text{finite difference equation}$$

Idea of the proof: using the connection

$$\begin{array}{rcl} \text{FBSDE} & \Longleftrightarrow & \text{semilinear heat equation} \\ \text{convergence} \quad \Uparrow & & \Uparrow \\ \text{FBS}\Delta\text{E} & \Longleftrightarrow & \text{finite difference equation} \end{array}$$

FBSDE $\iff$ semilinear heat equation

$$\begin{cases} \partial_t u(t,x) + \frac{1}{2}\Delta u(t,x) + f(t,x,u(t,x),\nabla u(t,x)) = 0, \\ u(T,\cdot) = g, \end{cases} \qquad (t,x) \in [0,T) \times \mathbb{R}$$

is associated to the FBSDE

$$Y_t = g(B_T) + \int_t^T f(B_s, Y_s, Z_s)\, ds - \int_t^T Z_s\, dB_s, \quad 0 \le t \le T.$$

$\Leftarrow$: $\quad Y_s = u(s, B_s)$ and $Z_s = \nabla u(s, B_s)$.

$\Rightarrow$:

$$u(t,x) = \mathbb{E}\left[g(B_T^{t,x}) + \int_t^T f\left(r, B_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\right) dr \right]$$

$$\nabla u(t,x) = \mathbb{E}\left[g(B_T^{t,x}) \frac{B_T - B_t}{T-t} + \int_t^T f\left(r, B_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\right) \frac{B_r - B_t}{r-t}\, dr \right],$$

$$(t,x) \in [0,T) \times \mathbb{R}.$$

For the semilinear heat equation

$$\begin{cases} \partial_t u(t,x) + \frac{1}{2}\Delta u(t,x) + f(t,x,u(t,x),\nabla u(t,x)) = 0, \\ u(T,\cdot) = g, \end{cases} \qquad (t,x) \in [0,T) \times \mathbb{R}$$

we define the finite difference scheme letting

$$\begin{aligned} L^h u(x) &:= \frac{1}{2h}\Big(u(x+\sqrt{h}) + u(x-\sqrt{h}) - 2u(x)\Big) \\ \nabla^h u(x) &:= \frac{1}{2\sqrt{h}}\Big(u(x+\sqrt{h}) - u(x-\sqrt{h})\Big) \\ \partial_t^h u(t) &:= \frac{1}{h}(u(t+h) - u(t)) \end{aligned}$$

and $t_k = kh$. Then

$$\begin{cases} \partial_t^h U^n(t_k,x) + L^h U^n(t_{k+1},x) + f(t_{k+1},x,U^n(t_k,x),\nabla^h U^n(t_{k+1},x)) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad k = 0,...,n-1 \\ U^n(t_n,x) = g(x). \end{cases}$$

FBSΔE $\iff$ finite difference equation

finite difference equation

$$\begin{cases} \partial_t^h U^n(t_k, x) + L^h U^n(t_{k+1}, x) + f(t_{k+1}, x, U^n(t_k, x), \nabla^h U^n(t_{k+1}, x)) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad k = 0, ..., n-1 \\ U^n(t_n, x) = g(x). \end{cases}$$

FBSΔE

$$\begin{cases} Y^n_{t_k} = Y^n_{t_{k+1}} + h\, f\left(t_{k+1}, B^n_{t_k}, Y^n_{t_k}, Z^n_{t_{k+1}}\right) - \sqrt{h}\, Z^n_{t_{k+1}} \varepsilon_{k+1} \\ Y^n_T = g(B^n_T), \quad k = 0, ..., n-1 \end{cases}$$

$\Leftarrow$: $\quad Y^n_{t_k} = U^n\big(t_k, B^n_{t_k}\big), \quad Z^n_{t_k} = \nabla^h U^n\big(t_k, B^n_{t_{k-1}}\big),$

$\Rightarrow$:

$$U^n(t,x) = \mathbb{E} g(B^{n,t,x}_T) + \mathbb{E} \int_{(t,T]} f(s, B^{n,t,x}_{s-}, Y^{n,t,x}_{s-}, Z^{n,t,x}_s) d[B^n]_s \quad 0 \le t \le T.$$

$$\nabla^h U^n(\underline{t} + h, x) = \mathbb{E}\left[g(B^{n,t,x}_T) \frac{B^{n,t,x}_T - x}{T - \underline{t}} + \int_{(t,T]} f(...) \frac{B^{n,t,x}_s - x}{\underline{s} - \underline{t}}\, d[B^n]_s \right]$$

Theorem 4

Under Assumption 1 there exists a constant $C > 0$ depending at most on $(T, \alpha, \varepsilon, f, g)$ such that for $k = 0, ..., n-1$

(i) $|u(t,x) - U^n(t_k, x)| \leq C\,(1+|x|)^{\varepsilon}\, n^{-(\alpha \wedge \frac{\varepsilon}{2})}$ *for all* $(t,x) \in \mathbb{R} \times [t_k, t_{k+1})$,

(ii) $|\nabla u(t,x) - \nabla^h U^n(t_{k+1}, x)| \leq C\, \frac{(1+|x|)^{\varepsilon}}{\sqrt{T-t}}\, n^{-(\alpha \wedge \frac{\varepsilon}{2})}$ *for all* $(t,x) \in \mathbb{R} \times [t_k, t_{k+1})$.

Open question: the path dependent case (How to construct Malliavin weights?)

First answer: approximation of the Brownian motion by random walk without Skorohod embedding.

Theorem 5 (Briand, C.G., S.Geiss, Labart)

$(B_t)_{t\in[0,T]}$ *on* $(\Omega, \mathcal{F}, \mathbb{P})$. *Then one can construct*

1. *an extension* $(\Omega', \mathcal{F}', \mathbb{P}')$
2. $\varepsilon_1, ..., \varepsilon_n$ *independent Rademacher rv such that for* $p \in (0, \infty)$

$$\left(\mathbb{E}_{\mathbb{P}'} \sup_{t\in[0,T]} |B_t - B_t^n|^p\right)^{1/p} \leq C\frac{\log n}{\sqrt{n}}.$$

P. Briand, B. Delyon, J. Memin, *Donsker-Type theorem for BSDEs.* Electron. Comm. Probab., 6 1–14 (2001).

P. Briand, C. Geiss, S. Geiss, C. Labart, *Donsker-Type Theorem for BSDEs: Rate of Convergence.* Bernoulli, 27(2): 899–929 (2021).

B. Bouchard and N. Touzi. *Discrete-time approximation and Monte-Carlo simulation of backward stochastic differential equations.* Stochastic Process. Appl.,111(2):175–206, 2004.

P. Cheridito and M. Stadje, *BSΔEs and BSDEs with non-Lipschitz drivers: Comparison, convergence and robustness* Bernoulli 19(3) (2013) 1047–1085.

C. Geiss, C. Labart and A. Luoto, *Random walk approximation of BSDEs with Hölder continuous terminal condition* Bernoulli 26 (1) pp. 159–190, 2020.

S. Geiss and J. Ylinen, *Decoupling on the Wiener Space, Related Besov Spaces, and Applications to BSDEs*,

E, W., Han, J., Jentzen, A.: Deep learning-based numerical methods for high-dimensional parabolic partial differential equations and backward stochastic differential equations. Commun. Math. Stat. 5(4), 349– 380 (2017)

- K. Jańczak, *Discrete approximations of reflected backward stochastic differential equations with random terminal time,* Probab. Math. Statistics 28 (2008) 41–74.
- K. Jańczak, *Discrete approximations of generalized rbsde with random terminal time,* Discussiones Mathematicae Probability and Statistics 32 (2012) 69 – 85.
- A. Lionnet, G. dos Reis, L. Szpruch , *Convergence and qualitative properties of modified explicit schemes for BSDEs with polynomial growth* Ann. Appl. Probab. 28(4): 2544–2591 (2018).
- T. Nakayama, *Approximation of BSDE's by Stochastic Difference Equation's* J. Math. Sci. Univ. Tokyo 9 (2002), 257–277.
- J. Ma, P. Protter, J. San Martín and S. Torres, *Numerical method for backward stochastic differential equations.* Ann. Appl. Probab. 12 (2002) 302–316.
- M. Martínez, J. San Martín and S. Torres, *Numerical Method for Reflected Backward Stochastic Differential Equations,* Stochastic Analysis and Applications Volume 29, 2011 - Issue 6.

- J. Mémin, S. Peng and M. Xu, *Convergence of solutions of discrete reflected backward SDE's and simulations.* Acta Math. Appl. Sin. (English Series) 24 (2008) 1 – 18.
- S. Peng and M. Xu, *Numerical Algorithms for Backward Stochastic Differential Equations with 1-d Brownian motion: Convergence and simulations.* Math. Model. Numer. Anal. 45 (2011) 335 – 360.
- E. Rio, *Upper bounds for minimal distances in the central limit theorem*, Ann. Inst. Henri Poincaré Probab. Stat. **45** (2009), no. 3, 802–817.
- D. Sun, G. Liang, and S. Tang, *Quantitative stability and numerical analysis of Markovian quadratic BSDEs with reflection*, PUQR, Vol. 7, No. 1, 2022, 13–30
- S. Toldo. *Stability of solutions of BSDEs with random terminal time*. ESAIM: Probability and Statistics, 10:141–163, 2006.
- S. Toldo. *Corrigendum to 'Stability of solutions of BSDEs with random terminal time'*. ESAIM: Probability and Statistics, 11:381–384, 2007.
- J. Zhang. *A numerical scheme for BSDEs.* Ann. Appl. Probab., 14(1):459–488, 2004.