Mean Field Games with Branching

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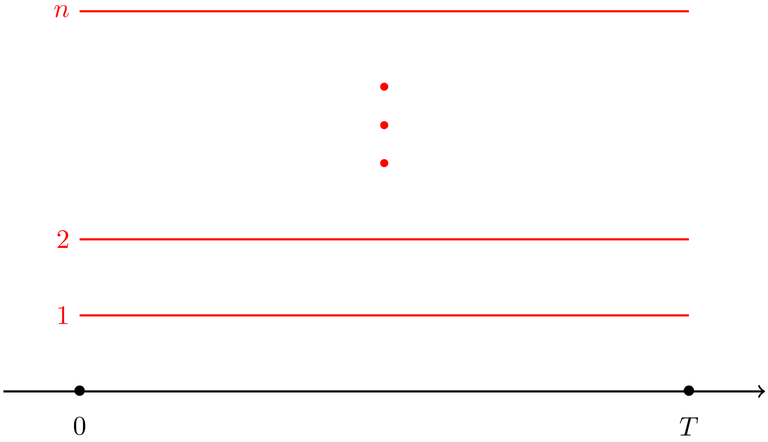
Motivations

- MFG corresponds to the limit of large population differential games with mean-field interaction
  - Nash equilibria for n-player games when n is large
- Introduced by *Lasry & Lions '06*
- Two broad lines of research
  - PDE approach
  - Probabilistic approach
- Applications in economy (economic growth), finance (price impact), crowd dynamics, ...
- However n might not be constant
  - Impact of demography in socio-economic games
  - Population dynamics

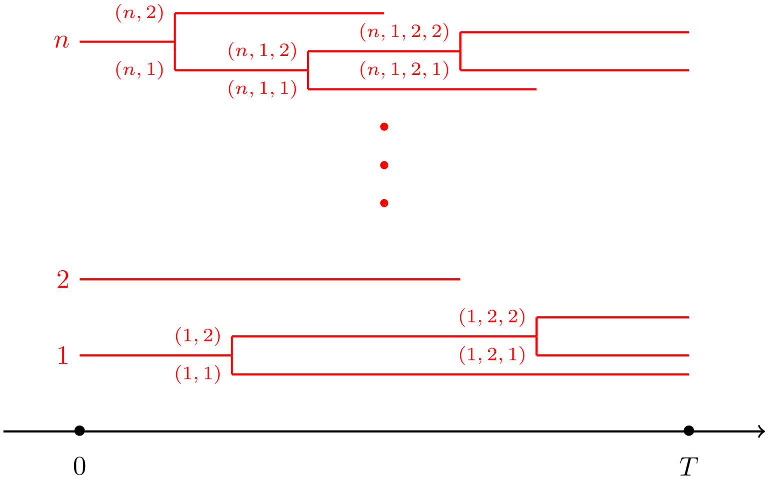
Outline

1. Toy Model
   - MFG with Branching
   - Numerical Example
2. Relaxed Formulation
   - Strong MFG with branching
   - Relaxed Control of Diffusions
   - Relaxed Control of Branching Diffusions
   - Relaxed MFG with Branching

Classical MFG



MFG with Branching



Branching Diffusion

- Consider a population where each agent $k \in \mathbb{K} = \bigcup_{n\geq 1} \mathbb{N}^n$
  - enters the game at time S_k
  - follows a diffusion

$$X_t^k = X_{S_k}^k + \int_{S_k}^t \alpha_s^k \, ds + \sqrt{2}\left(B_t^k - B_{S_k}^k\right), \quad B^k \text{ BM}$$

  - leaves the game at rate γ, *i.e.*, at time

$$T_k := (S_k + \tau_k) \wedge T, \quad \tau_k \sim \mathcal{E}(\gamma)$$

  - is replaced by $\ell \in \mathbb{N}$ agents with probability $p_\ell(X_{T_k}^k)$, *i.e.*, if

$$\sum_{i=0}^{\ell-1} p_i(X_{T_k}^k) \leq U_k < \sum_{i=0}^{\ell} p_i(X_{T_k}^k), \quad U_k \sim \mathcal{U}(0,1)$$

- $(B^k, \tau_k, U_k)_{k\in\mathbb{K}}$ are independent

Finite-Player Problem

- Start with n agents at position $(X_0^k)_{k=1,..,n}$ i.i.d.
- Each agent $k \in \mathbb{K}$ enters at time S_k, leaves at time T_k and follows

$$X_t^k = X_{S_k}^k + \int_{S_k}^t \alpha_s^k \, ds + \sqrt{2} \left(B_t^k - B_{S_k}^k \right)$$

while choosing α^k in order to minimize

$$\mathbb{E}\left[\frac{1}{2} \int_{S_k}^{T_k} \left| \alpha_t^k \right|^2 dt + g\left(X_T^k, \mu_T^n \right) \mathbf{1}_{T_k = T} \right] = J_k(\alpha^k, (\alpha^j)_{j \neq k})$$

where

$$\mu_T^n := \frac{1}{n} \sum_{k \in K_T^n} \delta_{X_T^k} \quad \text{with } K_T^n \text{ the set of agents at time } T$$

- Find a Nash equilibrium, *i.e.*, $(\hat{\alpha}^k)_{k \in \mathbb{K}}$ such that

$$J_k(\alpha^k, (\hat{\alpha}^j)_{j \neq k}) \geq J_k(\hat{\alpha}^k, (\hat{\alpha}^j)_{j \neq k}) \quad \forall\, k,\ \alpha^k$$

Mean Field Games with Branching

- As $n \to \infty$
  - a single player has no influence on μ_T^n
  - by symmetry and independence

$$\langle \mu_T^n, \varphi \rangle = \frac{1}{n} \sum_{k \in K_T^n} \varphi(X_T^k) \longrightarrow \mathbb{E}\Big[\sum_{k \in K_T^1} \varphi(X_T^k) \Big]$$

- MFG with branching
  1. Fix $\mu_T \in \mathcal{M}(\mathbb{R}^d)$
  2. Find an optimal branching diffusion $(\hat{X}_t^k)_{k \in \hat{K}_t^1}$ where each agent k plays the strategy $\hat{\alpha}^k$ to minimize

$$\inf_{\alpha^k} \mathbb{E}\left[\frac{1}{2} \int_{S_k}^{T_k} \left|\alpha_t^k\right|^2 dt + g\left(X_T^k, \mu_T\right) \mathbf{1}_{T_k = T} \right]$$

     where

$$dX_t^k = \alpha_s^k \, ds + \sqrt{2}\, dB_t^k, \quad T_k = (S_k + \tau_k) \wedge T$$

  3. The problem is then to find μ_T such that

$$\langle \mu_T, \varphi \rangle = \mathbb{E}\Big[\sum_{k \in \hat{K}_T^1} \varphi\left(\hat{X}_T^k\right) \Big]$$

PDE Formulation

- A solution to the MFG with branching is a couple (u, m) satisfying

$$\partial_t u + \Delta u - \frac{1}{2}|Du|^2 - \gamma u = 0 \qquad \text{in } [0,T) \times \mathbb{R}^d$$

$$\partial_t m - \Delta m - \operatorname{div}(mDu) - \gamma \sum_{\ell \in \mathbb{N}} (\ell - 1) p_\ell m = 0 \qquad \text{in } (0,T] \times \mathbb{R}^d$$

$$u_T = g(m_T), \; m_0 = \mu_0 \qquad \text{in } \mathbb{R}^d$$

- Existence of a solution as extension of *Cardaliaguet '11*
  1. Fix $\mu_T \in \mathcal{M}(\mathbb{R}^d)$
  2. Find u^μ (smooth) solution to

     $$\partial_t u^\mu + \Delta u^\mu - \frac{1}{2}|Du^\mu|^2 - \gamma u^\mu = 0, \quad u^\mu_T = g(\mu_T)$$

     Then $m^\mu = \mathcal{L}\left(\sum_{k \in \hat{K}^1_\cdot} \delta_{\hat{X}^k_\cdot}\right)$ (weak) solution to

     $$\partial_t m^\mu - \Delta m^\mu - \operatorname{div}(m^\mu Du^\mu) - \gamma \sum_{\ell \in \mathbb{N}} (\ell - 1) p_\ell m^\mu = 0, \quad m^\mu_0 = \mu_0$$

  3. Find a fixed point to $\psi : \mu_T \mapsto m^\mu_T$ by Schauder Theorem
     - Continuity of ψ on a convex compact subset of $\mathcal{M}(\mathbb{R}^d)$ into itself

ε-Nash Equilibrium

- Let (u, m) be a solution to the MFG with branching
- Consider the family of controls

$$\hat{\alpha}_t^k := -Du\big(t, \hat{X}_t^k\big)$$

where

$$d\hat{X}_t^k = -Du\big(t, \hat{X}_t^k\big)\,dt + \sqrt{2}\,dB_t^k$$

- For all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$

$$J_k(\alpha^k, (\hat{\alpha}^j)_{j\neq k}) + \varepsilon \geq J_k(\hat{\alpha}^k, (\hat{\alpha}^j)_{j\neq k}) \qquad \forall\, k,\ \alpha^k$$

where

$$J_k(\alpha^k, (\alpha^j)_{j\neq k}) := \mathbb{E}\left[\frac{1}{2}\int_{S_k}^{T_k} \left|\alpha_t^k\right|^2 dt + g\left(X_T^k, \mu_T^n\right)\mathbf{1}_{T_k=T}\right]$$

Numerical Example

- Consider the simple linear-quadratic model

$$\begin{aligned} \partial_t u + \Delta u - \frac{1}{2}|Du|^2 - \gamma u = 0 \qquad & \text{in } [0,T) \times \mathbb{R} \\ \partial_t m - \Delta m - \operatorname{div}(mDu) - \lambda x^2 m = 0 \qquad & \text{in } (0,T] \times \mathbb{R} \\ u_T = g(m_T), \; m_0 = \mu_0 \qquad & \text{in } \mathbb{R} \end{aligned}$$

  where $\lambda \geq 0$ and

$$g(x,\mu) := \frac{1}{2}(x-5)^2 + \frac{1}{4}\left(x - \frac{1}{\mu(\mathbb{R})}\int_{\mathbb{R}} y\,\mu(dy)\right)^2$$

- In particular

$$\sum_{\ell \in \mathbb{N}} \ell p_\ell(x) = 1 + \frac{\lambda}{\gamma} x^2$$

- If $m_0 = \mathcal{N}(0,1)$, then there exists a solution on short time horizon such that

$$\frac{m_t}{m_t(\mathbb{R})} = \mathcal{N}(\rho_t, v_t)$$

Numerical Result

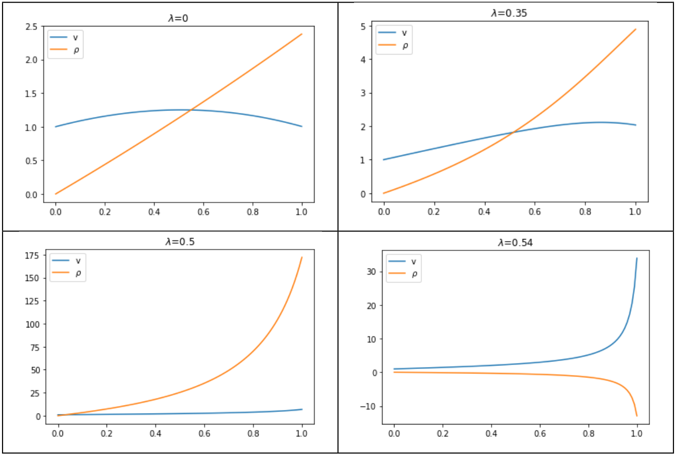


Figure: Change of equilibrium for different values of λ

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Strong MFG with Branching

1. Fix $(\mu_t)_{t\in[0,T]}, \mu_t \in \mathcal{M}(\mathbb{R}^d)$
2. Find an optimal branching diffusion $(\hat{X}^k_t)_{k\in\hat{K}^1_t}$ where each agent k plays the strategy $\hat{\alpha}^k$ to minimize

$$\inf_{\alpha^k} \mathbb{E}\Big[\int_{S_k}^{T_k} f(s, X^k_s, \mu_s, \alpha^k_s)\, ds + g(X^k_T, \mu_T)\mathbf{1}_{T_k=T}\Big]$$

 subject to the constraint

$$dX^k_s = b(s, X^k_s, \mu_s, \alpha^k_s)\, ds + \sigma(s, X^k_s, \mu_s, \alpha^k_s)\, dB^k_s$$

$$T_k = \inf\Big\{t > S_k;\ \int_{S_k}^{t} \gamma(s, X^k_s, \mu_s)\, ds \geq \tau^k\Big\} \wedge T, \quad \tau^k \sim \mathcal{E}(1)$$

$$\sum_{i=0}^{\ell-1} p_i(T_k, X^k_{T_k}, \mu_{T_k}) \leq U_k < \sum_{i=0}^{\ell} p_i(T_k, X^k_{T_k}, \mu_{T_k}), \quad U_k \sim \mathcal{U}(0,1)$$

3. Find $(\mu_t)_{t\in[0,T]}$ such that

$$\langle \mu_t, \varphi\rangle = \mathbb{E}\Big[\sum_{k\in\hat{K}^1_t} \varphi(\hat{X}^k_t)\Big], \quad t \in [0,T]$$

Relaxed Control of Diffusions [El Karoui et al. '87]

Let $\Omega := \mathcal{C} \times \mathcal{V}$ be the canonical space where

$$\mathcal{C} := \left\{x : [0,T] \to \mathbb{R}^d \text{ continuous}\right\}$$
$$\mathcal{V} := \left\{q : [0,T] \to \mathcal{P}(A)\right\} \ni s \mapsto \delta_{\alpha_s}(da)$$

Controlled Martingale Problem

An element $\mathbb{P} \in \mathcal{R}(\mu)$ is a probability on Ω such that the process

$$\varphi(x_s) - \int_0^s \int_A \mathcal{L}_r^{\mu,a} \varphi(x_r)\, q_r(da)\, dr$$

is a $\mathbb{P}$–martingale for all $\varphi \in C_b^2(\mathbb{R}^d)$ where

$$\mathcal{L}_s^{\mu,a}\varphi(x) = \frac{1}{2}\mathrm{tr}\left(\sigma\sigma^*\left(s,x,\mu_s,a\right)D^2\varphi\left(x\right)\right) + b\left(s,x,\mu_s,a\right) \cdot D\varphi\left(x\right)$$

Relaxed Control of Diffusions [El Karoui et al. '87]

Consider the problem

$$V(\mu) := \inf_{\mathbb{P} \in \mathcal{R}(\mu)} J(\mathbb{P}, \mu)$$

where

$$J(\mathbb{P}, \mu) := \mathbb{E}^{\mathbb{P}} \Big[\int_0^T \int_A e^{-\int_0^s \gamma(r, x_r, \mu_r)\, dr} f(s, x_s, \mu_s, a)\, q_s(da)\, ds + e^{-\int_0^T \gamma(r, x_r, \mu_r)\, dr} g(x_T, \mu_T) \Big]$$

Theorem

If b, σ, f, g are bounded continuous in (x, a), then there exists an optimal relaxed control, i.e., $\mathbb{P}^* \in \mathcal{R}(\mu)$ such that

$$V(\mu) = J(\mathbb{P}^*, \mu)$$

- $\mathbb{P} \mapsto J(\mathbb{P}, \mu)$ is (linear) continuous
- $\mathcal{R}(\mu)$ is compact

Relaxed Control of Branching Diffusions

Let $\bar{\Omega} := \mathcal{D} \times \bar{\mathcal{V}}$ be the canonical space where

$$\mathcal{D} := \left\{ z : [0,T] \to E \text{ càdlàg} \right\} \qquad \text{where } E := \left\{ e = \sum_{k \in K} \delta_{(k,x^k)} \right\}$$

$$\bar{\mathcal{V}} := \left\{ \bar{q} = (q^k)_{k \in \mathbb{K}},\ q^k : [0,T] \to \mathcal{P}(A) \right\}$$

Controlled Martingale Problem

An element $\bar{\mathbb{P}} \in \mathcal{T}(\mu)$ is a probability on $\bar{\Omega}$ such that the process

$$\Phi_{\bar{\varphi}}(z_s) - \int_0^s \int_{A^{\mathbb{K}}} \mathcal{H}_r^{\mu,\bar{a}} \Phi_{\bar{\varphi}}(z_r)\, \bar{q}_r(d\bar{a})\, dr$$

is a $\mathbb{P}$–martingale for all $\Phi \in C_b^2(\mathbb{R}), \bar{\varphi} \in C_b^2(\mathbb{K} \times \mathbb{R}^d)$, where $\Phi_{\bar{\varphi}}(e) = \Phi(\langle e, \bar{\varphi} \rangle)$ and

$$\mathcal{H}_s^{\mu,\bar{a}} \Phi_{\bar{\varphi}}(e) = \Phi'_{\bar{\varphi}}(e) \sum_{k \in K} \mathcal{L}_s^{\mu,a^k} \varphi^k(x^k) + \frac{1}{2} \Phi''_{\bar{\varphi}}(e) \sum_{k \in K} \left| D\varphi^k(x^k) \sigma(s, x^k, \mu_s, a^k) \right|^2$$

$$+ \sum_{k \in K} \gamma(s, x^k, \mu_s) \left(\sum_{\ell \geq 0} \Phi_{\bar{\varphi}} \Big(e - \delta_{(k,x^k)} + \sum_{i=1}^{\ell} \delta_{(ki,x^k)} \Big) p_\ell(s, x^k, \mu_s) - \Phi_{\bar{\varphi}}(e) \right)$$

Relaxed MFG with Branching

Proposition

If we assume that b, σ Lipschitz in x and $b, \sigma, f, g, \gamma, p_\ell$ bounded continuous in (x, μ, a). Then an optimal relaxed control $\bar{\mathbb{P}}^* \in \mathcal{T}(\mu)$, i.e., for all $k \in \mathbb{K}$

$$\mathbb{E}^{\bar{\mathbb{P}}^*}\left[\int_{S_k}^{T_k}\int_A f\left(s, X_s^k, \mu_s, a\right) q_s^k(da)\, ds + g\left(X_T^k, \mu_T\right)\mathbf{1}_{T_k=T}\right]$$
$$= \mathbb{E}^{\bar{\mathbb{P}}^*}\left[V\left(S_k, X_{S_k}^k, \mu\right)\right]$$

Theorem

Under the same assumption, there exists a solution to the relaxed MFG with branching, i.e., a probability μ on $\mathcal{D}$ such that there exists an optimal relaxed control $\bar{\mathbb{P}}^* \in \mathcal{T}(\mu)$ satisfying

$$\mu = \bar{\mathbb{P}}^* \circ Z^{-1} \quad \textit{where } Z(z, \bar{q}) = z$$

- Apply Kakutani Fixed Point Theorem to the set-valued map

$$\mu \in \mathcal{P}(\mathcal{D}) \mapsto \left\{\bar{\mathbb{P}}^* \circ Z^{-1};\ \bar{\mathbb{P}}^* \in \mathcal{T}(\mu) \text{ optimal}\right\} \subset \mathcal{P}(\mathcal{D})$$

Thank you for your attention!