Mean Field Games with Branching

Julien Claisse (with Z. Ren and X. Tan)

CEREMADE, Université Paris-Dauphine

9th International Colloquium on BSDEs and Mean Field Systems

June 28, 2022
Motivations

- MFG corresponds to the limit of large population differential games with mean-field interaction
  - Nash equilibria for $n$-player games when $n$ is large
- Introduced by Lasry & Lions '06
- Two broad lines of research
  - PDE approach
  - Probabilistic approach
- Applications in economy (economic growth), finance (price impact), crowd dynamics, …
- However $n$ might not be constant
  - Impact of demography in socio-economic games
  - Population dynamics
Outline

1. Toy Model
   - MFG with Branching
   - Numerical Example

2. Relaxed Formulation
   - Strong MFG with branching
   - Relaxed Control of Diffusions
   - Relaxed Control of Branching Diffusions
   - Relaxed MFG with Branching
Classical MFG

\[ n \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad 2 \quad \cdots \quad \cdots \quad 1 \quad \cdots \quad \cdots \quad \cdots \quad 0 \quad \cdots \quad \cdots \quad T \]
MFG with Branching
Branching Diffusion

Consider a population where each agent $k \in K = \bigcup_{n \geq 1} \mathbb{N}^n$

- enters the game at time $S_k$
- follows a diffusion

$$X^k_t = X^k_{S_k} + \int_{S_k}^t \alpha^k_s ds + \sqrt{2} \left( B^k_t - B^k_{S_k} \right), \quad B^k \text{ BM}$$

- leaves the game at rate $\gamma$, i.e., at time

$$T_k := (S_k + \tau_k) \land T, \quad \tau_k \sim \mathcal{E}(\gamma)$$

- is replaced by $\ell \in \mathbb{N}$ agents with probability $p_\ell(X^k_{T_k})$, i.e., if

$$\sum_{i=0}^{\ell-1} p_i(X^k_{T_k}) \leq U_k < \sum_{i=0}^\ell p_i(X^k_{T_k}), \quad U_k \sim \mathcal{U}(0, 1)$$

- $(B^k, \tau_k, U_k)_{k \in K}$ are independent
Finite-Player Problem

- Start with $n$ agents at position $(X^k_0)_{k=1,...,n}$ i.i.d.
- Each agent $k \in \mathbb{K}$ enters at time $S_k$, leaves at time $T_k$ and follows

$$X^k_t = X^k_{S_k} + \int_{S_k}^t \alpha^k_s ds + \sqrt{2} \left( B^k_t - B^k_{S_k} \right)$$

while choosing $\alpha^k$ in order to minimize

$$\mathbb{E} \left[ \frac{1}{2} \int_{S_k}^{T_k} |\alpha^k_t|^2 dt + g \left( X^k_T, \mu^T_n \right) 1_{T_k=T} \right] = J_k(\alpha^k, (\hat{\alpha}^j)_{j \neq k})$$

where

$$\mu^T_n := \frac{1}{n} \sum_{k \in K^T_n} \delta_X^k$$

- Find a Nash equilibrium, i.e., $(\hat{\alpha}^k)_{k \in \mathbb{K}}$ such that

$$J_k(\alpha^k, (\hat{\alpha}^j)_{j \neq k}) \geq J_k(\hat{\alpha}^k, (\hat{\alpha}^j)_{j \neq k}) \quad \forall \, k, \, \alpha^k$$
Mean Field Games with Branching

- As $n \to \infty$
  - a single player has no influence on $\mu^n_T$
  - by symmetry and independence

$$\langle \mu^n_T, \varphi \rangle = \frac{1}{n} \sum_{k \in K^n_T} \varphi(X^k_T) \to \mathbb{E} \left[ \sum_{k \in K^1_T} \varphi(X^k_T) \right]$$

- MFG with branching
  1. Fix $\mu_T \in \mathcal{M}(\mathbb{R}^d)$
  2. Find an optimal branching diffusion $(\hat{X}^k_t)_{k \in \hat{K}_1^1}$ where each agent $k$
     plays the strategy $\hat{\alpha}^k$ to minimize

$$\inf_{\alpha^k} \mathbb{E} \left[ \frac{1}{2} \int_{S_k}^{T_k} |\alpha^k|^2 dt + g(X^k_T, \mu_T) 1_{T_k=T} \right]$$

where

$$dX_t^k = \alpha_s^k ds + \sqrt{2} dB_t^k, \quad T_k = (S_k + \tau_k) \wedge T$$

3. The problem is then to find $\mu_T$ such that

$$\langle \mu_T, \varphi \rangle = \mathbb{E} \left[ \sum_{k \in \hat{K}_1^1} \varphi(\hat{X}^k_T) \right]$$
A solution to the MFG with branching is a couple \((u, m)\) satisfying

\[
\partial_t u + \Delta u - \frac{1}{2} |Du|^2 - \gamma u = 0 \quad \text{in } [0, T) \times \mathbb{R}^d
\]

\[
\partial_t m - \Delta m - \text{div} (m Du) - \gamma \sum_{\ell \in \mathbb{N}} (\ell - 1) p_\ell m = 0 \quad \text{in } (0, T] \times \mathbb{R}^d
\]

\[
u_T = g(m_T), \quad m_0 = \mu_0 \quad \text{in } \mathbb{R}^d
\]

Existence of a solution as extension of Cardaliaguet '11

1. Fix \(\mu_T \in \mathcal{M}(\mathbb{R}^d)\)
2. Find \(u^\mu\) (smooth) solution to

\[
\partial_t u^\mu + \Delta u^\mu - \frac{1}{2} |Du^\mu|^2 - \gamma u^\mu = 0, \quad u_T^\mu = g(\mu_T)
\]

Then \(m^\mu = \mathcal{L}\left( \sum_{k \in \hat{K}^1} \delta_{\hat{X}_k} \right)\) (weak) solution to

\[
\partial_t m^\mu - \Delta m^\mu - \text{div} (m^\mu Du^\mu) - \gamma \sum_{\ell \in \mathbb{N}} (\ell - 1) p_\ell m^\mu = 0, \quad m_0^\mu = \mu_0
\]

3. Find a fixed point to \(\psi : \mu_T \mapsto m_T^\mu\) by Schauder Theorem

   Continuity of \(\psi\) on a convex compact subset of \(\mathcal{M}(\mathbb{R}^d)\) into itself
\( \varepsilon \)-Nash Equilibrium

- Let \( (u, m) \) be a solution to the MFG with branching.
- Consider the family of controls:

\[
\hat{\alpha}^k_t := -Du(t, \hat{X}^k_t)
\]

where

\[
d\hat{X}^k_t = -Du(t, \hat{X}^k_t) \, dt + \sqrt{2} \, dB^k_t
\]

- For all \( \varepsilon > 0 \), there exists \( N \in \mathbb{N} \) such that for all \( n \geq N \)

\[
J_k(\alpha^k, (\hat{\alpha}^j)_j \neq k) + \varepsilon \geq J_k(\hat{\alpha}^k, (\hat{\alpha}^j)_j \neq k) \quad \forall k, \alpha^k
\]

where

\[
J_k(\alpha^k, (\alpha^j)_j \neq k) := \mathbb{E} \left[ \frac{1}{2} \int_{S_k}^{T_k} |\alpha^k_t|^2 \, dt + g(X^k_T, \mu^n_T) \mathbb{1}_{T_k=T} \right]
\]
Consider the simple linear-quadratic model

\[
\begin{align*}
\partial_t u + \Delta u - \frac{1}{2} |Du|^2 - \gamma u &= 0 & \text{in } [0, T) \times \mathbb{R} \\
\partial_t m - \Delta m - \text{div} (mDu) - \lambda x^2 m &= 0 & \text{in } (0, T] \times \mathbb{R} \\
u_T = g(m_T), \ m_0 = \mu_0 & \text{in } \mathbb{R}
\end{align*}
\]

where \( \lambda \geq 0 \) and

\[
g(x, \mu) := \frac{1}{2} (x - 5)^2 + \frac{1}{4} \left( x - \frac{1}{\mu(\mathbb{R})} \int_{\mathbb{R}} y \mu(dy) \right)^2
\]

In particular

\[
\sum_{\ell \in \mathbb{N}} \ell p_\ell(x) = 1 + \frac{\lambda}{\gamma} x^2
\]

If \( m_0 = \mathcal{N}(0, 1) \), then there exists a solution on short time horizon such that

\[
\frac{m_t}{m_t(\mathbb{R})} = \mathcal{N}(\rho_t, \nu_t)
\]
Numerical Result

Figure: Change of equilibrium for different values of $\lambda$
Outline

1 Toy Model
   - MFG with Branching
   - Numerical Example

2 Relaxed Formulation
   - Strong MFG with branching
   - Relaxed Control of Diffusions
   - Relaxed Control of Branching Diffusions
   - Relaxed MFG with Branching
Strong MFG with Branching

1. Fix \((\mu_t)_{t \in [0,T]}, \mu_t \in \mathcal{M}(\mathbb{R}^d)\)

2. Find an optimal branching diffusion \((\hat{X}^k_t)_{k \in \hat{K}_t}\) where each agent \(k\) plays the strategy \(\hat{\alpha}^k\) to minimize

\[
\inf_{\alpha^k} \mathbb{E} \left[ \int_{S_k}^{T_k} f(s, X^k_s, \mu_s, \alpha_s^k) \, ds + g(X^k_T, \mu_T) \mathbf{1}_{T_k = T} \right]
\]

subject to the constraint

\[
dX^k_s = b(s, X^k_s, \mu_s, \alpha^k_s) \, ds + \sigma(s, X^k_s, \mu_s, \alpha^k_s) \, dB^k_s
\]

\[
T_k = \inf \left\{ t > S_k; \int_{S_k}^{t} \gamma(s, X^k_s, \mu_s) \, ds \geq \tau^k \right\} \wedge T, \quad \tau^k \sim \mathcal{E}(1)
\]

\[
\sum_{i=0}^{\ell-1} \pi_i(T_k, X^k_{T_k}, \mu_{T_k}) \leq U_k < \sum_{i=0}^{\ell} \pi_i(T_k, X^k_{T_k}, \mu_{T_k}), \quad U_k \sim \mathcal{U}(0,1)
\]

3. Find \((\mu_t)_{t \in [0,T]}\) such that

\[
\langle \mu_t, \varphi \rangle = \mathbb{E} \left[ \sum_{k \in \hat{K}_t} \varphi(\hat{X}^k_t) \right], \quad t \in [0,T]
\]
Relaxed Control of Diffusions [El Karoui et al. ’87]

Let $\Omega := C \times V$ be the canonical space where

\[
C := \{ x : [0, T] \rightarrow \mathbb{R}^d \text{ continuous} \}
\]

\[
V := \{ q : [0, T] \rightarrow \mathcal{P}(A) \} \ni s \mapsto \delta_{\alpha_s}(da)
\]

Controlled Martingale Problem

An element $\mathbb{P} \in \mathcal{R}(\mu)$ is a probability on $\Omega$ such that the process

\[
\varphi(x_s) - \int_0^s \int_A \mathcal{L}_{r,a}^{\mu,a} \varphi(x_r) q_r(da) \, dr
\]

is a $\mathbb{P}$–martingale for all $\varphi \in C^2_b(\mathbb{R}^d)$ where

\[
\mathcal{L}_{s,a}^{\mu,a} \varphi(x) = \frac{1}{2} \text{tr} \left( \sigma \sigma^*(s, x, \mu_s, a) D^2 \varphi(x) \right) + b(s, x, \mu_s, a) \cdot D\varphi(x)
\]
Consider the problem

\[ V(\mu) := \inf_{P \in \mathcal{R}(\mu)} J(P, \mu) \]

where

\[ J(P, \mu) := \mathbb{E}^P \left[ \int_0^T \int_A e^{-\int_0^s \gamma(r, x_r, \mu_r) \, dr} f(s, x_s, \mu_s, a) q_s(da) \, ds \right. \]
\[ + \left. e^{-\int_0^T \gamma(r, x_r, \mu_r) \, dr} g(x_T, \mu_T) \right] \]

**Theorem**

*If \( b, \sigma, f, g \) are bounded continuous in \( (x, a) \), then there exists an optimal relaxed control, i.e., \( P^* \in \mathcal{R}(\mu) \) such that*

\[ V(\mu) = J(P^*, \mu) \]

- \( P \mapsto J(P, \mu) \) is (linear) continuous
- \( \mathcal{R}(\mu) \) is compact
Relaxed Control of Branching Diffusions

Let $\tilde{\Omega} := D \times \tilde{V}$ be the canonical space where

$$D := \{ z : [0, T] \to E \text{ càdlàg} \} \quad \text{where} \quad E := \{ e = \sum_{k \in K} \delta(k, x^k) \}$$

$$\tilde{V} := \{ \tilde{q} = (q^k)_{k \in K}, \ q^k : [0, T] \to \mathcal{P}(A) \}$$

### Controlled Martingale Problem

An element $\tilde{P} \in \mathcal{T}(\mu)$ is a probability on $\tilde{\Omega}$ such that the process

$$\Phi \tilde{\varphi}(z_s) - \int_0^s \int_{A^K} \mathcal{H}_{s, a}^{\mu, \tilde{a}} \Phi \tilde{\varphi}(z_r) \tilde{q}_r( da ) \, dr$$

is a $\mathbb{P}$–martingale for all $\Phi \in C_2^b(\mathbb{R})$, $\tilde{\varphi} \in C_2^b(K \times \mathbb{R}^d)$, where $\Phi \tilde{\varphi}(e) = \Phi(\langle e, \tilde{\varphi} \rangle)$ and

$$\mathcal{H}_{s, a}^{\mu, \tilde{a}} \Phi \tilde{\varphi}(e) = \Phi' \tilde{\varphi}(e) \sum_{k \in K} \mathcal{L}_{s, k}^{\mu, a^k} \varphi^k(x^k) + \frac{1}{2} \Phi'' \tilde{\varphi}(e) \sum_{k \in K} \left| D \varphi^k(x^k) \sigma(s, x^k, \mu_s, a^k) \right|^2$$

$$+ \sum_{k \in K} \gamma(s, x^k, \mu_s) \left( \sum_{\ell \geq 0} \Phi \tilde{\varphi}(e - \delta(k, x^k) + \sum_{i=1}^{\ell} \delta(k_i, x^k)) p_\ell(s, x^k, \mu_s) - \Phi \tilde{\varphi}(e) \right)$$
Relaxed MFG with Branching

**Proposition**

If we assume that \( b, \sigma \) Lipschitz in \( x \) and \( b, \sigma, f, g, \gamma, p_L \) bounded continuous in \( (x, \mu, a) \). Then an optimal relaxed control \( \bar{P}^* \in \mathcal{T}(\mu) \), i.e., for all \( k \in K \)

\[
\mathbb{E}^{\bar{P}^*} \left[ \int_{T_k}^T \int_{S_k} A f(s, X^k_s, \mu_s, a) q^k_s(da) ds + g(X^k_T, \mu_T) \mathbf{1}_{T_k=T} \right] = \mathbb{E}^{\bar{P}^*} \left[ V(S_k, X^k_{S_k}, \mu) \right]
\]

**Theorem**

Under the same assumption, there exists a solution to the relaxed MFG with branching, i.e., a probability \( \mu \) on \( \mathcal{D} \) such that there exists an optimal relaxed control \( \bar{P}^* \in \mathcal{T}(\mu) \) satisfying

\[
\mu = \bar{P}^* \circ Z^{-1} \quad \text{where} \quad Z(z, \bar{q}) = z
\]

- Apply Kakutani Fixed Point Theorem to the set-valued map

\[
\mu \in \mathcal{P}(\mathcal{D}) \mapsto \{ \bar{P}^* \circ Z^{-1}; \bar{P}^* \in \mathcal{T}(\mu) \ \text{optimal} \} \subset \mathcal{P}(\mathcal{D})
\]
Thank you for your attention!