Infinite dimensional McKean-Vlasov processes. Application to fluid dynamics models

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- Motivation: ”Climate is what you expect, weather is what you get”
- Mathematical framework: ~~two~~ one coupled Atmosphere-Ocean models
- Well-posedness results
- Main steps of the proof
- Final remarks

◦ [1] D. Crisan, D. D. Holm, P. Korn, An Implementation of Hasselmann’s Paradigm for Stochastic Climate Modelling based on Stochastic Lie Transport, arXiv:2205.04560

Consider the infinite system

$$dX_i(t) = \sigma(X_i(t), V(t))dW_i(t) + b(X_i(t), V(t))dt, \quad X_i \in \mathbb{R}^d \tag{1}$$

where

$$V(t) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \delta_{X_i(t)}.$$

Under certain condition, there is a unique solution of equation (1), that is the infinite dimensional system X_i is well defined on $[0, \infty)$. X_i satisfy the McKean-Vlasov equation

$$dX(t) = \sigma(X(t), P_{X(t)})dW(t) + b(X_i(t), P_{X(t)})dt, \tag{2}$$

What if the system is *not globally* well-defined ? It may be that with positive probability the process 'blows up' before some time $T > 0$ (possible not at a deterministic time T. What if state space is *not finite dimensional*?

Ongoing work with Tom Kurtz.

Hasselmann's Paradigm

Stochastic climate models

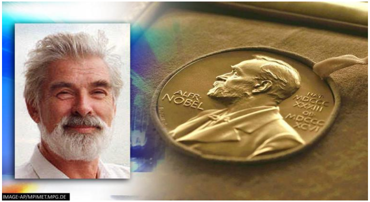
Part I. Theory

By K. HASSELMANN, *Max-Planck-Institut für Meteorologie, Hamburg, FRG*

(Manuscript received January 19; in final form April 5, 1976)

ABSTRACT

A stochastic model of climate variability is considered in which slow changes of climate are explained as the integral response to continuous random excitation by short period "weather" disturbances. The coupled ocean–atmosphere–cryosphere–land system is divided into a rapidly varying "weather" system (essentially the atmosphere) and a slowly responding "climate" system (the ocean, cryosphere, land vegetation, etc.). In the usual Statistical Dynamical Model (SDM) only the average transport effects of the rapidly varying weather components are parameterised in the climate system. The resultant prognostic equations are deterministic, and climate variability can normally arise only through variable external conditions. The essential feature of stochastic climate models is that the non-averaged "weather" components are also retained. They appear formally as random forcing terms. The climate system, acting as an integrator of this short-period excitation, exhibits the same random-walk response characteristics as large particles interacting with an ensemble of much smaller particles in the analogous Brownian motion problem. The model predicts "red" variance



Prize motivation: For the physical modelling of Earths climate, quantifying variability and reliably predicting global warming.

Klaus Hasselmann - Nobel prize 2021

Progress in Probability, Vol. 49

Hasselmann's program revisited: the analysis of stochasticity in deterministic climate models

Ludwig Arnold

Abstract. In his seminal 1976 paper on "Stochastic Climate Models", K. Hasselmann proposed to improve deterministic models for the "climate" (slow variables) by incorporating the influence of the "weather" (fast variables) in the form of random noise.

We will recast this program in the language of modern probability theory as follows: While the transition from a GCM (general circulation model) to an SDM (statistical dynamical model) (both deterministic) is facilitated by the *method of averaging*, stochasticity comes into the picture when studying the error made in the averaging procedure, provided that the fast variables are sufficiently "chaotic".

The study of *normal deviations* from the averaged system is described by the central limit theorem, while the study of *large deviations* from the average (events happening on an exponential time scale) is done by the theory of large deviations. We feel that the latter should be particularly appealing to meteorologists, as one can, for example, describe the "hopping" of the climate between its various local attractors due to the forcing by chaotic weather.

We believe that the dwindling interest in Hasselmann's program was caused, on the one hand, by disillusionment and frustration as the program did not seem to live up to its original expectations, and, on the other hand, by the availability of computing power that made qualitative research less pressing.

We claim that Hasselmann's program has never really been implemented. We believe that this is also caused by the fact that, to our knowledge, no mathematician has ever really worked on it. This paper is an attempt to change this situation as we are convinced that the program has not lost its importance and significance, and that there are today new promising ways and means which are worth being tried.

Imkeller, P., von Storch, JS. (eds) Stochastic Climate Models. Progress in Probability, Vol 49. Birkhuser, 2001.

- $\mathbf{u}^a \backslash \mathbf{u}^o$ atmospheric\oceanic velocity field
- $\theta^a \backslash \theta^o$ atmospheric\oceanic temperature
- $Ro^a \backslash Ro^o$, $Re^a \backslash Re^o$, $Pe^a \backslash Pe^o$ atmospheric\oceanic Rossby ($\approx 10^{-1}, \approx 10^{-2}$), Reynolds $>> 1$, Peclet $>> 1$ numbers
- $\gamma, \sigma < 0$ regulate the interaction between $(\mathbf{u}^a, \theta^a)$ and $(\mathbf{u}^o, \theta^o)$
- p^o, ocean pressure, q^a, atmos velocity potential
- $(\xi_i)_i$ sufficiently smooth vector fields
- $(W^i)_i$ independent Brownian motions
- $\operatorname{curl} \mathbf{R}(\mathbf{x}) = 2\Omega(\mathbf{x})$ the Coriolis parameter

Lagrangian-Averaged Stochastic Advection by Lie Transport Model

Atmosphere

$$d\mathbf{u}^a + (d\mathbf{x}_t^a \cdot \nabla)\mathbf{u}^a + \frac{1}{Ro^a} d\mathbf{x}_t^{a\perp} + \sum_i \left(u_j^a \nabla \xi_i^j + \frac{1}{Ro^a} \nabla \Big(R_j(\mathbf{x}) \xi_i^j \Big) \right) \circ dW_t^i$$

$$+ u_j^a \nabla \mathbb{E}[u^{aj}] dt + \frac{1}{Ro^a} \nabla (\mathbb{E}[\mathbf{u}^a] \cdot \mathbf{R}) dt + \frac{1}{Ro^a} \nabla \theta^a \, dt = \frac{1}{Re^a} \triangle \mathbf{u}^a \, dt \, ,$$

$$d\theta^a + d\mathbf{x}_t^a \cdot \nabla \theta^a = -\gamma(\theta^o - \theta^a) + \frac{1}{Pe^a} \triangle \theta^a,$$

$$d\mathbf{x}_t^a = \mathbb{E}[\mathbf{u}^a] dt + \sum_i \xi_i \circ dW_l^i$$

Ocean:

$$\frac{\partial \mathbf{u}^o}{\partial t} + (\mathbf{u}^o \cdot \nabla)\mathbf{u}^o + \frac{1}{Ro^o}\mathbf{u}^{o\perp} + \frac{1}{Ro^o}\nabla p^o = \sigma(\mathbf{u}^o - \mathbb{E}\bar{\mathbf{u}}^a) + \frac{1}{Re^o}\triangle \mathbf{u}^o,$$

$$\frac{\partial \theta^o}{\partial t} + (\mathbf{u}^o \cdot \nabla)\theta^o = \frac{1}{Pe^o}\triangle \theta^o,$$

$$div(\mathbf{u}^o) = 0$$

- $\bar{\mathbf{u}} := \mathbf{u} - \frac{1}{|\Omega|}\int_\Omega \mathbf{u} dx$ the oceanic velocity remains in the space of periodic flows with vanishing average. Used to determine the oceanic pressure.
- The rapid mean velocity of the atmosphere relative to the slower ocean velocity in the frame of motion of the Earth's rotation is removed.
- The ocean momentum responds to the shear force, which is proportional to the difference between the local ocean velocity at a given time and the local deviation of the atmospheric velocity away from its mean velocity.
- This is an *intermediate coupled model*: simpler than the coupled general circulation models of the atmosphere-ocean system. Still, it allows the study of fundamental aspects of the atmosphere-ocean interaction, e.g. the *El Niño-Southern Oscillation (ENSO)* in the tropical Pacific.

Stronger trade winds increase the upwelling in the east Pacific, thereby creating a temperature gradient in the sea-surface temperature that amplifies the trade winds. This interaction between the trade winds and sea surface temperature in the tropical Pacific generates a quasi-periodic oscillation between the three ENSO-phases: the neutral phase, El Niño and La Niña. Intermediate coupled models have been used successfully to shed light on the fundamental principle of ENSO.

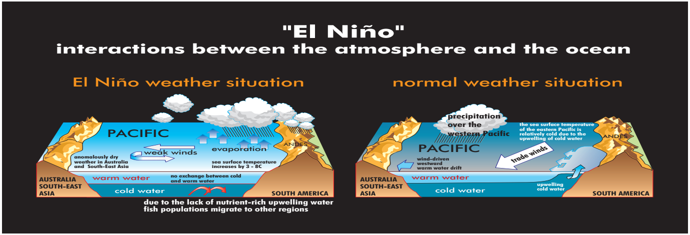


Figure: *Illustration of the dynamics and feedbacks of the atmosphere-ocean model that generate the El Niño-Southern Oscillation (ENSO). The trade winds that are part of the Walker circulation in the tropical Pacific interact with the cold/warm pools of the sea surface temperature. Local heating creates wind anomalies that in turn change the thermocline and the upwelling. During these process Rossby and Kelvin waves are emitted. The feedback of the ocean weakens the trade winds.*

Domain and Boundary Conditions: The spatial domain is a two dimensional square $\Omega := [0, L] \times [0, L]$ with $L \in \mathbb{R}^+$. Periodic boundary conditions.
Operators and Spaces: By $W^s(\Omega)$ we denote the L^2-Sobolev space of order $s \in \mathbb{Z}_+ \cup \{0\}$ that is defined as the set of functions $f \in L^2(\Omega)$ such that its derivatives in the distributional sense $\mathcal{D}^\alpha f(x, y) = \partial_x^{\alpha_1} \partial_y^{\alpha_2} f(x, y)$ are in $L^2(\Omega)$ for all $|\alpha| \leq s$, with multi-index $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$, and degree $|\alpha| := \alpha_1 + \alpha_2$. The scalar product in $W^s(\Omega)$ is defined by

$$\langle f, g \rangle_{W^s} := \sum_{|\alpha| \leq s} \int_\Omega \mathcal{D}^\alpha f \cdot \mathcal{D}^\alpha g \, dx.$$

The vectorial counterpart of the Sobolev space $W^s(\Omega)$ is denoted by $\mathbf{W}^s(\Omega)$. We define the scalar space

$$V := \{f : \mathbb{R}^2 \to \mathbb{R} : f \text{ is a trigonometric polynomial with period L}\},$$

and its vector-valued equivalents for atmosphere and ocean component

$$\mathbf{V}^a := \{u : \mathbb{R}^2 \to \mathbb{R}^2 : \ u \text{ is a vector-valued trig polynomial with period L}\}$$

$$\mathbf{V}^o := \{u : \mathbb{R}^2 \to \mathbb{R}^2 : \ u \text{ is a vector-valued trig polynomial with period L and } \int_\Omega u \, dx = 0\}.$$

We define now the following function spaces

$H^s(\Omega) :=$ the closure of V in $W^s(\Omega)$, $\quad \mathbf{H}^{s,a}(\Omega) :=$ the closure of $\mathbf{V}^a$ in $\mathbf{W}^s(\Omega)$,
$\mathbf{H}^{s,o}(\Omega) :=$ the closure of $\mathbf{V}^o$ in $\mathbf{W}^s(\Omega)$, $\quad \mathbf{H}^s_{div}(\Omega) := \{u \in \mathbf{H}^{s,o}(\Omega) : div(u) = 0\}$.

We define for $s \in \mathbb{N} \cup \{0\}$ the *Sobolev space of state vectors* by

$$\mathcal{H}^s(\Omega) := \mathbf{H}^{s,a}(\Omega) \times H^s(\Omega) \times \mathbf{H}^{s,o}_{div}(\Omega) \times H^s(\Omega),$$

in which the norm of $\psi = (\mathbf{u}^a, \theta^a, \mathbf{u}^o, \theta^o) \in \mathcal{H}^s$ is given by

$$||\psi||_{\mathcal{H}^s} := (||\mathbf{u}^a||^2_{\mathbf{H}^s} + ||\theta^a||^2_{H^s} + ||\mathbf{u}^o||^2_{\mathbf{H}^s} + ||\theta^o||^2_{H^s})^{1/2}.$$

We use an analogous notation for the Lebesgue spaces and denote by $L^2, \mathbf{L}^2, \mathcal{L}^2$ square-integrable scalar functions, vector and state vectors resp.

$H^s(\Omega) :=$ the closure of V in $W^s(\Omega)$, $\quad \mathbf{H}^{,a}(\Omega) :=$ the closure of $\mathbf{V}^a$ in $\mathbf{W}^s(\Omega)$,
$\mathbf{H}^{s,o}(\Omega) :=$ the closure of $\mathbf{V}^o$ in $\mathbf{W}^s(\Omega)$, $\quad \mathbf{H}^s_{div}(\Omega) := \{u \in \mathbf{H}^{s,o}(\Omega) : div(u) = 0\}$.

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$$\mathcal{H}^s(\Omega) := \mathbf{H}^{s,a}(\Omega) \times H^s(\Omega) \times \mathbf{H}^{s,o}_{div}(\Omega) \times H^s(\Omega),$$

in which the norm of $\psi = (\mathbf{u}^a, \theta^a, \mathbf{u}^o, \theta^o) \in \mathcal{H}^s$ is given by

$$||\psi||_{\mathcal{H}^s} := (||\mathbf{u}^a||^2_{\mathbf{H}^s} + ||\theta^a||^2_{H^s} + ||\mathbf{u}^o||^2_{\mathbf{H}^s} + ||\theta^o||^2_{H^s})^{1/2}.$$

Crucial remark: The equation satisfied by the coupled system $(\mathbb{E}[\psi^a], \psi^o)$ has a closed form and has a unique maximal solution:

Theorem

Let $s \geq 2$ and suppose the initial condition of the LASALT system satisfies $\psi_0 = (\widehat{\psi}_0^a, \psi_0^o) = (\psi_0^a, \psi_0^o) \in \mathcal{H}^s(\Omega)$. Then there exists a unique time $t_{e,1}^* \in (t_0, \infty]$ such that a local regular solution $(\widehat{\psi}^a, \widehat{\psi}^o)$ exists and is unique on any interval $T := [t_0, t_1]$, where $t_0 < t_1 < t_{e,1}^*$ and that, if $t_{e,1}^* < \infty$, then

$$\lim_{t \nearrow t_{e,1}^*} ||(\widehat{\psi}^a, \widehat{\psi}^o)||_{\mathcal{H}^s} = \infty.$$

The fact that the equation satisfied by the coupled system $(\mathbb{E}[\psi^a], \psi^o)$ has a closed form and, following Theorem 1, has a local regular solution enables us to show that there exists a solution of the system LASALT up to a time $t_{e,2}^* \in (t_0, \infty]$. We have the following:

Theorem

Let $s \geq 2$ and suppose the initial condition of the LASALT system satisfies $\psi_0 \in \mathcal{H}^s(\Omega)$. Then there exists a unique time $t^*_{e,2} \in (t_0, \infty]$, such that on any interval $T := [t_0, t_1]$, where $t_0 < t_1 < t^*_{e,2}$, the system has a unique solution with the property that

$$\psi^a \in L^2\left(\Xi; C(T; \mathbf{H}^{s-2,a}(\Omega) \times H^{s-2}(\Omega)\right) \cup L^2\left(\Xi; L^2(T; \mathbf{H}^{s-1,a}(\Omega) \times H^{s-1}(\Omega)\right),$$

$$(\mathbb{E}[\psi^a], \psi^o) \in C(T; \mathcal{H}^s(\Omega) \cup L^2(T; \mathcal{H}^{s+1}(\Omega).$$

and that, if $t^*_{e,2} < \infty$, then

$$\lim_{t \nearrow t^*_{e,2}} ||(\mathbb{E}[\psi^a], \psi^o)||_{\mathcal{H}^s} = \infty.$$

Remark

The loss of regularity in the atmospheric component ψ^a as compared to the regularity of the pair $(\mathbb{E}[\psi^a], \psi^o)$ is an artifact of our proof.

In classical fluid mechanics, Kelvin's circulation theorem (1869) states In a barotropic ideal fluid with conservative body forces, the circulation around a closed curve (which encloses the same fluid elements) moving with the fluid remains constant with time.

Theorem (Kelvin theorem for the LA-SALT atmospheric model)

$$d \oint_{c(\mathrm{d}\mathbf{X}_t)} (\boldsymbol{u} + \boldsymbol{R}(\boldsymbol{x})) \cdot d\boldsymbol{x} = \frac{1}{Re} \oint_{c(\mathrm{d}\mathbf{X}_t)} \triangle \mathbf{u}\, dt \cdot d\boldsymbol{x} \tag{3}$$

where

$$\mathrm{d}\mathbf{x}_t^a := \mathbb{E}[\mathbf{u}^a](\boldsymbol{x}, t)dt + \sum_i \xi_i^a \circ dW_i(t).$$

The LA-SALT model is grounded in fundamental geometric mechanics.

- DD Holm, Variational principles for stochastic fluid dynamics, Proc. R. Soc, 2015.
- O Street, DC, Semi-martingale driven variational principles, Proc. R. Soc, 2021.
- DC DD Holm, JM Leahy, T Nilssen, Variational principles for fluid dynamics on rough paths, Advances in Mathematics 404, 2022.

- The equation (deterministic) satisfied by the coupled system $(\mathbb{E}[\psi^a], \psi^o)$ is considered first.
- A truncation procedure is applied to the nonlinear terms.
- The truncated equation is approximated by using a Galerkin system
- The system is shown to be relatively compact
- Any limit points is show to satisfy the truncated equation
- The truncation is lifted leading to maximal solution of the original equation.
- We use Theorems 1 and 2 in Rozovskii, Chapter 4, to justify the existence of the original system. This requires additional regularity on the coefficients. This can only be ensured by defining the solution in a lower Sobolev space.
- Uniqueness is proved in a standard way.

- While following Hasselmann's view in a general sense, stochasticity is incorporated in the LA SALT in a manner that deviates from common practice: The path of a fluid element in the Lagrangian sense is assumed to be stochastic.
- Stochasticity is injected directly into the transport velocity of the atmospheric fluid dynamics. The stochastic models are transparently related to the deterministic model by the Kelvin Circulation Theorem.
- The stochastic Lagrangian path of the material loop is a McKean-Vlasov process in the LA SALT approach.
- We show local well-posedness theorem that the proposed stochastic climate models.
- Stochastic climate models we derive here are related to a class of idealized climate models that target the study of El-Nino-Southern Oscillation (ENSO). ENSO is an instability of the coupled atmosphere-ocean system that occurs with frequency of 5-7 years.
- We hope that these results open the door to a whole slew of additional results, both theoretical and practical (Bayesian inference, stability properties, theoretical results), and help bring Hasselmann's programme to the attention of the Mathematics community.