Weak error for the uniform propagation of chaos on the torus

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Part I. Introduction

Weakly interacting particle system

- Throughout, we consider an N-particle system of the type

$$dX_t^i = b(X_t^i, \bar{\mu}_t^N)dt + dW_t^i$$

for i an index in $\{1, \cdots, N\}$ where $\bar{\mu}_t^N = \frac{1}{N}\sum_{j=1}^{N} \delta_{X_t^j}$

  - $(W_t^1)_{t\geq 0}, \cdots, (W_t^N)_{t\geq 0}$ are independent Brownian motions
  - i.i.d. initial conditions $X_0^1, \cdots, X_0^N$
  - model is said to be mean-field

- Main question: long time and large N behavior of the model (absolutely not a new question)

$$dX_t = b(X_t, \mathcal{L}(X_t))dt + dW_t$$

- X_t^i may take values with in dimension d

$$b : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}^d$$

Large N, large t

- Long time behaviour of McKean-Vlasov equations

  ◦ convergence toward ! invariant mesure: e.g. Benachour et al. (98), Carrillo (03), Cattiaux et al. (08),... , for gradient flows with convex potentials

  ◦ uniqueness may be lost: see e.g. Bertini et al. (09), Giacomin (12)... for periodic Kuramoto model

Large N, large t

- Uniform propagation of chaos: convergence as $N \nearrow \infty$ unif. in t

  - may fail even if unique attractive invariant measure, see e.g. Malrieu (03) in $\mathbb{R}^d$

  - hierarchy $\sup_{t\geq 0} W_1\Big(\mathcal{L}(X_1^1, \cdots, X_t^k), \mathcal{L}(X_t)^{\otimes k}\Big)$

Durmus et al., (18), Salem (18), $\mathbb{R}^d$: confinement + small or convex potential $\rightsquigarrow N^{-1/2}$ for $k = 1$

Lacker and LeFlem (22) : similar conditions, but k/N

  - empirical measure $\sup_{t\geq 0}\Big|\mathbb{E}[\Phi(\mu_t^N)] - \Phi(m_t^\mu)\Big|$

Chassagneux et al. (19): finite horizon $\rightsquigarrow N^{-1}$ (+ error expansion),

Mischler et al., (15), Arnaudon et al., (20): $\Phi(\mu) = \langle f, \mu\rangle \rightsquigarrow N^{-1}$

$\rightsquigarrow$ week error with suitable choice of Φ

Part II. Semi-group of the MKV SDE

Semi-group

- Introduce the semi-group of a standard McKean-Vlasov equation

$$dX_t = b(X_t, \mathcal{L}(X_t))dt + dW_t$$

  - $(m_t = \mathcal{L}(X_t))_{t\geq 0}$ solves Fokker-Planck equation

$$\partial_t m_t - \tfrac{1}{2}\Delta m_t + \mathrm{div}(m_t b(\cdot, m_t)) = 0, \quad m_0 = \mathcal{L}(X_0)$$

  - if $\exists! \Rightarrow \mathcal{L}(X_t)$ only depends on $\mathcal{L}(X_0)$

- define the semi-group

$$(P_t\Phi)(\mathcal{L}(X_0)) = \Phi(\mathcal{L}(X_t)), \quad t \geq 0, \quad \Phi : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$$

  - master equation

$$\partial_t(P_t\Phi)(\mu) - \int_{\mathbb{R}^d} b(v,\mu)\cdot \partial_\mu(P_t\Phi)(\mu, v)d\mu(v)$$
$$- \frac{1}{2}\int_{\mathbb{R}^d} \mathrm{Trace}\Big[\partial_v\partial_\mu(P_t\Phi)(\mu, v)\Big]d\mu(v) = 0, \quad (P_0\Phi)(\mu) = \phi(\mu)$$

Overview of differentiation on $\mathcal{P}(\mathbb{T}^d)$

- We say that $\mathcal{V} : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ is C^1 if

$$\frac{d}{d\varepsilon}_{|\varepsilon=0+} \mathcal{V}((1-\varepsilon)\mu + \varepsilon\mu') = \int_{\mathbb{T}^d} \frac{\delta \mathcal{V}}{\delta m}(\mu)(v) d(\mu' - \mu)(v)$$

for a continuous map $\frac{\delta \mathcal{V}}{\delta m} : \mathcal{P}(\mathbb{T}^d) \times \mathbb{T}^d \to \mathbb{R}$

  - unique up to an additive constant $\rightsquigarrow$ impose zero mean under m

- Wasserstein derivative $\partial_\mu \mathcal{V}(\mu)(v) = \partial_v \frac{\delta \mathcal{V}}{\delta m}(\mu)(v)$

- Finite-dimensional projection

$$\partial_{x_i}\Big[\mathcal{V}\Big(\frac{1}{N}\sum_{j=1}^N \delta_{x_j}\Big)\Big] = \frac{1}{N}\partial_\mu \mathcal{V}\Big(\frac{1}{N}\sum_{j=1}^N \delta_{x_j}\Big)(x_i), \quad x_1, \ldots, x_N \in \mathbb{R}^d$$

- $\mathcal{V}$ is C^2 if $\mathcal{P}(\mathbb{T}^d) \ni \mu \mapsto \frac{\delta \mathcal{V}}{\delta m}(\mu)(v)$ is C^1. Second order derivatives read $(v, v') \mapsto \frac{\delta^2 \mathcal{V}}{\delta m^2}(\mu)(v, v')$.

Revisiting propagation of chaos

- Back to the particle system

$$dX_t^i = b(X_t^i, \mu_t^N)dt + dW_t^i, \quad \mu_t^N = \frac{1}{N}\sum_j \delta_{X_t^j}$$

- How to test proximity with the limiting semi-group?
  - notice that

$$(P_{t-s}\Phi)\Big(\mathcal{L}(X_s^\mu)\Big) = \Phi\Big(\mathcal{L}(X_{t-s}^{\mathcal{L}(X_s^\mu)})\Big) = \Phi\Big(\mathcal{L}(X_t^\mu)\Big)$$

  - expansion of $(P_{t-s}\Phi(\mu_s^N))_{0\le s\le t}$

$$\mathbb{E}\Big[\Phi(\mu_t^N)\Big] - \Phi(\mathcal{L}(X_t^\mu)) = \mathbb{E}\Big[(P_t\Phi)(\mu_0^N)\Big] - (P_t\Phi)(\mu)$$
$$+ \frac{1}{2N^2}\sum_{i=1}^N \mathrm{Trace}\int_0^t \mathbb{E}\Big[\partial_\mu^2 P_{t-s}\Phi(\mu_s^N)(X_s^i, X_s^i)\Big]ds$$

  - as for the initial condition

$$\Big|\mathbb{E}\Big[(P_t\Phi)(\bar\mu_0^N)\Big] - (P_t\Phi)(\mu)\Big| \le \frac{C}{N}\sup_{m\sim\mu}\Big\|\frac{\delta^2 P_t\Phi}{\delta m^2}(m)(\cdot,\cdot)\Big\|_\infty$$

Choice of Φ

- In short, we want to get integrable bounds on the various derivatives and then get uniform propagation of chaos at rate $1/N$.

- Using smoothing effect of the diffusion, we just need bounds and Hölder regularity on

$$\Phi(m), \qquad \frac{\delta\Phi}{\delta m}(m)(x), \qquad \frac{\delta^2\Phi}{\delta m^2}(m)(x,x')$$

  - $\Phi(m) = \langle f, m\rangle \Rightarrow \dfrac{\delta\Phi}{\delta m}(m)(x) = f(x)$

- Example

  - for given $\varepsilon \in (0,1]$ and $\mu_0 \in \mathcal{P}(\mathbb{T}^d)$, choose

$$\Phi(\mu) = \left\|\mu - \mu_0\right\|^2_{-(d+\varepsilon)/2}$$

where

$$\|m\|^2_{-(d+\varepsilon)/2} = \sum_{n\in\mathbb{Z}^d} \frac{1}{(1+|n|^2)^{(d+\varepsilon)/2}} \left| \int_{\mathbb{T}^d} \exp(i2\pi n\cdot\theta)dm(\theta)\right|^2$$

Part III. Long time derivatives of the semi-group

Road map to the regularity of the semi-group

- Goal is to address $\dfrac{\delta P_t \Phi}{\delta m}(\mu)(x)$
- Make use of a flow/characteristics method

$$\frac{d}{d\epsilon} P_t \Phi((1-\epsilon)\mu + \epsilon\mu')_{|\epsilon=0+} = \frac{d}{d\epsilon}\Big[\Phi\big(m_t^{(1-\epsilon)\mu+\epsilon\mu'}\big)\Big]_{|\epsilon=0+}$$

$$= \int_{\mathbb{T}^d} \frac{\delta\Phi}{\delta m}(m_t^\mu)(x) \frac{d}{d\epsilon} m_t^{(1-\epsilon)\mu+\epsilon\mu'}(dx)$$

- Fomally, $\dfrac{d}{d\epsilon} m_t^{(1-\epsilon)\mu+\epsilon\mu'}(dx)$ solves linearized equation

$$\partial_t m_t^{(1)} - \tfrac{1}{2}\Delta m_t^{(1)} + \mathrm{div}\Big(m_t^{(1)} b(\cdot, m_t^\mu)\Big) + \mathrm{div}\Big(m_t \Big\langle \frac{\delta b}{\delta m}(\cdot, m_t^\mu), m_t^{(1)}\Big\rangle\Big) = 0$$

with $m_0^{(1)} = \mu' - \mu$

- Replace $m_0^{(1)}$ by δ_z: $\dfrac{\delta P_t \Phi}{\delta m}(\mu)(z) = \displaystyle\int_{\mathbb{T}^d} \frac{\delta\Phi}{\delta m}(m_t^\mu)(x) m_t^{(1)}(\delta_z)(dx)$
  - similarly for the derivative w.r..t z, focus on $m_t^{(1)}\Big(\dfrac{d}{dz_i}\delta_z\Big)$

Second order derivatives

- Call

$$L_m(q) = \tfrac{1}{2}\Delta q + \mathrm{div}\Big(qb(\cdot, m)\Big) + \mathrm{div}\Big(m\Big\langle \frac{\delta b}{\delta m}(\cdot, m), q\Big\rangle\Big)$$

  - first order derivatives of $P_t\Phi$ obey the long run behavior of zeros of $L_{m_t(\mu)}$

- But, most of all, we need second order derivatives

$$\partial_{z_i}\partial_{z_j}\frac{\delta^2 P_t\Phi}{\delta m^2}(\mu)(z, z') = \int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\frac{\delta^2\Phi}{\delta m^2}(m_t^\mu)(x, y)d_t^{(1),i}(z)(dx)d_t^{(1),j}(z)(dy) + \int_{\mathbb{T}^d}\frac{\delta\Phi}{\delta m}(m_t^\mu)(x)d_t^{(2),i,j}(z, z')(dx)$$

  - $d^{(2),i,j}$ solves same equation but with perturbation

$$\partial_t d_t^{(2),i,j} - L_{m_t(\mu)}d_t^{(2),i,j} + R\Big(t, d_t^{(1),i}(z), d_t^{(1),j}(z')\Big) = 0$$

  and 0 initial condition

Bounds that look needed

- Uniform exponential decay (within suitable distribution space) of solutions of

$$\partial_t d_t^{(1),i} - L_{m(t,\mu)} d_t^{(1),i} = 0$$

  - should be enough to imply bounded of $\partial_x \dfrac{\delta\Phi}{\delta m}$

- Uniform exponential decay (within suitable distribution space) of solutions of

$$\partial_t d_t^{(2),i,j} - L_{m(t,\mu)} d_t^{(1),i,j} + r_t = 0$$

  - given a perturbation that itself tends to 0 exponentially fast
  - should be enough to imply bounded of $\partial^2_{x,x'} \dfrac{\delta\Phi^2}{\delta m^2}$

- Roadmap is clear
  - address the long time behavior of the linearized operator
  - question: locally or uniformly with respect to initial μ?
  - + smoothing in small time

Part IV. Examples

Small enough McKean-Vlasov interaction

- Isolate the mean field dependence in linearized operator

$$L_{m_t(\mu)}(q) = \tfrac{1}{2}\Delta q + \mathrm{div}\Big(q b(\cdot, m_t(\mu))\Big) + \mathrm{div}\Big(m_t(\mu)\Big\langle \frac{\delta b}{\delta m}(\cdot, m_t(\mu)), q\Big\rangle\Big)$$

  - perturbation of

$$L^0_{m_t(\mu)}(q) = \tfrac{1}{2}\Delta q + \mathrm{div}\Big(q b(\cdot, m_t(\mu))\Big)$$

- $L^0_{m_t(\mu)}$ adjoint of

$$(L^0_{m_t(\mu)})^\dagger \psi = \tfrac{1}{2}\Delta\psi - \nabla\psi \cdot b(\cdot, m_t(\mu))$$

  - spectral gap $\Rightarrow$ exponential convergence towards a constant
  - if $\int_{\mathbb{T}^d} q_0 = 0 \Rightarrow$ exponential decay

- If $\delta b/\delta m$ small enough with respect to the rate at which exponential decay occurs $\Rightarrow$ method works! (Recover Arnaudon, Guillin...)

Potential case

- Consider the case

$$b(x,m) = -\kappa \int_{\mathbb{T}^d} \nabla W(x-y)dm(y), \quad \kappa > 0$$

- Key observation

$$b(x, \mathrm{Leb}_{\mathbb{T}^d}) = 0$$

  - Lebesgue measure is always invariant!

- Positive definiteness condition (Ruelle, Carrillo...)

$$\forall \boldsymbol{k} \in \mathbb{Z}^d, \quad \widehat{W}^{\boldsymbol{k}} \geq 0$$

  - Lebesgue measure is the only invariant measure and exponentially stable

  - the linearized operator, at the Lebesgue measure, has spectral gap!

  $\Rightarrow$ by combining boths, we get the required results for the linearized operator, for any initial condition

Kuramoto model I

- Same as before but $d = 1$ and $W(x) = \pm\cos(2\pi x)$

  $\rightsquigarrow$ if $+\cos \Rightarrow$ same as above (interaction has repulsive effect)

  $\rightsquigarrow$ if $-\cos \Rightarrow$ interaction becomes attractive

- From now on, we focus on the attractive case

  ◦ known fact: there exists a threshold κ_c such that many invariant measures for $\kappa > \kappa_c$

  $\Rightarrow$ no uniform propagation of chaos! for the simple reason that

$$d\Big(\frac{1}{N}\sum_{i=1}^{N} X_t^i\Big) = -\frac{\kappa}{N^2}\underbrace{\sum_{i,j=1}^{N}\sin(X_t^i - X_t^j)}_{0}dt + \frac{1}{N}\sum_{i=1}^{N} dB_t^i$$

  ◦ two types of invariant measures: Leb_T and $(p_\infty(\cdot - \varphi))_{\varphi\in\mathbb{R}}$

  ◦ counter-example obtained by initializing form p_∞

Kuramoto model II

• Best result (Coppini) says that, if the initial distribution is close enough to $(p_\infty(\cdot - \varphi))_\varphi$

$\Rightarrow \bar{\mu}_t^N$ stays, with large probability, close to

$$(p_\infty(\cdot - \varphi))_\varphi$$

up until time $\exp(N^{1-})$

• Prompts us to assume that Φ is invariant by translation

$$\Phi\Big(m \circ (x \mapsto x + \theta)^{-1}\Big) = \Phi(m)$$

∘ same question as before but for an initial condition μ different from $\mathrm{Leb}_{\mathbb{T}}$

$$\mu \ll\gg \mathrm{Leb}_{\mathbb{T}} \quad : \quad \left| \int_{\mathbb{T}} \cos(2\pi\theta) d\mu(\theta) \right| \geq \eta > 0$$

Sktech of proof

- Exponential convergence of m_t^μ to some $p(\cdot - \varphi)$ for $\mu \ll\gg \mathrm{Leb}_{\mathbb{T}}$
- Study of the linearized operator works well when $\mu = p(\cdot - \varphi)$
- Combining the two

$$\left| d_t^{(1),i}(z) - c\frac{dm_t^\mu}{dx} \right| \leq C\exp(-\lambda t)$$

when μ is away from $\mathrm{Leb}_{\mathbb{T}}$ (and similarly for $d^{(2),i,j}(z)$

- Bounds for $\partial_x \frac{\delta P_t \Phi}{\delta m}(\mu)(x)$ and $\partial_x \partial_y \frac{\delta P_t \Phi}{\delta m^2}(\mu)(x,y)$ for $\mu \ll\gg \mathrm{Leb}_{\mathbb{T}}$
  - for $\bar{\mu}_0^N \ll\gg \mathrm{Leb}_{\mathbb{T}}$

$$\mathbb{E}\Big[\Phi\big(\mu_t^N\big) - \underbrace{\Phi\big(m_t^{\bar{\mu}_0^N}\big)}_{\sim_{t\nearrow\infty}\Phi(p)}\Big] \leq \frac{C}{N}, \quad t \leq \exp(N^{1-})$$

  - remains for t large because $\bar{\mu}_t^N$ cannot stay close to $\mathrm{Leb}_{\mathbb{T}}$