Optimal control of the Fokker-Planck equation with state constraint in the Wasserstein space

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Presentation Overview

1. Formulation of the problem, main results and assumptions
2. A penalized problem
3. From the penalized problem to the constrained one
4. Conclusion and related litterature

$$\inf_{(\alpha,m)} \int_0^T \int_{\mathbb{R}^d} L(x,\alpha(t,x))dm(t)(x)dt + \int_0^T f(m(t))dt + g(m(T))$$

where $(m,\alpha) \in \mathcal{C}([0,T],\mathcal{P}_2(\mathbb{R}^d)) \times L^2_{m(t)\otimes dt}([0,T]\times\mathbb{R}^d,\mathbb{R}^d)$ satisfy the Fokker-Planck equation

$$\begin{cases} \partial_t m + \mathrm{div}(\alpha m) - \Delta m = 0 & \text{in } (0,T)\times\mathbb{R}^d \\ m(0) = m_0 \in \mathcal{P}_2(\mathbb{R}^d) \end{cases}$$

and the state constraint $\Psi(m(t)) \leqslant 0$ for all $t \in [0,T]$.

Notations:

- T is a finite horizon
- running cost $L : \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R}$, convex in the second variable
- $f : \mathcal{P}_2(\mathbb{R}^d) \mapsto \mathbb{R}$ mean-field running cost
- $g : \mathcal{P}_2(\mathbb{R}^d) \mapsto \mathbb{R}$ mean field terminal cost
- $\Psi : \mathcal{P}_2(\mathbb{R}^d) \mapsto \mathbb{R}$ is the constraint

- This is motivated by the problem of quantile hedging in financial mathematics: see Föllmer and Leukert '1999, Bouchard et al. '2009.
- Typically $\Psi(m) = \int_{\mathbb{R}^d} h(x)dm(x)$ for some $h : \mathbb{R}^d \to \mathbb{R}$ (expectation constraint) or $\Psi(m) = F(\int_{\mathbb{R}^d} h_1(x)dm(x), \dots, \int_{\mathbb{R}^d} h_k(x)dm(x))$

Main questions:

- Existence of solutions (compactness / controllability)
- Characterization of solutions
- Regularity of optimal controls / optimal trajectories

Link with Mean Field Game theory

When there is no constraint ($\Psi = 0$) Lasry and Lions proved the following, where $H(x,p) := \sup_{q \in \mathbb{R}^d} -p.q - L(x,q)$

Theorem (Lasry/Lions 2007)

Under standing assumptions (precised later) optimal solutions exist and satisfy

$$\alpha(t,x) = -D_p H(x, Du(t,x))$$

for some (strong) solution (u,m) *of the mean field game system of pdes*

$$\begin{cases} -\partial_t u(t,x) + H(x, Du(t,x)) - \Delta u(t,x) = \dfrac{\delta f}{\delta m}(m(t),x) & \text{in } (0,T) \times \mathbb{R}^d \\ \partial_t m - \mathrm{div}(D_p H(x, Du(t,x))m) - \Delta m = 0 & \text{in } (0,T) \times \mathbb{R}^d \\ m(0) = m_0, \qquad u(T,x) = \dfrac{\delta g}{\delta m}(m(T),x) & \text{in } \mathbb{R}^d \end{cases}$$

For the associated mean-field game, u is the value function of an infinitesimal player
m is the density of the players at equilibrium

We define the Hamiltonian of the system by $H(x,p) := \sup_{q\in\mathbb{R}^d} \{-p.q - L(x,q)\}$.

H belongs to $\mathcal{C}^3(\mathbb{R}^d \times \mathbb{R}^d)$ and there exists a positive constant C such that, for all $(x,p) \in \mathbb{R}^d \times \mathbb{R}^d$

$$\begin{cases} -C + \frac{1}{C}|p|^2 \leqslant H(x,p) \leqslant C + C|p|^2 \\ H \text{ and its derivatives are bounded on sets of the form } \mathbb{R}^d \times B(0,R). \\ |D_x H(x,p)| \leqslant C(1+|p|) \\ \frac{1}{C} I_d \leqslant D^2_{pp} H(x,p) \leqslant C I_d \end{cases}$$

For $U = f, g, \Psi$, U is C^1 with bounded linear derivative and $x \mapsto \dfrac{\delta U}{\delta m}(m,x)$ belongs to $C^3_b(\mathbb{R}^d)$.

Ψ is a convex function with two (smooth in the space variables) linear derivatives and satisfies the geometric condition

$$\int_{\mathbb{R}^d} |D_m\Psi(m,x)|^2 dm(x) \geqslant \eta_1, \quad \text{whenever } |\Psi(m)| \leqslant \eta_2$$

for some $\eta_1, \eta_2 > 0$.

Recall that $\Psi : \mathcal{P}_2(\mathbb{R}^d) \mapsto \mathbb{R}$ has a (bounded) linear derivative at m if there exists a bounded continuous map $(m,x) \in \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \mapsto \frac{\delta \Psi}{\delta m}(m,x)$ such that, for all $m' \in \mathcal{P}_2(\mathbb{R}^d)$,

$$\lim_{\epsilon \to 0} \frac{\Psi((1-\epsilon)m + \epsilon m') - \Psi(m)}{\epsilon} = \int_{\mathbb{R}^d} \frac{\delta \Psi}{\delta m}(m,x) dm'(x).$$

We define the intrinsic derivative $D_m\Psi(m,x) := D_x \frac{\delta \Psi}{\delta m}(m,x)$ which corresponds to the Wasserstein gradient $\nabla_w \Psi$ from optimal transport.

In this case

$$\int_{\mathbb{R}^d} |D_m\Psi(m,x)|^2 dm(x) = \|\nabla_w \Psi(m)\|^2_{Tan_m(\mathcal{P}_2(\mathbb{R}^d))}.$$

Theorem (D. 2021)

Assume that $\Psi(m_0) < 0$ and the above assumptions hold. Then, optimal solutions exist and satisfy $\alpha = -D_pH(x, Du)$ for some solution (u, m, ν) of the PDE system

$$\begin{cases} -\partial_t u(t,x) + H(x, Du(t,x)) - \Delta u(t,x) \\ \qquad\qquad = \nu(t)\dfrac{\delta \Psi}{\delta m}(m(t), x) + \dfrac{\delta f}{\delta m}(m(t), x) & \text{in } (0,T)\times\mathbb{R}^d, \\ \partial_t m - \operatorname{div}(D_pH(x, Du(t,x))m) - \Delta m = 0 & \text{in } (0,T)\times\mathbb{R}^d, \\ u(T,x) = \nu(T)\dfrac{\delta\Psi}{\delta m}(m(T), x) + \dfrac{\delta g}{\delta m}(m(T), x) & \text{in } \mathbb{R}^d, \\ m(0) = m_0. \end{cases}$$

where $u \in W^{1,\infty}([0,T]\times\mathbb{R}^d)\bigcap\mathcal{C}([0,T], \mathcal{C}^3_b(\mathbb{R}^d))$, $m \in \mathcal{C}([0,T], \mathcal{P}_2(\mathbb{R}^d))$ *and* $\nu \in L^\infty([0,T])$ *satisfies*

$$\nu(t) = \begin{cases} 0 & \text{if } \Psi(m(t)) < 0 \\ \nu(t) \in \mathbb{R}^+ & \text{if } \Psi(m(t)) = 0. \end{cases}$$

Theorem (continued)

Moreover:

- *The value of the problem is*

$$\int_{\mathbb{R}^d} u(0,x)dm_0(x) + \int_0^T f(m(t))dt + g(m(T))$$

- *the optimal control is Lipschitz continuous in time and space.*
- *the map* $t \mapsto \Psi(m(t))$ *is* C^1 *in* $[0,T]$ *and* C^2 *in* $[0,T]\bigcap\{t, \Psi(m(t)) < 0\}$.
- *If* f, g *and* Ψ *are convex functions the conditions are sufficient: if* (u, m, ν) *is a solution of the above system and* $\Psi(m(t)) \leqslant 0$ *for all* $t \in [0,T]$ *then* $(m, -D_pH(x, Du))$ *is optimal.*

A penalized Problem

For small parameters $\epsilon, \delta > 0$ we consider the penalized problem

$$\inf_{(m,\alpha)} J_{\epsilon,\delta}(\alpha, m) \tag{$P_{\epsilon,\delta}$}$$

where the infimum is taken over the solutions (α, m) of the FP equation

$$\partial_t m + \mathrm{div}(\alpha m) - \Delta m = 0, \qquad m(0) = m_0$$

and $J_{\epsilon,\delta}$ is defined by

$$\begin{aligned} J_{\epsilon,\delta}(\alpha, m) &:= \int_0^T \int_{\mathbb{R}^d} L(x, \alpha(t,x)) dm(t)(x) dt + \int_0^T f(m(t)) dt \\ &\quad + \frac{1}{\epsilon} \int_0^T \Psi^+(m(t)) dt + g(m(T)) + \frac{1}{\delta} \Psi^+(m(T)) \\ &= J(\alpha, m) + \frac{1}{\epsilon} \int_0^T \Psi^+(m(t)) dt + \frac{1}{\delta} \Psi^+(m(T)). \end{aligned}$$

and $\Psi^+(m) = \Psi(m) \vee 0 = \max(\Psi(m), 0)$.

Remark

Not a standard problem because $r \mapsto \max(0, r)$ *is not differentiable at* 0.

Proposition

Problem $P_{\epsilon,\delta}$ *admits at least one solution and, for any solution* (α, m) *of* $P_{\epsilon,\delta}$ *there exist* $u \in \mathcal{C}([0,T], \mathcal{C}^3_b(\mathbb{R}^d))$, $\lambda \in L^\infty([0,T])$ *and* $\beta \in [0,1]$ *such that* $\alpha = -D_pH(x, Du(t,x))$ *and*

$$\begin{cases} -\partial_t u(t,x) + H(x, Du(t,x)) - \Delta u(t,x) \\ \qquad\qquad = \dfrac{\lambda(t)}{\epsilon}\dfrac{\delta \Psi}{\delta m}(m(t),x) + \dfrac{\delta f}{\delta m}(t, m(t), x) & \text{in } (0,T)\times\mathbb{R}^d, \\ \partial_t m - \mathrm{div}(D_pH(x, Du(t,x))m) - \Delta m = 0 & \text{in } (0,T)\times\mathbb{R}^d, \\ m(0) = m_0, \qquad u(T,x) = \dfrac{\beta}{\delta}\dfrac{\delta \Psi}{\delta m}(m(T),x) + \dfrac{\delta g}{\delta m}(m(T),x) & \text{in } \mathbb{R}^d., \end{cases}$$

Moreover, λ *and* β *satisfy*

$$\lambda(t) \begin{cases} = 0 & \text{if } \Psi(m(t)) < 0 \\ \in [0,1] & \text{if } \Psi(m(t)) = 0 \\ = 1 & \text{if } \Psi(m(t)) > 0 \end{cases}$$

$$\beta \begin{cases} = 0 & \text{if } \Psi(m(T)) < 0 \\ \in [0,1] & \text{if } \Psi(m(T)) = 0 \\ = 1 & \text{if } \Psi(m(T)) > 0. \end{cases}$$

Uniform estimates with respect to ϵ and δ

Question: How can we pass to the limit when $\epsilon, \delta \to 0$?

Recall that Ψ satisfies $\int_{\mathbb{R}^d} |D_m\Psi(m,x)|^2 dm(x) \geqslant \eta_1$ whenever $|\Psi(m)| \leqslant \eta_2$.

Lemma (Construction of admissible strategies for the constrained problem)

There is some trajectory (α, m) starting from m_0 satisfying $J(m,\alpha) < +\infty$ and $\Psi(m(t)) \leqslant \max(\Psi(m_0), -\eta_2)$ for all $t \in [0,T]$.

Proof.

- By a fixed point argument build a solution to

$$-\partial_t m - C\mathrm{div}(D_m\Psi(m(t),x)m) - \Delta m = 0, \quad m(0) = m_0$$

  for $C > 0$ sufficiently large.
- Use Itô's formula for flows of probability measures to find

$$\frac{d}{dt}\Psi(m(t)) = -C\int_{\mathbb{R}^d} |D_m\Psi(m(t),x)|^2 dm(t)(x) + \int_{\mathbb{R}^d} \mathrm{div}_x D_m\Psi(m(t),x) dm(t)(x)$$

- conclude that $\frac{d}{dt}\Psi(m(t)) \leqslant 0$ whenever $\Psi(m(t)) \geqslant \max(\Psi(m_0), -\eta_2)$ and therefore $\Psi(m(t)) \leqslant \max(\Psi(m_0), -\eta_2)$ for all $t \in [0,T]$.

Using the previous lemma and the convexity of Ψ we can prove the key estimate

Lemma

There is a constant $M > 0$ such that, for all $\epsilon, \delta > 0$ and for all tuple (u, m, λ, β) satisfying the optimality conditions for the penalized problem it holds

$$\frac{1}{\epsilon}\int_0^T \lambda(t)dt + \frac{\beta}{\delta} \leqslant M.$$

As a consequence we have the uniform estimates

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |D^k u(t,x)| \leqslant C(M)$$

for some $C(M) > 0$ and $k = 0, ..., 3$.

- with the above estimates we could pass to the limit and find a solution to the constrained problem with ν a priori in $\mathcal{M}([0, T])$ (finite measures over $[0, T]$).
- we can actually do better

- Goal: Show the existence of ϵ_0, δ_0 such that $\Psi(m^{\epsilon,\delta}(t)) \leqslant 0, \quad \forall t \in [0, T]$ whenever $m^{\epsilon,\delta}$ is a solution to the penalized problem with $\epsilon \leqslant \epsilon_0$ and $\delta \leqslant \delta_0$.
- Strategy: look at $t \mapsto \dfrac{d^2}{dt^2}\Psi(m^{\epsilon,\delta}(t))$ at maximum points of $t \mapsto \Psi(m^{\epsilon,\delta}(t))$

Proposition

Suppose that (m, u, λ, β) is a solution of the optimality conditions for the penalized problem for some $\epsilon, \delta > 0$. Then the map $t \to \Psi(m(t))$ is $\mathcal{C}^1$ in $[0, T]$ and $\mathcal{C}^2$ in $[0, T] \bigcap \{\Psi(m(t)) \neq 0\}$ with

$$\frac{d^2}{dt^2}\Psi(m(t)) = \frac{\lambda(t)}{\epsilon}\int_{\mathbb{R}^d} D_m\Psi(m(t), x).D^2_{pp}H(x, Du(t,x))D_m\Psi(m(t), x)dm(t)(x) \\ + F(Du(t), D^2u(t), D\Delta u(t), m(t))$$

for some functional $F : \mathcal{C}_b(\mathbb{R}^d, \mathbb{R}^d) \times \mathcal{C}_b(\mathbb{R}^d, \mathbb{S}^d(\mathbb{R})) \times \mathcal{C}_b(\mathbb{R}^d, \mathbb{R}^d) \times \mathcal{P}_2(\mathbb{R}^d)$ independent of ϵ and δ and bounded in sets of the form $\mathcal{A} \times \mathcal{P}_2(\mathbb{R}^d)$ for bounded subsets $\mathcal{A}$ of $\mathcal{C}_b(\mathbb{R}^d, \mathbb{R}^d) \times \mathcal{C}_b(\mathbb{R}^d, \mathbb{S}^d(\mathbb{R})) \times \mathcal{C}_b(\mathbb{R}^d, \mathbb{R}^d)$.

- We show that $\Psi(m^{\epsilon,\delta}(t)) \leqslant 0$ for all $t \in [0, T]$ whenever ϵ and δ are small enough.
- Suppose that $t \mapsto \Psi(m^{\epsilon,\delta}(t))$ has a maximum point at $\bar{t} \in (0, T)$ such that $\Psi(m^{\epsilon,\delta}(\bar{t})) > 0$. By second order condition it must hold

$$\frac{d^2}{dt^2}\Psi(m^{\epsilon,\delta}(\bar{t})) \leqslant 0.$$

- Using the previous proposition we get

$$\begin{aligned}
&\frac{d^2}{dt^2}\Psi(m^{\epsilon,\delta}(t)) \\
&\quad \sim \frac{1}{\epsilon}\int_{\mathbb{R}^d} D_m\Psi(m^{\epsilon,\delta}(t),x).D^2_{pp}H(x,Du^{\epsilon,\delta}(t,x))D_m\Psi(m^{\epsilon,\delta}(t),x)dm^{\epsilon,\delta}(t)(x) \\
&\quad \geqslant \frac{C}{\epsilon}\int_{\mathbb{R}^d} |D_m\Psi(m^{\epsilon,\delta}(t),x)|^2 dm^{\epsilon,\delta}(t)(x) \geqslant \frac{C}{\epsilon}
\end{aligned}$$

which leads to a contradiction if ϵ is small enough.

Conclusion

- We proved existence of optimal solutions for the problem with constraint
- We characterized the optimal solutions/controls with a MFG system associated with an exclusion condition
- We proved that optimal controls are Lipshitz-continuous in time and space

Related Literature

The paper on arkiv: D. Optimal control of the Fokker-Planck equation under state constraint in the Wasserstein space, 2021 arXiv:2109.14978

- Stochastic Control with probability constraints Föllmer/Leukert 1999', Bouchard/Elie/Imbert 2009', Bouchard/Elie/Touzi 2010', Tan/Touzi 2013',
- Constraints in law/Expectation constraints Pfeiffer 2020', Pfeiffer/Tan/Zhou 2020'; Chow/Yu/Zhou 2020'; D. 2020', Germain/Pham/Warin 2021'
- Applications in financial mathematics Guo/Langrené/Loeper/Ning 2019', Guo/Loeper/Wang 2019'
- Optimal control in the Wasserstein space Jimenez/Marigonda/Quincampoix 2020', Bonnet 2019', Bonnet/Frankowska 2021'
- Mean field game theory Lasry/Lions 2007'; Briani/Cardaliaguet 2018'
- First order mean field game with state constraint: Cannarsa/Capuani/Cardaliaguet 2019', 2021'

Thank you for your attention !