Introduction
000

Setting
00000

Solution to the MFG
00000000

Approximate NE for N-player game
00000

Conclusions
00

Mean-field games of finite-fuel capacity expansion with singular controls

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joint work with L. Campi (Milan), M. Ghio (Exprivia) and G. Livieri (SNS Pisa)

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CCA
Research
Education
Outreach

Introduction

A classical problem in stochastic singular control theory:
(see, e.g., Baldursson-Karatzas (1996), El Karoui-Karatzas (1988,1991))

- A firm produces a single good which is sold on the market at a price $(X_t)_{t\geq 0}$ that evolves as

$$\mathrm{d}X_t = a(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \quad X_0 = x.$$

- Revenues from sales are measured via a running profit $f(X_t, Y_t)$, where Y is a controlled process

$$Y_t^\xi = y + \xi_t, \quad \xi \text{ non-decreasing, right-cont., } \xi_{0-} = 0, \text{ s.t. } Y^\xi \in [0,1].$$

- $(Y_t)_{t\geq 0}$ measures the cumulative investment in, e.g., advertising, productive capacity, etc., and the firm's manager chooses $(\xi_t)_{t\geq 0}$ to maximise

$$\mathbb{E}\left[\int_0^T e^{-rt} f(X_t, y + \xi_t)\mathrm{d}t - \int_{[0,T]} e^{-rt} c_0 \mathrm{d}\xi_t\right]$$

where $c_0 > 0$ cost of investment and $r \geq 0$ subjective discount rate of the manager, $T > 0$ time horizon of the investment.

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Introduction
○○●

Setting
○○○○○

Solution to the MFG
○○○○○○○○

Approximate NE for N-player game
○○○○○

Conclusions
○○

Methods of solution: (Finding an optimal control ξ^*)

- Solving a (parabolic degenerate) free boundary problem with gradient constraint;

- First order conditions and Bank-El Karoui's representation theorem (2004);

- Connections with optimal stopping;

- ...

Extensions to multi-agent settings (including impulse controls):
Steg (2012), De Angelis-Ferrari (2018), Guo-Tang-Xu (2022), Basei-Cao-Guo (2022),
Aïd-Campi-Ludkovski (2021), Aïd-Basei-Campi-Callegaro-Vargiolu (2020).

N-player games are not very tractable! $\implies$ we adopt a MFG approach

MFG with singular controls

- Zhang (2012), Hu-Øksendal-Sulem (2014), Fu-Horst (2017), Guo-Xu (2019),
  Cao-Guo-Lee (2017), Guo-Lee (2018), Fu (2019)

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CCA
Research
Education
Outreach

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UNIVERSITÀ
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DI TORINO

CCA

Introduction
○○○

Setting
●○○○○

Solution to the MFG
○○○○○○○○

Approximate NE for N-player game
○○○○○

Conclusions
○○

Setting

Introduction
○○○

Setting
○●○○○○

Solution to the MFG
○○○○○○○○

Approximate NE for N-player game
○○○○○

Conclusions
○○

The N-player game:

- *N firms produce the same good.* The i-th firm's sale price is $(X_t^{N,i})_{t\geq0}$ and the level of investment is $(Y_t^{N,i})_{t\geq0}$, with $(X_0^{N,i}, Y_{0-}^{N,i}) \sim \nu$.

- The dynamics of the prices are coupled via the average investment across the sector, i.e.,

$$m_t^N := \frac{1}{N} \sum_{i=1}^{N} Y_t^{N,i}$$

and

$$dX_t^{N,i} = a(X_t^{N,i}, m_t^N)dt + \sigma(X_t^{N,i})dW_t^i,$$

with $(W^1, \ldots, W^N)$ a vector of 1D indep. standard Brownian motions, also indep. of $(X_0^{N,i}, Y_{0-}^{N,i})$.

- The i-th firm's manager chooses $(\xi_t^{N,i})_{t\geq0}$ to maximise

$$J^{N,i}(\xi^N) = E\left[\int_0^T e^{-rt} f(X_t^{N,i}, y + \xi_t^{N,i})dt - \int_{[0,T]} e^{-rt} c_0 d\xi_t^{N,i}\right]$$

under the finite-fuel constraint $Y^{N,i} \in [0,1]$.

UNIVERSITÀ
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DI TORINO

CCA

Introduction
000

Setting
0●0000

Solution to the MFG
00000000

Approximate NE for N-player game
00000

Conclusions
00

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Introduction
000

Setting
0●0000

Solution to the MFG
00000000

Approximate NE for N-player game
00000

Conclusions
00

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Education
Outreach
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The Mean-Field Game:
Letting $N \to \infty$ we expect $m_t^N \to m(t)$ where m is a measurable function $m : [0, T] \to [0, 1]$.

The dynamics in our MFG read

$$X_t = X_0 + \int_0^t a(X_s, m(s))\mathrm{d}s + \int_0^t \sigma(X_s)\mathrm{d}W_s, \quad Y_t^\xi = Y_{0-} + \xi_t, \quad t \in [0, T].$$

where $(W_t)_{t \geq 0}$ is a 1D Brownian motion, $(X_0, Y_{0-}) \sim \nu$ are indep. of W, and $(\xi_t)_{t \geq 0}$ is non-decr., right-cont., with $\xi_{0-} = 0$ and s.t. $Y^\xi \in [0, 1]$.

The goal of the "representative player" is to choose ξ that maximises

$$J(\xi) = \mathbb{E}\left[\int_0^T e^{-rt} f(X_t, Y_t^\xi)dt - \int_{[0,T]} e^{-rt} c_0 \, d\xi_t \right],$$

and at the same time $m(t) = \mathbb{E}[Y_t^\xi]$.

Remark: The MF feature is via the control process and it feeds into the drift of X.

Introduction
000

Setting
00●00

Solution to the MFG
00000000

Approximate NE for N-player game
00000

Conclusions
00

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UNIVERSITÀ
DEGLI STUDI
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CCA
Research
Education
Outreach

Introduction
000

Setting
○○●○○

Solution to the MFG
○○○○○○○○

Approximate NE for N-player game
○○○○○

Conclusions
○○

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Introduction
000

Setting
00●00

Solution to the MFG
00000000

Approximate NE for N-player game
00000

Conclusions
00

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Definition(Solution of the MFG of capacity expansion)

A solution of the MFG of capacity expansion with initial condition ν is a pair (m^*, ξ^*) with $m^* : [0, T] \to [0, 1]$ measurable and ξ^* (admissible) s.t.

(i) (Optimality) ξ^* is optimal, i.e.,

$$J(\xi^*) = V^\nu = \sup_\xi \mathbb{E}\left[\int_0^T e^{-rt} f(X_t^*, Y_t^\xi) \mathrm{d}t - \int_{[0,T]} e^{-rt} c_0 \mathrm{d}\xi_t \right],$$

where (X^*, Y^ξ) is the dynamics associated to (m^*, ξ).

(ii) (Consistency) Let (X^*, Y^*) be the dynamics associated to (m^*, ξ^*), then

$$m^*(t) = \mathbb{E}[Y_t^*], \quad t \in [0, T].$$

We will say that a MFG solution ξ^* is in *feedback form* if $\xi_t^* = \eta(t, X, Y_{0-})$, $t \in [0, T]$, for some non-anticipative mapping η.

Introduction
000

Setting
00000

Solution to the MFG
00000000

Approximate NE for N-player game
00000

Conclusions
00

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Introduction
000

Setting
00000

Solution to the MFG
00000000

Approximate NE for N-player game
00000

Conclusions
00

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UNIVERSITÀ
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DI TORINO

CCA

Introduction
000

Setting
○○○○●

Solution to the MFG
○○○○○○○○

Approximate NE for N-player game
○○○○○

Conclusions
○○

Assumptions:
There is a set of mild technical assumptions on $(x, m) \mapsto a(x, m)$, $x \mapsto \sigma(x)$ and $(x, y) \mapsto f(x, y)$. For simplicity in this talk let us consider

$$a(x, m) = (\mu + m)x, \ \sigma(x) = \sigma x \quad \text{with } \mu \in \mathbb{R}, \sigma > 0,$$

and

$$f(x, y) = x \cdot y^{\alpha}, \ \alpha \in (0, 1).$$

The structural conditions are $m \mapsto a(x, m)$ non-decreasing and $y \mapsto f(x, y)$ concave.

Introduction
ooo

Setting
ooooo

Solution to the MFG
●ooooooo

Approximate NE for N-player game
ooooo

Conclusions
oo

Solution to the MFG

Introduction
000

Setting
00000

Solution to the MFG
0●000000

Approximate NE for N-player game
00000

Conclusions
00

Theorem (Existence of solutions)
There exists an upper-semi continuous (u.s.c.) function $c : [0, T] \times \mathbb{R}_+ \to [0, 1]$, with $t \mapsto c(t, x)$ and $x \mapsto c(t, x)$ both non-decreasing, s.t. (m^*, ξ^*) given by

$$\xi_t^* := \sup_{0 \leq s \leq t} (c(s, X_s^*) - Y_{0-})^+, \quad m^*(t) := \mathbf{E}[Y_{0-} + \xi_t^*], \quad t \in [0, T],$$

is a solution of the MFG.

Remark. Here we are able not only to prove existence of a solution but also to construct the optimal control in terms of an u.s.c., monotone surface in the state space $[0, T] \times \mathbb{R}_+ \times [0, 1]$.

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Introduction
000

Setting
00000

Solution to the MFG
0000●00000

Approximate NE for N-player game
00000

Conclusions
00

Iterative construction of the solution

- Initialisation: $m^{[-1]}(t) \equiv 1$, for $t \in [0, T]$.

- n-th step, $n \geq 0$: fix a non-decreasing, right-cont. function $m^{[n-1]} : [0, T] \to [0, 1]$.

  For $(t, x, y) \in [0, T] \times \mathbb{R}_+ \times [0, 1]$, consider the dynamics

  $$X_{t+s}^{[n]} = x + \int_0^s a(X_{t+u}^{[n]}, m^{[n-1]}(t + u))du + \int_0^s \sigma(X_{t+u}^{[n]})dW_{t+u}.$$

  We define the singular control problem $SC_{t,x,y}^{[n]}$ as:

  $$v_n(t, x, y) := \sup_{\xi} E_{t,x}\left[\int_0^{T-t} e^{-rs} f(X_{t+s}^{[n]}, y + \xi_s)ds - \int_{[0,T-t]} e^{-rs} c_0 d\xi_s\right]$$

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Iterative construction of the solution

- Initialisation: $m^{[-1]}(t) \equiv 1$, for $t \in [0, T]$.

- n-th step, $n \geq 0$: fix a non-decreasing, right-cont. function $m^{[n-1]} : [0, T] \to [0, 1]$.

  For $(t, x, y) \in [0, T] \times \mathbb{R}_+ \times [0, 1]$, consider the dynamics

  $$X_{t+s}^{[n]} = x + \int_0^s a(X_{t+u}^{[n]}, m^{[n-1]}(t+u))du + \int_0^s \sigma(X_{t+u}^{[n]})dW_{t+u}.$$

  We define the singular control problem $\mathbf{SC}_{t,x,y}^{[n]}$ as:

  $$v_n(t, x, y) := \sup_{\xi} \mathbb{E}_{t,x} \left[\int_0^{T-t} e^{-rs} f(X_{t+s}^{[n]}, y + \xi_s)ds - \int_{[0,T-t]} e^{-rs} c_0 d\xi_s \right]$$

Introduction
000

Setting
00000

Solution to the MFG
000●0000

Approximate NE for N-player game
00000

Conclusions
00

- $(n+1)$-th step: assume there exists an opt. control $\xi^{[n]*}$ for $\mathbf{SC}_{0,x,y}^{[n]}$ for each $(x,y) \in \mathbb{R}_+ \times [0,1]$ (with $(x,y) \mapsto \xi^{[n]*}(x,y)$ measurable). Then, define

$$m^{[n]}(t) := \mathbb{E}\left[\gamma_{0-} + \xi_t^{[n]*}\right].$$

- The map $t \mapsto m^{[n]}(t)$ is non-decreasing and right-continuous (by dom. convergence) with values in $[0,1]$, so we can use it to define

$$X^{[n+1]} \quad \text{and} \quad v_{n+1}$$

by iterating the above construction.

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CCA

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Introduction
000

Setting
00000

Solution to the MFG
000●0000

Approximate NE for N-player game
00000

Conclusions
00

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Introduction
○○○

Setting
○○○○○

Solution to the MFG
○○○○○●○○○

Approximate NE for N-player game
○○○○○

Conclusions
○○

Solution of $SC^{[n]}$ via optimal stopping

One can prove that $\partial_y v_n(t, x, y) = u_n(t, x, y)$, where

$$u_n(t, x, y) := \inf_{0 \leq \tau \leq T-t} \mathbb{E}_{t,x} \left[\int_0^\tau e^{-rs} \partial_y f(X_{t+s}^{[n]}, y) ds + c_0 e^{-r\tau} \right],$$

which is an easier problem to solve.

There exists a unique u.s.c. function $c_n : [0, T] \times \mathbb{R}_+ \to [0, 1]$, with $t \mapsto c_n(t, x)$ and $x \mapsto c_n(t, x)$ non-decreasing, s.t. the minimal OS time is

$$\tau_*^{[n]}(t, x, y) = \inf\{s \in [0, T-t] : c_n(t+s, X_{t+s}^{[n]}) \geq y\}.$$

We prove that the (unique) optimal control in $SC^{[n]}$ reads

$$\xi_{t+s}^{[n]*} := \sup_{0 \leq u \leq s} \left(c_n(t+u, X_{t+u}^{[n]}) - y \right)^+, \quad \xi_{t-}^{[n]*} = 0.$$

UNIVERSITÀ
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CCA

Introduction
000

Setting
00000

Solution to the MFG
00000●000

Approximate NE for N-player game
00000

Conclusions
00

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CCA

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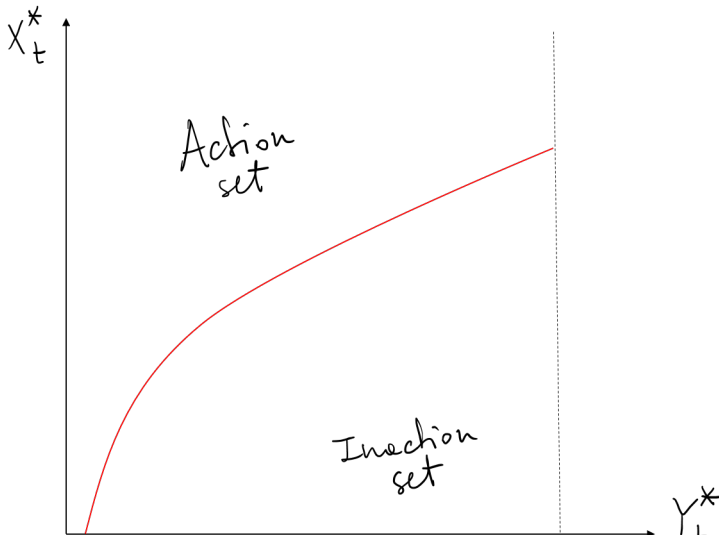
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UNIVERSITÀ
DEGLI STUDI
DI TORINO

CCA

Introduction
000

Setting
00000

Solution to the MFG
0000000●0

Approximate NE for N-player game
00000

Conclusions
00

Proposition (Monotonicity of the scheme)

For $n \geq 0$ we have

$$u_n \geq u_{n+1} \implies c_n \geq c_{n+1} \implies \xi^{[n]*} \geq \xi^{[n+1]*}$$
$$\implies m^{[n]} \geq m^{[n+1]} \implies X^{[n]} \geq X^{[n+1]}.$$

Then, defining

$$c := \lim_{n \to \infty} c_n, \quad m^* := \lim_{n \to \infty} m^{[n]}, \quad X^* := \lim_{n \to \infty} X^{[n]}, \quad \xi^* := \lim_{n \to \infty} \xi^{[n]}$$

we have $m^*(t) = \mathbb{E}[Y_{0-} + \xi_t^*]$ (consistency),

$$X_{t+s}^* = x + \int_0^s a(X_{t+u}^*, m^*(t+u))du + \int_0^s \sigma(X_{t+u}^*)dW_{t+u},$$

$$\xi_{t+s}^* = \xi_{t+s}^*(t, x, y) = \sup_{0 \leq u \leq s} (c(t+u, X_{t+u}^*) - y)^+.$$

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Introduction
000

Setting
00000

Solution to the MFG
0000000●

Approximate NE for N-player game
00000

Conclusions
00

For $(t, x, y) \in [0, T] \times \mathbb{R}_+ \times [0, 1]$ the process ξ^* is the unique maximiser of

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Integrating the payoff over the distribution ν of (X_0, Y_{0-}) the pair (m^*, ξ^*) translates into the solution of MFG.

Introduction
000

Setting
00000

Solution to the MFG
0000000●

Approximate NE for N-player game
00000

Conclusions
00

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Introduction
ooo

Setting
ooooo

Solution to the MFG
oooooooo

Approximate NE for N-player game
●oooo

Conclusions
oo

Approximate NE for N-player game

Definition (ε-Nash equilibrium)
Let $\varepsilon \geq 0$, a strategy vector ξ^{ε} is a ε-NE for the N-player game if for every $i = 1, \ldots, N$, and for every strategy vector ξ^i,

$$J^{N,i}(\xi^{\varepsilon}) \geq J^{N,i}([\xi^{\varepsilon,-i}, \xi^i]) - \varepsilon.$$

Theorem
Assume $x \mapsto c(t, x)$ is Lipschitz unif. for $t \in [0, T]$. Let (m^*, ξ^*) be the feedback MFG solution above, i.e., $\xi_t^* = \eta^*(t, X^*, Y_{0-})$, where

$$\eta^*(t, \varphi, y) := \sup_{0 \leq s \leq t} \left(c(s, \varphi(s)) - y \right)^+.$$

Set

$$\hat{\xi}^{N,i} := \eta^*(t, X^{N,i}, Y_{0-}^i),$$

then $\hat{\xi}^N$ is a ε_N-Nash eq for the N-player game of capacity expansion with $\varepsilon_N \to 0$ as $N \to \infty$.
(In the example the rate of convergence is of order $1/\sqrt{N}$)

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Research
Education
Outreach
CCA

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Sketch of the proof: three main steps

(i) We prove that $J^{N,1}(\hat{\xi}^N) \to J(\xi^*)$ as $N \to \infty$.

(ii) Recalling the notation $[\hat{\xi}^{N,-1}, \xi] = (\xi, \hat{\xi}^{N,2}, \ldots, \hat{\xi}^{N,N})$, we prove

$$\limsup_{N \to \infty} \sup_{\xi} J^{N,1}([\hat{\xi}^{N,-1}, \xi]) \le J(\xi^*) = V^\nu. \tag{1}$$

(iii) Combining (i) and (ii), for any $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that

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Remark. Lipschitz $c(t, \cdot)$ is used to show that $X^{N,1} \to X^*$ as $N \to \infty$ with Gronwall's type estimates.

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UNIVERSITÀ
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DI TORINO

CCA

Introduction
000

Setting
00000

Solution to the MFG
00000000

Approximate NE for N-player game
000●0

Conclusions
00

We give sufficient conditions for Lipschitz continuity of $x \mapsto c(t, x)$. For example

$$\mathrm{d}X_t = (\mu + m(t))X_t \mathrm{d}t + \sigma X_t \mathrm{d}W_t \quad \text{and} \quad f(x, y) = x \cdot y^\alpha, \ \alpha \in (0, 1).$$

Introduction
ooo

Setting
ooooo

Solution to the MFG
oooooooo

Approximate NE for N-player game
ooooo●

Conclusions
oo

Economic interpretation for the N-player game
A game version of the goodwill problem (Buratto-Viscolani (2002), Marinelli (2007), Jack et al. (2008)):

- Players are firms producing the same good (e.g. mobile phones), which they want to advertise;

- $X^{N,i}$ is market price of i-th firm's product; $Y^{N,i}$ is i-th firm's cumulative marketing investment (finite-fuel $\rightarrow$ maximum budget for advertising);

- $c_0 d\xi$ is the (proportional) cost of advertising;

- investing $\Delta\xi^{N,i} > 0$ has a cost $c_0\Delta\xi^{N,i}$ with two effects:

  - increases firm's popularity $\Rightarrow$ increases firm's profit (due to $y \mapsto f(x,y)$ increasing);
  - increases visibility of the type of product $\Rightarrow$ demand and price increase for all players ($m \mapsto a(x,m)$ nondecreasing).

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UNIVERSITÀ
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DI TORINO

CCA
Research
Education
Outreach

Introduction
000

Setting
00000

Solution to the MFG
00000000

Approximate NE for N-player game
0000●

Conclusions
00

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Research
Education
Outreach

CCA

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Research
Education
Outreach
CCA

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UNIVERSITÀ
DEGLI STUDI
DI TORINO

CCA

Introduction
○○○

Setting
○○○○○

Solution to the MFG
○○○○○○○○

Approximate NE for N-player game
○○○○●

Conclusions
○○

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CCA

Our contribution

- We formulate and solve a MFG of finite-fuel capacity expansion with *singular controls* associated to a N-player game

- Under mild assumptions, we construct a feedback solution of the MFG of capacity expansion

- Our constructive approach allows us to determine the MFG optimal control in terms of an optimal boundary $(t, x) \mapsto c(t, x)$ splitting the state space into *action* and *inaction* regions

- The MFG solution induces a sequence of approximate ε_N-Nash for the N-player games with vanishing error at rate $O(1/\sqrt{N})$ as $N \to \infty$.

- Finally, we provide sufficient conditions ensuring the boundary's Lipschitz continuity and give examples.

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CCA

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UNIVERSITÀ DEGLI STUDI DI TORINO

CCA
Research Education Outreach

Introduction
000

Setting
00000

Solution to the MFG
00000000

Approximate NE for N-player game
00000

Conclusions
●○

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UNIVERSITÀ
DEGLI STUDI
DI TORINO

CCA

Introduction
000

Setting
00000

Solution to the MFG
00000000

Approximate NE for N-player game
00000

Conclusions
●○

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- Finally, we provide sufficient conditions ensuring the boundary's Lipschitz continuity and give examples.

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Thank you

1. Campi, De Angelis, Ghio, Livieri
   Mean-field games of finite-fuel capacity expansion with singular controls
   arXiv:2006.02074
   To appear in *Ann. Appl. Probab.* 2022