On the Weak Representation Property in Progressively Enlarged Filtrations

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9th Colloquium on BSDEs and Mean Field Systems
Annecy 2022

Definition of the Weak Representation Property

- Let $\mathbb{F}$ be a filtration satisfying the usual conditions.
- Let X be a d-dimensional $\mathbb{F}$-semimartingale with characteristics $(B^{\mathbb{F},X}, C^{\mathbb{F},X}, \nu^{\mathbb{F},X})$ w.r.t. the standard truncation function:

$$h(x) := x1_{\{|x|\leq 1\}}, \quad x \in \mathbb{R}^d.$$

- X^c is the continuous martingale part of X.

Definition (weak predictable martingale representation property)

X has the WRP with respect to $\mathbb{F}$ if every $\mathbb{F}$-local martingale M can be represented as follows:

$$M_t = M_0 + \int_0^t K_s \mathrm{d}X_s^c + \int_0^t \int_{\mathbb{R}^d} W(s,x)(\mu^X - \nu^X)(\mathrm{d}s, \mathrm{d}x), \quad t \geq 0.$$

where K is an $\mathbb{F}$-predictable process and W an $\mathbb{F}$-predictable function.

Examples

- X is a local martingale with the PRP: $M = M_0 + \int_0^t K_s \mathrm{d}X_s$.
  - X is a Brownian motion and $\mathbb{F} = \mathbb{F}^X$.
  - X is a compensated Poisson process and $\mathbb{F} = \mathbb{F}^X$ (or $X = Y - Y^p$ is a compensated Point process and $\mathbb{F} = \mathbb{F}^X \vee \mathscr{R}$).
  - X is a solution of $[X, X]_t - t = \beta \int_0^t X_{s-} \mathrm{d}X_s$ with $\beta \in [-2, 0]$, $\mathbb{F} = \mathbb{F}^X$.
- X is a Lévy process and $\mathbb{F} = \mathbb{F}^X$.
- X has conditionally independent increments and $\mathbb{F} = \mathbb{F}^X \vee \mathscr{R}$.
- X is a step process $X = \sum_{n=1}^{\infty} \xi_n 1_{[T_n, \infty)}$ and $\mathbb{F} = \mathbb{F}^X \vee \mathscr{R}$.

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- In general

$$\text{PRP} \Longrightarrow \text{WRP}$$

$$\text{WRP} \not\Longleftarrow\!\!\!\Longrightarrow \text{PRP}$$

WRP and Filtrations 1

If X has the WRP wrt $\mathbb{F}$ and $\mathbb{G} \supseteq \mathbb{F}$, then the WRP is not preserved in $\mathbb{G}$!

Examples

Take a Brownian motion B with respect to $\mathbb{F}^B$.

- Define $\mathbb{G} = (\mathscr{G}_t)_{t \geq 0}$ by $\mathscr{G}_t = \mathscr{F}^B_t \vee \mathscr{F}^B_\infty$. Then an $\mathbb{F}^B \subseteq \mathbb{G}$ and $\mathbb{F}^B$-semimartingale X is a $\mathbb{G}$-semimartingale iff X is $\mathbb{F}^B$-adapted and has paths of finite variation. So B is even not a $\mathbb{G}$-semimartingale.

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- Define $\mathbb{G}' := \mathbb{F}^B \vee \mathbb{F}^N$, where N is an independent standard Poisson process. Then B is again a $\mathbb{G}'$-Brownian motion but B does not possess the PRP with respect to $\mathbb{G}'$ because the $\mathbb{G}'$-martingale $(N_t - t)_{t \geq 0}$ is orthogonal to B.

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- Less trivial example by Jeluin and Yor (Faux-Amis) for initial enlargement, adapted by Aksamit to progressive enlargement.

WRP and Filtrations 2

If $\mathbb{F}$ is enlarged to $\mathbb{G}$, then the following natural questions arise

1. How do $\mathbb{F}$-martingales change, when the filtration $\mathbb{F}$ is changed to $\mathbb{G}$?
2. If X has the WRP wrt $\mathbb{F}$, is it possible to establish a new predictable WRP in the enlarged filtration $\mathbb{G}$? With respect to which semimartingales?

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Because of the WRP in $\mathbb{G}$ the problem up to $T \wedge \tau$ can be solved as the one up to T.

General Formulation of the Problem

Meta-Theorem

- $(X, \mathbb{F})$ $\mathbb{R}^d$-valued semimartingale with the WRP wrt $\mathbb{F}$.
- $(Y, \mathbb{H})$ $\mathbb{R}^\ell$-valued semimartingale with the WRP wrt $\mathbb{H}$.
- Define $\mathbb{G} = (\mathscr{G}_t)_{t\geq 0}$ by

$$\mathscr{G}_t := \bigcap_{\varepsilon>0} (\mathscr{F}_{t+\varepsilon} \vee \mathscr{H}_{t+\varepsilon}).$$

Add sufficient conditions to:

- Ensure that X and Y are $\mathbb{G}$ semimartingales;
- Show that the $\mathbb{R}^{d+\ell}$-valued $\mathbb{G}$-semimartingale $Z = (X, Y)$ has the WRP wrt $\mathbb{G}$: Every $\mathbb{G}$-local martingale M has the representation

$$M = M_0 + \int_0^\cdot K_s \mathrm{d}Z^c + \int_0^\cdot \int_{\mathbb{R}^{d+\ell}} W(s, x_1, x_2)(\mu^Z - \nu^{Z,\mathbb{G}})(\mathrm{d}s, \mathrm{d}x_1, \mathrm{d}x_2)$$

Main Idea

- Filtration $\mathbb{F}$ with an $\mathbb{R}^d$-valued semimartingale X with a WRP in $\mathbb{F}$.
- Filtration $\mathbb{H}$ with an $\mathbb{R}^\ell$-valued Semimartingale Y with a WRP in $\mathbb{H}$.
- $\mathbb{G}$ smallest right continuous filtration containing both $\mathbb{F}$ and $\mathbb{H}$.

Plan

- Ensure that X and Y are both $\mathbb{G}$-semimartingales.
- Regard the progressive enlargement as increase of the dimension: Put $Z = (X, Y)$

$$(X, \mathbb{F}) \text{ in } \mathbb{R}^d, (Y, \mathbb{H}) \text{ in } \mathbb{R}^\ell \xrightarrow{\quad\mathbb{G}\quad} (Z, \mathbb{G}) \text{ in } \mathbb{R}^{d+\ell}.$$

- Use semimartingale calculus for Z in $\mathbb{G}$ and, if possible, compute the characteristics of Z in $\mathbb{G}$.

Progressively Enlargement by a Random Time 1

$\tau : \Omega \longrightarrow (0, +\infty]$ is a random variable but not an $\mathbb{F}$ stopping time.

Progressive enlargement of $\mathbb{F}$

Set $Y = 1_{[\tau,+\infty)}$ and $\mathbb{H} = (\mathscr{H}_t)_{t\geq 0}$ with $\mathscr{H}_t = \sigma(\tau \wedge t)$, $t \geq 0$.

$$\mathbb{G} = (\mathscr{G}_t)_{t\geq 0}, \qquad \mathscr{G}_t := \bigcap_{\varepsilon>0} \mathscr{F}_{t+\varepsilon} \vee \mathscr{H}_{t+\varepsilon}, \qquad t \geq 0.$$

$\mathbb{G}$ is the smallest right-con. filtration: $\mathbb{G} \supseteq \mathbb{F}$ and τ is $\mathbb{G}$-stopping time.

Interpretation

τ: Occurrence time of an external event E not known from the info in $\mathbb{F}$.
In $\mathbb{G}$ at $t \geq 0$ we recognize if E has occurred or not ($\{\tau \leq t\} \in \mathscr{G}_t$).

Progressively Enlargement by a Random Time 2

- **Credit risk:** A random time τ describes the time-of-default of part of the market. To make inference about τ the information available in the market could be not sufficient. Therefore it is convenient to enlarge the market-information in such a way that at every time t one can say if $\{\tau \leq t\}$ has occurred or not (default up to t).

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- **Life insurance**: τ models the death-time of an agent. A contract in this context has a maturity $T > 0$ such that $\mathbb{P}[\tau \leq T] > 0$. If the agent wants to maximize her expected utility function U from a wealth process $W^{x,\theta}$, then she has to solve $\sup_{\theta \in \Theta} \mathbb{E}[U(W^{x,\theta}_{T\wedge\tau} - \xi)]$.

Random Times under Avoidance and Immersion

Avoidance and Immersion

Let $\mathbb{F}$ be a filtration and let $\tau : \Omega \longrightarrow (0, +\infty]$ be a random time. Let $\mathbb{G}$ denote the progressive enlargement of $\mathbb{F}$ by τ.

Immersion and Avoidance

1. Immersion: $\mathbb{F}$-martingales remain $\mathbb{G}$-martingales.
2. Avoidance: Let σ be an $\mathbb{F}$ stopping time. Then $\mathbb{P}[\tau = \sigma < +\infty] = 0$.

Examples of Immersion: τ is independent of $\mathbb{F}$; τ is constructed via the Cox-method.

Meaning of Avoidance: τ is not an $\mathbb{F}$ stopping time and we add to $\mathbb{F}$ a completely new information.

Why Avoidance

- We have to exclude the trivial case in which τ is an $\mathbb{F}$-stopping time.
- Simpler computations: X and $Y = 1_{[\tau, +\infty)}$ have no common jumps.

Propagation of the Weak Representation Property

τ random time, $Y := 1_{[\tau,\infty)}$ and $Y - \Lambda^{\mathbb{G}}$ $\mathbb{G}$-martingale ($\Lambda^{\mathbb{G}}$ compensator).

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Theorem 1 (DT 2020)

If $\mathbb{F}$ is immersed in $\mathbb{G}$, τ avoids $\mathbb{F}$-stopping times and X has the WRP with respect to $\mathbb{F}$, then

- X is a $\mathbb{G}$-semimartingale with characteristics $(B^{\mathbb{F},X}, C^{\mathbb{F},X}, \nu^{\mathbb{F},X})$.

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- The $\mathbb{G}$-compensator $\nu^{\mathbb{G},Z}$ of the jump-measure μ^Z of $Z = (X, H)$ is

$$\nu^{\mathbb{G},Z}(\mathrm{d}t, \mathrm{d}x_1, \mathrm{d}x_2) = \nu^{\mathbb{F},X}(\mathrm{d}t, \mathrm{d}x_1)\delta_0(\mathrm{d}x_2) + \mathrm{d}\Lambda_t^{\mathbb{G}}\delta_1(\mathrm{d}x_2)\delta_0(\mathrm{d}x_1)$$

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Generalizations

Avoidance and immersion are quite strong assumptions.

- Avoidance means that τ in $\mathbb{G}$ is a totally inaccessible time. However, there are cases in which τ has an accessible component.

- Immersion is widely used in credit risk but it has some drawbacks: It is not preserved by equivalent changes of measure.

Question

Is it possible to weaken these assumptions?

Enlargement by a whole process without avoidance

Independent Enlargement

Theorem 2 (DT 2021)

- $(X, \mathbb{F})$ is an $\mathbb{R}^d$-valued (semi)martingale with a WRP in $\mathbb{F}$.
- $(Y, \mathbb{H})$ is a $\mathbb{R}^\ell$-valued (semi)martingale with a WRP in $\mathbb{H}$.
- $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$ and $\mathbb{F}$ and $\mathbb{H}$ are independent.

Set $Z = (X, Y)$. Every $\mathbb{G}$-local martingale M can be represented as

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- If $Y = 1_{[\tau, +\infty)}$ Thm. 2 is not a special case of Thm. 1 (avoidance).
- If τ satisfies Jacod's **equivalence** assumption, then we can deduce the WRP of $Z = (X, 1_{[\tau,+\infty)})$ in $\mathbb{G}$ as a corollary of Thm. 2 also in case the uncoditional law of τ is not continuous (extending Callegaro et al. 2013).

Idea of the Proof for the Independent Enlargement

$(M, \mathbb{F})$ and $(N, \mathbb{H})$ bounded martingales. Independence $\Longrightarrow$ MN $\mathbb{G}$-mart.

Aim

It is enough to represent MN is terms of the WRP of $Z = (X, Y)$ since the random variables of the form $M_\infty N_\infty$ generate $\mathscr{G}_\infty$.

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- If X or Y are quasi-left continuous $[M, N] = 0$ (Xue 1993).
- If X and Y may charge the same predictable jumps, then $[M, N] \neq 0$ is a $\mathbb{G}$-local martingale. Integration by parts and WRP yield

$$M_t N_t = \textbf{Friendly part} + [M, N]_t.$$

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Problem

The difficulty is to find an adequate representation of $[M, N]$ as a stochastic integral wrt $\mu^{(X,Y)} - \nu^{\mathbb{G},(X,Y)}$.

Enlargement by a whole process for step processes

Step Processes

Definition

An adapted $\mathbb{R}^d$-valued process X is called a *step process* w.r.t. $\mathbb{F}$ if $X_0 = 0$

$$X_t = \sum_{n=1}^{\infty} \xi_n 1_{\{\tau_n \le t\}}, \quad t > 0$$

where

- $(\tau_n)_n$ $\mathbb{F}$-stopping times with $\tau_n \uparrow +\infty$ and $\tau_n < \tau_{n+1}$ on $\{\tau_n < +\infty\}$;
- ξ_n is $\mathbb{F}_{\tau_n}$-measurable and $\xi_n \neq 0$ if and only if $\tau_n \neq 0$.

If $\xi_n \equiv 1$ we call X a point process.

A step process is always a semimartingale, provided it is adapted!

An example

Example

- X is a Poisson process with respect to $\mathbb{F}^X$, $\mathscr{R}$ is assumed trivial.
- $\tau = X_T + 1$, where $T > 0$ is arbitrary but fixed.
- $H = 1_{[\tau,+\infty)}$ is a point process in $\mathbb{H}$.

- τ does not avoid $\mathbb{F}^X$-stopping times: $\mathbb{P}[\tau = n] > 0$, $\forall n$.
- τ is not independent of $\mathbb{F}^X$ (even not under an equivalent measure).
- $\mathbb{F}^X$ is not immersed in $\mathbb{G} = \mathbb{F}^X \vee \mathbb{H}$.
- τ does not satisfies Jacod's equivalence condition (but it satisfies Jacod's absolute continuity)!

Summary

None of the results from the literature can be applied!!!

WRP for Point and Step Processes

Theorem (DT, Jeanblanc 2021)

Let $(X, \mathbb{F})$ and $(Y, \mathbb{H})$ be point processes, where $\mathbb{F} = \mathbb{F}^X \vee \mathscr{R}$ and $\mathbb{H} = \mathbb{F}^Y$. Let $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$. Define the $\mathbb{R}^2$-valued semimartingale $Z = (X, Y)$. Then every $\mathbb{G}$-local martingale M can be represented as

$$M_t = M_0 + \int_0^t \int_{\mathbb{R}^2} W(s,x)\mu^Z(\mathrm{d}s, \mathrm{d}x) - \int_0^t \int_{\mathbb{R}^2} W(s,x)\nu^{Z,\mathbb{G}}(\mathrm{d}s, \mathrm{d}x)$$

where W is a $\mathbb{G}$-predictable function such that $\int_0^\cdot \int_{\mathbb{R}^2} |W(s,x)|\mu^Z(\mathrm{d}s, \mathrm{d}x)$ is a locally integrable process.

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Step Processes

Replacing in the above theorem the word *point* by *step*, an analogous result holds (Bandini, Confortola, DT '21).

Representation by Local Martingales for Point Processes

Theorem (DT, Jeanblanc 2021)

Let $(X, \mathbb{F})$ and $(H, \mathbb{H})$ be point processes, where $\mathbb{F} = \mathbb{F}^X \vee \mathscr{R}$ and $\mathbb{H} = \mathbb{F}^H$. Let $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$. Define the $\mathbb{G}$-local martingales

- $Z^1 := X - [X, H] - (X - [X, H])^{p,\mathbb{G}}$.
- $Z^2 := H - [X, H] - (H - [X, H])^{p,\mathbb{G}}$.
- $Z^3 := [X, H] - [X, H]^{p,\mathbb{G}}$.

Then, every $\mathbb{G}$-local martingale Y can be represented as

$$Y_t - Y_0 = \int_0^t K_s^1 \mathrm{d}Z_s^1 + \int_0^t K_s^2 \mathrm{d}Z_s^2 + \int_0^t K_s^3 \mathrm{d}Z_s^3$$

where K^i is a $\mathbb{G}$-predictable process, $i = 1, 2, 3$. If in addition $Y \in \mathscr{H}^2_{\mathrm{loc}}(\mathbb{G})$ and Z^1, Z^2 and Z^3 are pairwise orthogonal, then this is an orthogonal representation of Y.

Thank you for your attention!!!