Submodular mean field games with common noise: Existence and approximation of strong solutions

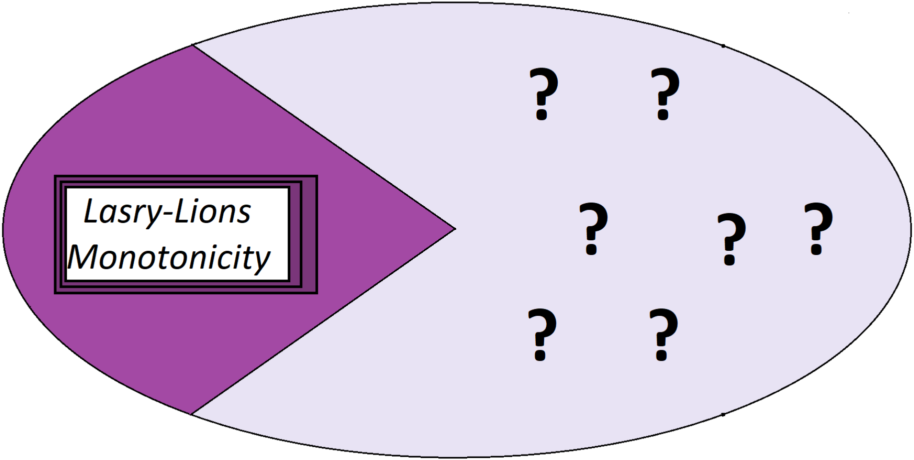
Jodi Dianetti[1]

joint work with Giorgio Ferrari[1], Markus Fischer[2] and Max Nendel[1]

[1] Center for Mathematical Economics, Bielefeld University
Collaborative Research Center 1283
[2] Dipartimento di Matematica, Università degli Studi di Padova

9th International Colloquium on BSDEs and Mean Field Systems
Annecy, June 27, 2022

Structural conditions on MFG models?



Motivations

Multiple solutions in MFGs?

- [Bardi-Fischer (2019)] presented a class of MFGs with multiple solutions
- [Cecchin-Dai Pra-Fischer-Pelino (2019)] convergence problem in a two state model without uniqueness
- [Delarue-Tchuendom (2019)] selection of equilibria in a linear quadratic MFG

Motivations

Multiple solutions in MFGs?

- [Bardi-Fischer (2019)] presented a class of MFGs with multiple solutions
- [Cecchin-Dai Pra-Fischer-Pelino (2019)] convergence problem in a two state model without uniqueness
- [Delarue-Tchuendom (2019)] selection of equilibria in a linear quadratic MFG

Strong solutions to MFGs with COMMON NOISE under lack of uniqueness?

- [Carmona-Delarue-Lacker 16] probabilistic formulation of MFGs with common noise: weak solutions via topological fixed point theorem, strong solutions via UNIQUENESS

Motivations

Multiple solutions in MFGs?

- [Bardi-Fischer (2019)] presented a class of MFGs with multiple solutions
- [Cecchin-Dai Pra-Fischer-Pelino (2019)] convergence problem in a two state model without uniqueness
- [Delarue-Tchuendom (2019)] selection of equilibria in a linear quadratic MFG

Strong solutions to MFGs with COMMON NOISE under lack of uniqueness?

- [Carmona-Delarue-Lacker 16] probabilistic formulation of MFGs with common noise: weak solutions via topological fixed point theorem, strong solutions via UNIQUENESS

Learning procedures in MFGs?

- [Cardaliaguet-Hadikhanloo (2017)] convergence of the *fictitious play* in *potential* MFGs
- [Dumitrescu-Leutscher-Tankov (2022)] fictitious play for MFGs of stopping and MFGs with absorption
- [Perrin-Perolat-Laurière-Geist-Elie-Pietquin (2020), Xie-Yang-Wang-Minca (2020), etc.] Fictitious play combined with machine learning techniques

Motivations

Multiple solutions in MFGs?

- [Bardi-Fischer (2019)] presented a class of MFGs with multiple solutions
- [Cecchin-Dai Pra-Fischer-Pelino (2019)] convergence problem in a two state model without uniqueness
- [Delarue-Tchuendom (2019)] selection of equilibria in a linear quadratic MFG

Strong solutions to MFGs with COMMON NOISE under lack of uniqueness?

- [Carmona-Delarue-Lacker 16] probabilistic formulation of MFGs with common noise: weak solutions via topological fixed point theorem, strong solutions via UNIQUENESS

Learning procedures in MFGs?

- [Cardaliaguet-Hadikhanloo (2017)] convergence of the *fictitious play* in *potential* MFGs
- [Dumitrescu-Leutscher-Tankov (2022)] fictitious play for MFGs of stopping and MFGs with absorption
- [Perrin-Perolat-Laurière-Geist-Elie-Pietquin (2020), Xie-Yang-Wang-Minca (2020), etc.] Fictitious play combined with machine learning techniques

Submodularity

in games: economic literature: [Topkis (1979)], [Milgrom-Roberts (1990)], [Vives (1990)]

Motivations

Multiple solutions in MFGs?

- [Bardi-Fischer (2019)] presented a class of MFGs with multiple solutions
- [Cecchin-Dai Pra-Fischer-Pelino (2019)] convergence problem in a two state model without uniqueness
- [Delarue-Tchuendom (2019)] selection of equilibria in a linear quadratic MFG

Strong solutions to MFGs with COMMON NOISE under lack of uniqueness?

- [Carmona-Delarue-Lacker 16] probabilistic formulation of MFGs with common noise: weak solutions via topological fixed point theorem, strong solutions via UNIQUENESS

Learning procedures in MFGs?

- [Cardaliaguet-Hadikhanloo (2017)] convergence of the *fictitious play* in *potential* MFGs
- [Dumitrescu-Leutscher-Tankov (2022)] fictitious play for MFGs of stopping and MFGs with absorption
- [Perrin-Perolat-Laurière-Geist-Elie-Pietquin (2020), Xie-Yang-Wang-Minca (2020), etc.] Fictitious play combined with machine learning techniques

Submodularity
in games: economic literature: [Topkis (1979)], [Milgrom-Roberts (1990)], [Vives (1990)]
in MFGs: - [Adlakha -Johari (2013)] discrete time MFG with infinite horizon discounted costs
- [Carmona-Delarue-Lacker (2017)] for MFGs of timing to study a model of bank runs in a continuous time setting

Submodular MFGs: Itô type dynamics, regular controls, common noise

- Let $T > 0$ be a fixed time horizon, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})$ complete filtered probability space

- Let $T > 0$ be a fixed time horizon, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ complete filtered probability space
- Let W and B be independent Brownian motions, B is the COMMON NOISE

- Let $T > 0$ be a fixed time horizon, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ complete filtered probability space
- Let W and B be independent Brownian motions, B is the COMMON NOISE
- $\mathbb{F}^B = (\mathcal{F}_t^B)_{t\in[0,T]}$ the (extended) filtration generated by B

- Let $T > 0$ be a fixed time horizon, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ complete filtered probability space
- Let W and B be independent Brownian motions, B is the COMMON NOISE
- $\mathbb{F}^B = (\mathcal{F}_t^B)_{t\in[0,T]}$ the (extended) filtration generated by B
- for a closed control set $U \subset \mathbb{R}$, consider the set of admissible controls

$$\mathcal{A} := \{\alpha \colon \Omega \times [0,T] \to U \,|\, \alpha \text{ is progr. meas. with } \mathbb{E}[\int_0^T \alpha_s ds] < \infty\}$$

- Let $T > 0$ be a fixed time horizon, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ complete filtered probability space
- Let W and B be independent Brownian motions, B is the COMMON NOISE
- $\mathbb{F}^B = (\mathcal{F}_t^B)_{t\in[0,T]}$ the (extended) filtration generated by B
- for a closed control set $U \subset \mathbb{R}$, consider the set of admissible controls

$$\mathcal{A} := \{\alpha\colon \Omega \times [0,T] \to U \,|\, \alpha \text{ is progr. meas. with } \mathbb{E}[\textstyle\int_0^T \alpha_s ds] < \infty\}$$

For any given a $\mathcal{P}_1(\mathbb{R})$-valued $\mathbb{F}^B$-progressively measurable flow $\mu = (\mu_t)_{t\in[0,T]}$, consider the control problem $\inf_{\alpha\in\mathcal{A}} J(\alpha,\mu)$ where

$$J(\alpha,\mu) := \mathbb{E}\left[\int_0^T \left[f(t, X_t^\alpha, \mu_t) + l(t, X_t^\alpha, \alpha_t)\right] dt + g(X_T^\alpha, \mu_T)\right],$$

where X^α denotes the unique solution to the dynamics

$$dX_t = b(t, X_t, \alpha_t)dt + \sigma_t dW_t + \sigma^o dB_t, \quad t \in [0,T], \quad X_0 = \xi.$$

- Let $T > 0$ be a fixed time horizon, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ complete filtered probability space
- Let W and B be independent Brownian motions, B is the COMMON NOISE
- $\mathbb{F}^B = (\mathcal{F}_t^B)_{t\in[0,T]}$ the (extended) filtration generated by B
- for a closed control set $U \subset \mathbb{R}$, consider the set of admissible controls

$$\mathcal{A} := \{\alpha\colon \Omega \times [0,T] \to U \,|\, \alpha \text{ is progr. meas. with } \mathbb{E}[\textstyle\int_0^T \alpha_s ds] < \infty\}$$

For any given a $\mathcal{P}_1(\mathbb{R})$-valued $\mathbb{F}^B$-progressively measurable flow $\mu = (\mu_t)_{t\in[0,T]}$, consider the control problem $\inf_{\alpha\in\mathcal{A}} J(\alpha, \mu)$ where

$$J(\alpha,\mu) := \mathbb{E}\left[\int_0^T \Big[f(t, X_t^\alpha, \mu_t) + l(t, X_t^\alpha, \alpha_t)\Big] dt + g(X_T^\alpha, \mu_T)\right],$$

where X^α denotes the unique solution to the dynamics

$$dX_t = b(t, X_t, \alpha_t)dt + \sigma_t dW_t + \sigma^o dB_t, \quad t \in [0,T], \quad X_0 = \xi.$$

Assumption: existence & uniqueness of optimal controls (EU). For any $\mathbb{F}^B$-progr.meas. μ, there exists a unique optimal pair (X^μ, α^μ) with $-\infty < J(\alpha^\mu, \mu) < \infty$ and

$$J(\alpha^\mu, \mu) = \inf_{\alpha\in\mathcal{A}} J(\alpha, \mu) \quad \text{and} \quad X^\mu = X^{\alpha^\mu}.$$

- Let $T > 0$ be a fixed time horizon, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ complete filtered probability space
- Let W and B be independent Brownian motions, B is the COMMON NOISE
- $\mathbb{F}^B = (\mathcal{F}^B_t)_{t\in[0,T]}$ the (extended) filtration generated by B
- for a closed control set $U \subset \mathbb{R}$, consider the set of admissible controls

$$\mathcal{A} := \{\alpha\colon \Omega \times [0,T] \to U \,|\, \alpha \text{ is progr. meas. with } \mathbb{E}[\textstyle\int_0^T \alpha_s ds] < \infty\}$$

For any given a $\mathcal{P}_1(\mathbb{R})$-valued $\mathbb{F}^B$-progressively measurable flow $\mu = (\mu_t)_{t\in[0,T]}$, consider the control problem $\inf_{\alpha\in\mathcal{A}} J(\alpha,\mu)$ where

$$J(\alpha,\mu) := \mathbb{E}\left[\int_0^T \Big[f(t, X_t^\alpha, \mu_t) + l(t, X_t^\alpha, \alpha_t)\Big] dt + g(X_T^\alpha, \mu_T)\right],$$

where X^α denotes the unique solution to the dynamics

$$dX_t = b(t, X_t, \alpha_t)dt + \sigma_t dW_t + \sigma^o dB_t, \quad t \in [0,T], \quad X_0 = \xi.$$

Assumption: existence & uniqueness of optimal controls (EU). For any $\mathbb{F}^B$-progr.meas. μ, there exists a unique optimal pair (X^μ, α^μ) with $-\infty < J(\alpha^\mu,\mu) < \infty$ and

$$J(\alpha^\mu,\mu) = \inf_{\alpha\in\mathcal{A}} J(\alpha,\mu) \quad \text{and} \quad X^\mu = X^{\alpha^\mu}.$$

STRONG SOLUTION: fix $\mu \longrightarrow$ optimize against $\mu \longrightarrow$ search for $\mu = \mathbb{P}[X^\mu \in \cdot | \mathcal{F}^B_T]$

Assumption: a priori estimate (APE). There exists $p > 1$ and a constant $M > 0$ such that

$$\mathbb{E}\big[|X_t^\mu|^p\big] \leq M \quad \text{for all measurable flows of probabilities } \mu \text{ and } t \in [0, T].$$

Assumption: a priori estimate (APE). There exists $p > 1$ and a constant $M > 0$ such that

$$\mathbb{E}\big[|X_t^\mu|^p\big] \leq M \quad \text{for all measurable flows of probabilities } \mu \text{ and } t \in [0, T].$$

First order stochastic dominance. For ν, $\bar{\nu} \in \mathcal{P}(\mathbb{R})$, we say that

$$\nu \leq^{\text{st}} \bar{\nu} \text{ if and only if } \nu((-\infty, x]) \geq \bar{\nu}((-\infty, x]) \ \forall\, x \in \mathbb{R}.$$

By **(APE)**, there exist $\mathcal{P}(\mathbb{R})$-valued, $\mathcal{F}_T^B$-adapted r.v.'s μ^{Min}, μ^{Max} with

$$\mu^{\mathrm{Min}} \leq^{\text{st}} \mathbb{P}[X_t^\mu \in \cdot | \mathcal{F}_T^B] \leq^{\text{st}} \mu^{\mathrm{Max}} \quad \text{for all measurable flow } \mu \text{ and } t \in [0, T], \ \mathbb{P}\text{-a.s.}$$

Assumption: a priori estimate (APE). There exists $p > 1$ and a constant $M > 0$ such that

$$\mathbb{E}\big[|X_t^\mu|^p\big] \leq M \quad \text{for all measurable flows of probabilities } \mu \text{ and } t \in [0, T].$$

First order stochastic dominance. For $\nu, \bar{\nu} \in \mathcal{P}(\mathbb{R})$, we say that

$$\nu \leq^{\text{st}} \bar{\nu} \text{ if and only if } \nu((-\infty, x]) \geq \bar{\nu}((-\infty, x]) \ \forall\, x \in \mathbb{R}.$$

By **(APE)**, there exist $\mathcal{P}(\mathbb{R})$-valued, $\mathcal{F}_T^B$-adapted r.v.'s μ^{Min}, μ^{Max} with

$$\mu^{\mathrm{Min}} \leq^{\text{st}} \mathbb{P}[X_t^\mu \in \cdot | \mathcal{F}_T^B] \leq^{\text{st}} \mu^{\mathrm{Max}} \quad \text{for all measurable flow } \mu \text{ and } t \in [0, T],\ \mathbb{P}\text{-a.s.}$$

Lattice structure. Defining on $[0, T]$ the measure $\pi := \delta_0 + dt + \delta_T$, define the lattice of *feasible flows* $(L, \leq^L)$ as

$$L := \left\{ \begin{array}{c} \text{equivalence classes (w.r.t. } \mathbb{P} \otimes \pi) \ \mathbb{F}^B\text{-progr.meas. flows } (\mu_t)_{t \in [0,T]} \text{ s.t.} \\ \mu_t \in [\mu^{\mathrm{Min}}, \mu^{\mathrm{Max}}] \text{ for } \pi\text{-almost all } t \in (0, T] \text{ and } \mu_0 = \mathbb{P} \circ (\xi)^{-1},\ \mathbb{P}\text{-a.s.} \end{array} \right\}$$

endowed with the order relation given by

$$\mu \leq^L \nu \text{ if and only if } \mu_t \leq^{\text{st}} \nu_t \text{ for } \mathbb{P} \otimes \pi\text{-a.a. } (\omega, t) \in \Omega \times [0, T].$$

Assumption: a priori estimate (APE). There exists $p > 1$ and a constant $M > 0$ such that

$$\mathbb{E}\big[|X_t^\mu|^p\big] \leq M \quad \text{for all measurable flows of probabilities } \mu \text{ and } t \in [0, T].$$

First order stochastic dominance. For $\nu, \bar{\nu} \in \mathcal{P}(\mathbb{R})$, we say that

$$\nu \leq^{\text{st}} \bar{\nu} \text{ if and only if } \nu((-\infty, x]) \geq \bar{\nu}((-\infty, x]) \ \forall\ x \in \mathbb{R}.$$

By **(APE)**, there exist $\mathcal{P}(\mathbb{R})$-valued, $\mathcal{F}_T^B$-adapted r.v.'s μ^{Min}, μ^{Max} with

$$\mu^{\mathrm{Min}} \leq^{\text{st}} \mathbb{P}[X_t^\mu \in \cdot | \mathcal{F}_T^B] \leq^{\text{st}} \mu^{\mathrm{Max}} \quad \text{for all measurable flow } \mu \text{ and } t \in [0, T],\ \mathbb{P}\text{-a.s.}$$

Lattice structure. Defining on $[0, T]$ the measure $\pi := \delta_0 + dt + \delta_T$, define the lattice of *feasible flows* $(L, \leq^L)$ as

$$L := \left\{ \begin{array}{l} \text{equivalence classes (w.r.t. } \mathbb{P} \otimes \pi) \ \mathbb{F}^B\text{-progr.meas. flows } (\mu_t)_{t \in [0,T]} \text{ s.t.} \\ \mu_t \in [\mu^{\mathrm{Min}}, \mu^{\mathrm{Max}}] \text{ for } \pi\text{-almost all } t \in (0, T] \text{ and } \mu_0 = \mathbb{P} \circ (\xi)^{-1},\ \mathbb{P}\text{-a.s.} \end{array} \right\}$$

endowed with the order relation given by

$$\mu \leq^L \nu \text{ if and only if } \mu_t \leq^{\text{st}} \nu_t \text{ for } \mathbb{P} \otimes \pi\text{-a.a. } (\omega, t) \in \Omega \times [0, T].$$

Now, the best reply map $R : L \to L$, $R(\mu) := (\mathbb{P}[X_t^\mu \in \cdot | \mathcal{F}_t^B])_{t \in [0,T]}$ is well defined and

$$\mu \in L \text{ is a MFG solution if and only if } R(\mu) = \mu$$

Assumption: submodularity (Sub). For $\mathbb{P} \otimes dt$ a.a. $(\omega, t) \in \Omega \times [0, T]$, the functions $f(t, \cdot, \cdot)$ and g have decreasing differences in (x, μ); that is, for $\varphi \in \{f(t, \cdot, \cdot), g\}$,

$$\varphi(\bar{x}, \bar{\mu}) - \varphi(x, \bar{\mu}) \leq \varphi(\bar{x}, \mu) - \varphi(x, \mu),$$

for all $\bar{x}, x \in \mathbb{R}$ and $\bar{\mu}, \mu \in \mathcal{P}(\mathbb{R})$ s.t. $\bar{x} \geq x$ and $\bar{\mu} \geq^{\text{st}} \mu$.

Assumption: submodularity (Sub). For $\mathbb{P} \otimes dt$ a.a. $(\omega, t) \in \Omega \times [0, T]$, the functions $f(t, \cdot, \cdot)$ and g have decreasing differences in (x, μ); that is, for $\varphi \in \{f(t, \cdot, \cdot), g\}$,

$$\varphi(\bar{x}, \bar{\mu}) - \varphi(x, \bar{\mu}) \leq \varphi(\bar{x}, \mu) - \varphi(x, \mu),$$

for all $\bar{x}, x \in \mathbb{R}$ and $\bar{\mu}, \mu \in \mathcal{P}(\mathbb{R})$ s.t. $\bar{x} \geq x$ and $\bar{\mu} \geq^{\text{st}} \mu$.

If φ satisfies **(Sub)**, then it satisfies the Lasry-Lions **anti**-monotonicity condition; that is,

$$\int_{\mathbb{R}} (\varphi(x, \bar{\mu}) - \varphi(x, \mu)) d(\bar{\mu} - \mu)(x) \leq 0, \quad \forall\, \bar{\mu}, \mu \in \mathcal{P}(\mathbb{R}).$$

Assumption: submodularity (Sub). For $\mathbb{P} \otimes dt$ a.a. $(\omega, t) \in \Omega \times [0, T]$, the functions $f(t, \cdot, \cdot)$ and g have decreasing differences in (x, μ); that is, for $\varphi \in \{f(t, \cdot, \cdot), g\}$,

$$\varphi(\bar{x}, \bar{\mu}) - \varphi(x, \bar{\mu}) \leq \varphi(\bar{x}, \mu) - \varphi(x, \mu),$$

for all $\bar{x}, x \in \mathbb{R}$ and $\bar{\mu}, \mu \in \mathcal{P}(\mathbb{R})$ s.t. $\bar{x} \geq x$ and $\bar{\mu} \geq^{\text{st}} \mu$.

If φ satisfies **(Sub)**, then it satisfies the Lasry-Lions **anti**-monotonicity condition; that is,

$$\int_{\mathbb{R}} (\varphi(x, \bar{\mu}) - \varphi(x, \mu)) d(\bar{\mu} - \mu)(x) \leq 0, \quad \forall\, \bar{\mu}, \mu \in \mathcal{P}(\mathbb{R}).$$

Examples:

- mean-field interaction of scalar type:

$$\varphi(x, \mu) = \gamma\big(x, \textstyle\int h\, d\mu\big) \quad \text{if } h \text{ is increasing and } \gamma \in \mathcal{C}^2(\mathbb{R}^2) \text{ with } \frac{\partial^2 \gamma}{\partial x \partial y} \leq 0$$

- mean-field interactions of order-1:

$$\varphi(x, \mu) = \int_{\mathbb{R}} \gamma(x, y) d\mu(y) \quad \text{if } \gamma \in \mathcal{C}^2(\mathbb{R}^2) \text{ with } \frac{\partial^2 \gamma}{\partial x \partial y} \leq 0$$

- **quadratic cost:** $\varphi(x, \mu) = (x - \text{Mean}(\mu))^2$
- **multiplicative cost:** $\varphi(x, \mu) = -x\, \text{Mean}(\mu)$

Theorem (Existence and Structure)

Under **(EU)**, **(APE)** *and* **(Sub)**, *the set of MFG solutions* $(\mathcal{M}, \leq^L)$ *is a non-empty complete lattice: in particular there exist a minimal and a maximal MFG solution*

Remarks:

- proof: Tarski's fixed point theorem to the best reply map $R : L \to L$

Theorem (Existence and Structure)

Under **(EU)**, **(APE)** *and* **(Sub)**, *the set of MFG solutions* $(\mathcal{M}, \leq^L)$ *is a non-empty complete lattice: in particular there exist a minimal and a maximal MFG solution*

Remarks:

- proof: Tarski's fixed point theorem to the best reply map $R : L \to L$
- if J is nondecreasing with respect to μ, the same lattice structure is inherited by the values at the equilibria, giving a criterion for the minimal cost selection

Theorem (Existence and Structure)

Under **(EU)**, **(APE)** *and* **(Sub)**, *the set of MFG solutions* $(\mathcal{M}, \leq^L)$ *is a non-empty complete lattice: in particular there exist a minimal and a maximal MFG solution*

Remarks:

- proof: Tarski's fixed point theorem to the best reply map $R : L \to L$
- if J is nondecreasing with respect to μ, the same lattice structure is inherited by the values at the equilibria, giving a criterion for the minimal cost selection
- the continuity of f, g in μ is NOT NEEDED

Theorem (Existence and Structure)

Under **(EU)**, **(APE)** *and* **(Sub)**, *the set of MFG solutions* $(\mathcal{M}, \leq^L)$ *is a non-empty complete lattice: in particular there exist a minimal and a maximal MFG solution*

Remarks:

- proof: Tarski's fixed point theorem to the best reply map $R : L \to L$
- if J is nondecreasing with respect to μ, the same lattice structure is inherited by the values at the equilibria, giving a criterion for the minimal cost selection
- the continuity of f, g in μ is NOT NEEDED
- **STRONG SOLUTIONS to MFGs with COMMON NOISE** (see [Carmona-Delarue-Lacker (2016)])

Theorem (Existence and Structure)

Under **(EU)**, **(APE)** *and* **(Sub)**, *the set of MFG solutions* $(\mathcal{M}, \leq^L)$ *is a non-empty complete lattice: in particular there exist a minimal and a maximal MFG solution*

Remarks:

- proof: Tarski's fixed point theorem to the best reply map $R : L \to L$
- if J is nondecreasing with respect to μ, the same lattice structure is inherited by the values at the equilibria, giving a criterion for the minimal cost selection
- the continuity of f, g in μ is NOT NEEDED
- **STRONG SOLUTIONS to MFGs with COMMON NOISE** (see [Carmona-Delarue-Lacker (2016)])
- using *relaxed controls*, the uniqueness of the optimal control in assumption **(EU)** can be removed

Theorem (Existence and Structure)

Under **(EU)**, **(APE)** *and* **(Sub)**, *the set of MFG solutions* $(\mathcal{M}, \leq^L)$ *is a non-empty complete lattice: in particular there exist a minimal and a maximal MFG solution*

Remarks:

- proof: Tarski's fixed point theorem to the best reply map $R : L \to L$
- if J is nondecreasing with respect to μ, the same lattice structure is inherited by the values at the equilibria, giving a criterion for the minimal cost selection
- the continuity of f, g in μ is NOT NEEDED
- **STRONG SOLUTIONS to MFGs with COMMON NOISE** (see [Carmona-Delarue-Lacker (2016)])
- using *relaxed controls*, the uniqueness of the optimal control in assumption **(EU)** can be removed
- geometric dynamics: $dX_t = b(t, X_t, \alpha_t)X_t dt + \sigma_t X_t dW_t, \qquad t \in [0, T], \quad X_0 = \xi,$

Theorem (Existence and Structure)

Under **(EU)**, **(APE)** *and* **(Sub)**, *the set of MFG solutions* $(\mathcal{M}, \leq^L)$ *is a non-empty complete lattice: in particular there exist a minimal and a maximal MFG solution*

Remarks:

- proof: Tarski's fixed point theorem to the best reply map $R : L \to L$
- if J is nondecreasing with respect to μ, the same lattice structure is inherited by the values at the equilibria, giving a criterion for the minimal cost selection
- the continuity of f, g in μ is NOT NEEDED
- **STRONG SOLUTIONS to MFGs with COMMON NOISE** (see [Carmona-Delarue-Lacker (2016)])
- using *relaxed controls*, the uniqueness of the optimal control in assumption **(EU)** can be removed
- geometric dynamics: $dX_t = b(t, X_t, \alpha_t) X_t dt + \sigma_t X_t dW_t, \qquad t \in [0, T], \quad X_0 = \xi,$
- for particular costs (as e.g. $\varphi(x, \mu) = -x\text{Mean}(\mu)$ and $l(a) = a^2$), we can treat mean-field dependent dynamics as

$$dX_t = X_t(\alpha_t + m(\mu_t))dt + \sigma X_t dW_t, \quad t \in [0, T], \quad X_0 = \xi \geq 0,$$
$$dX_t = \big(\kappa X_t + \alpha_t + m(\mu_t)\big)dt + \sigma dW_t, \quad t \in [0, T], \quad X_0 = \xi, \quad \kappa \in \mathbb{R}, \ \sigma \geq 0,$$

for bounded monotone $m : (\mathcal{P}(\mathbb{R}), \leq^{\text{st}}) \to \mathbb{R}$

Assumption: linear-convex case (LC).

- The control set $U \subset \mathbb{R}$ is convex and compact. The volatility σ is constant.
- $b(t, x, a) = c_t + p_t x + q_t a$, where c, p and q are deterministic continuous functions of t.
- The cost functions f, g and l are bounded and continuous, l is convex.

Assumption: linear-convex case (LC).

- The control set $U \subset \mathbb{R}$ is convex and compact. The volatility σ is constant.
- $b(t, x, a) = c_t + p_t x + q_t a$, where c, p and q are deterministic continuous functions of t.
- The cost functions f, g and l are bounded and continuous, l is convex.

The learning algorithm

- $\mathbf{n = 0}$: $\underline{\mu}^0 := \inf L$;
- $\mathbf{n \geq 1}$: $\underline{\mu}^{n+1} = R(\underline{\mu}^n)$.

Assumption: linear-convex case (LC).

- The control set $U \subset \mathbb{R}$ is convex and compact. The volatility σ is constant.
- $b(t, x, a) = c_t + p_t x + q_t a$, where c, p and q are deterministic continuous functions of t.
- The cost functions f, g and l are bounded and continuous, l is convex.

The learning algorithm

- $\mathbf{n = 0}$: $\underline{\mu}^0 := \inf L$;
- $\mathbf{n \geq 1}$: $\underline{\mu}^{n+1} = R(\underline{\mu}^n)$.

Theorem (Approximation)

Under **(EU)**, **(APE)**, **(Sub)** *and* **(LC)**, *the sequence* $\{\underline{\mu}^n\}_{n \in \mathbb{N}}$ *is increasing in* $(L, \leq^L)$ *and it weakly converges to the minimal MFG solution at any time* $t \in [0, T]$, $\mathbb{P}$*-a.s.*

Assumption: linear-convex case (LC).

- The control set $U \subset \mathbb{R}$ is convex and compact. The volatility σ is constant.
- $b(t,x,a) = c_t + p_t x + q_t a$, where c, p and q are deterministic continuous functions of t.
- The cost functions f, g and l are bounded and continuous, l is convex.

The learning algorithm

- $\mathbf{n = 0}$: $\underline{\mu}^0 := \inf L$;
- $\mathbf{n \geq 1}$: $\underline{\mu}^{n+1} = R(\underline{\mu}^n)$.

Theorem (Approximation)

Under **(EU)**, **(APE)**, **(Sub)** *and* **(LC)**, *the sequence* $\{\underline{\mu}^n\}_{n \in \mathbb{N}}$ *is increasing in* $(L, \leq^L)$ *and it weakly converges to the minimal MFG solution at any time* $t \in [0,T]$, $\mathbb{P}$*-a.s.*

Remarks:

- the maximal MFG solution can be reached by the same algorithm, starting from $\overline{\mu}^0 := \sup L$

Assumption: linear-convex case (LC).

- The control set $U \subset \mathbb{R}$ is convex and compact. The volatility σ is constant.
- $b(t,x,a) = c_t + p_t x + q_t a$, where c, p and q are deterministic continuous functions of t.
- The cost functions f, g and l are bounded and continuous, l is convex.

The learning algorithm

- $\mathbf{n = 0}$: $\underline{\mu}^0 := \inf L$;
- $\mathbf{n \geq 1}$: $\underline{\mu}^{n+1} = R(\underline{\mu}^n)$.

Theorem (Approximation)

Under **(EU)**, **(APE)**, **(Sub)** *and* **(LC)**, *the sequence* $\{\underline{\mu}^n\}_{n\in\mathbb{N}}$ *is increasing in* $(L, \leq^L)$ *and it weakly converges to the minimal MFG solution at any time* $t \in [0,T]$, $\mathbb{P}$*-a.s.*

Remarks:

- the maximal MFG solution can be reached by the same algorithm, starting from $\overline{\mu}^0 := \sup L$
- using *relaxed controls*, the uniqueness of the optimal control in assumption **(EU)** and assumption **(LC)** can be removed

Submodular MFGs and related McKean-Vlasov FBSDEs (working progress)

For any given a $\mathcal{P}_1(\mathbb{R})$-valued $\mathbb{F}^B$-progressively measurable flow $\mu = (\mu_t)_{t\in[0,T]}$, consider the FBSDE($\mu$)

$$dX_t = b(t, X_t, \hat{\alpha}(t, X_t, Y_t))dt + \sigma dW_t + \sigma^\circ dB_t, \quad X_0 = \xi,$$
$$dY_t = -D_x H(t, X_t, \mu_t, Y_t, \hat{\alpha}(t, X_t, Y_t))dt + Z_t dW_t + Z_t^\circ dB_t, \quad Y_T = D_x g(X_T, \mu_T)$$

where

$$H(t,x,\nu,y,a) = b(t,x,a)y + h(t,x,\nu) + l(t,x,a), \quad (t,x,\nu,y,a) \in [0,T] \times \mathbb{R} \times \mathcal{P}(\mathbb{R}) \times \mathbb{R} \times U,$$

and

$$\hat{\alpha}(t,x,y) := \operatorname*{argmin}_{a\in U} H(t,x,\nu,y,a) \in U, \quad (t,x,\nu,y) \in [0,T] \times \mathbb{R} \times \mathcal{P}(\mathbb{R}) \times \mathbb{R}.$$

For any given a $\mathcal{P}_1(\mathbb{R})$-valued $\mathbb{F}^B$-progressively measurable flow $\mu = (\mu_t)_{t\in[0,T]}$, consider the FBSDE($\mu$)

$$dX_t = b(t, X_t, \hat{\alpha}(t, X_t, Y_t))dt + \sigma dW_t + \sigma^\circ dB_t, \quad X_0 = \xi,$$
$$dY_t = -D_x H(t, X_t, \mu_t, Y_t, \hat{\alpha}(t, X_t, Y_t))dt + Z_t dW_t + Z_t^\circ dB_t, \quad Y_T = D_x g(X_T, \mu_T)$$

where

$$H(t, x, \nu, y, a) = b(t, x, a)y + h(t, x, \nu) + l(t, x, a), \quad (t, x, \nu, y, a) \in [0, T] \times \mathbb{R} \times \mathcal{P}(\mathbb{R}) \times \mathbb{R} \times U,$$

and

$$\hat{\alpha}(t, x, y) := \operatorname*{argmin}_{a\in U} H(t, x, \nu, y, a) \in U, \quad (t, x, \nu, y) \in [0, T] \times \mathbb{R} \times \mathcal{P}(\mathbb{R}) \times \mathbb{R}.$$

Denote by $(X^\mu, Y^\mu, Z^\mu, Z^{\circ,\mu})$ its solution.

For any given a $\mathcal{P}_1(\mathbb{R})$-valued $\mathbb{F}^B$-progressively measurable flow $\mu = (\mu_t)_{t\in[0,T]}$, consider the FBSDE($\mu$)

$$dX_t = b(t, X_t, \hat{\alpha}(t, X_t, Y_t))dt + \sigma dW_t + \sigma^\circ dB_t, \quad X_0 = \xi,$$
$$dY_t = -D_x H(t, X_t, \mu_t, Y_t, \hat{\alpha}(t, X_t, Y_t))dt + Z_t dW_t + Z_t^\circ dB_t, \quad Y_T = D_x g(X_T, \mu_T)$$

where

$$H(t, x, \nu, y, a) = b(t, x, a)y + h(t, x, \nu) + l(t, x, a), \quad (t, x, \nu, y, a) \in [0, T] \times \mathbb{R} \times \mathcal{P}(\mathbb{R}) \times \mathbb{R} \times U,$$

and

$$\hat{\alpha}(t, x, y) := \operatorname*{argmin}_{a\in U} H(t, x, \nu, y, a) \in U, \quad (t, x, \nu, y) \in [0, T] \times \mathbb{R} \times \mathcal{P}(\mathbb{R}) \times \mathbb{R}.$$

Denote by $(X^\mu, Y^\mu, Z^\mu, Z^{\circ,\mu})$ its solution.

Solution to the MKV FBSDE: $(X^\mu, Y^\mu, Z^\mu, Z^{\circ,\mu})$ such that $\mu = (\mathbb{P}[X_t^\mu \in \cdot | \mathcal{F}_t^B])_{t\in[0,T]}$.

For any given a $\mathcal{P}_1(\mathbb{R})$-valued $\mathbb{F}^B$-progressively measurable flow $\mu = (\mu_t)_{t\in[0,T]}$, consider the FBSDE($\mu$)

$$dX_t = b(t, X_t, \hat{\alpha}(t, X_t, Y_t))dt + \sigma dW_t + \sigma^\circ dB_t, \quad X_0 = \xi,$$
$$dY_t = -D_x H(t, X_t, \mu_t, Y_t, \hat{\alpha}(t, X_t, Y_t))dt + Z_t dW_t + Z_t^\circ dB_t, \quad Y_T = D_x g(X_T, \mu_T)$$

where

$$H(t, x, \nu, y, a) = b(t, x, a)y + h(t, x, \nu) + l(t, x, a), \quad (t, x, \nu, y, a) \in [0, T] \times \mathbb{R} \times \mathcal{P}(\mathbb{R}) \times \mathbb{R} \times U,$$

and

$$\hat{\alpha}(t, x, y) := \operatorname*{argmin}_{a\in U} H(t, x, \nu, y, a) \in U, \quad (t, x, \nu, y) \in [0, T] \times \mathbb{R} \times \mathcal{P}(\mathbb{R}) \times \mathbb{R}.$$

Denote by $(X^\mu, Y^\mu, Z^\mu, Z^{\circ,\mu})$ its solution.

Solution to the MKV FBSDE: $(X^\mu, Y^\mu, Z^\mu, Z^{\circ,\mu})$ such that $\mu = (\mathbb{P}[X_t^\mu \in \cdot | \mathcal{F}_t^B])_{t\in[0,T]}$.

Lemma (Comparison principle)

Under **(EU)**, **(APE)**, **(Sub)** *and* **(LC)**, *if* $\mu \leq \bar{\mu}$, *then* $X^\mu \leq X^{\bar{\mu}}$.

For any given a $\mathcal{P}_1(\mathbb{R})$-valued $\mathbb{F}^B$-progressively measurable flow $\mu = (\mu_t)_{t\in[0,T]}$, consider the FBSDE($\mu$)

$$dX_t = b(t, X_t, \hat{\alpha}(t, X_t, Y_t))dt + \sigma dW_t + \sigma^\circ dB_t, \quad X_0 = \xi,$$
$$dY_t = -D_x H(t, X_t, \mu_t, Y_t, \hat{\alpha}(t, X_t, Y_t))dt + Z_t dW_t + Z_t^\circ dB_t, \quad Y_T = D_x g(X_T, \mu_T)$$

where

$$H(t, x, \nu, y, a) = b(t, x, a)y + h(t, x, \nu) + l(t, x, a), \quad (t, x, \nu, y, a) \in [0, T] \times \mathbb{R} \times \mathcal{P}(\mathbb{R}) \times \mathbb{R} \times U,$$

and

$$\hat{\alpha}(t, x, y) := \operatorname*{argmin}_{a\in U} H(t, x, \nu, y, a) \in U, \quad (t, x, \nu, y) \in [0, T] \times \mathbb{R} \times \mathcal{P}(\mathbb{R}) \times \mathbb{R}.$$

Denote by $(X^\mu, Y^\mu, Z^\mu, Z^{\circ,\mu})$ its solution.

Solution to the MKV FBSDE: $(X^\mu, Y^\mu, Z^\mu, Z^{\circ,\mu})$ such that $\mu = (\mathbb{P}[X_t^\mu \in \cdot|\mathcal{F}_t^B])_{t\in[0,T]}$.

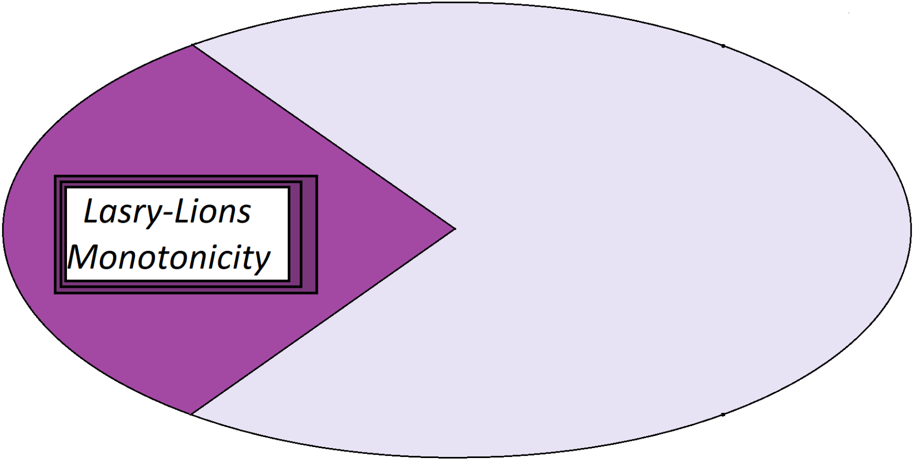
Lemma (Comparison principle)

Under **(EU)**, **(APE)**, **(Sub)** *and* **(LC)**, *if* $\mu \leq \bar{\mu}$, *then* $X^\mu \leq X^{\bar{\mu}}$.

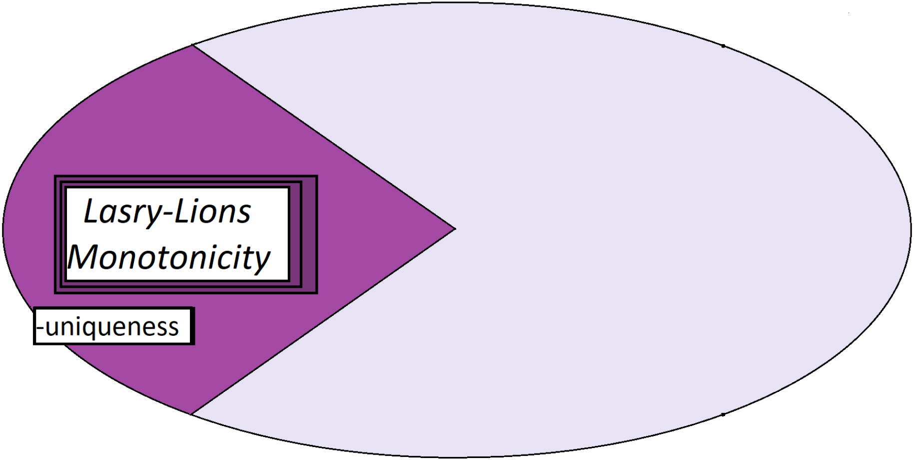
Theorem (Solution to MKV FBSDEs)

Under **(EU)**, **(APE)**, **(Sub)** *and* **(LC)**, *the set of solution to the MKV FBSDE is a non-empty complete lattice: in particular there exist a minimal and a maximal MFG solution .*

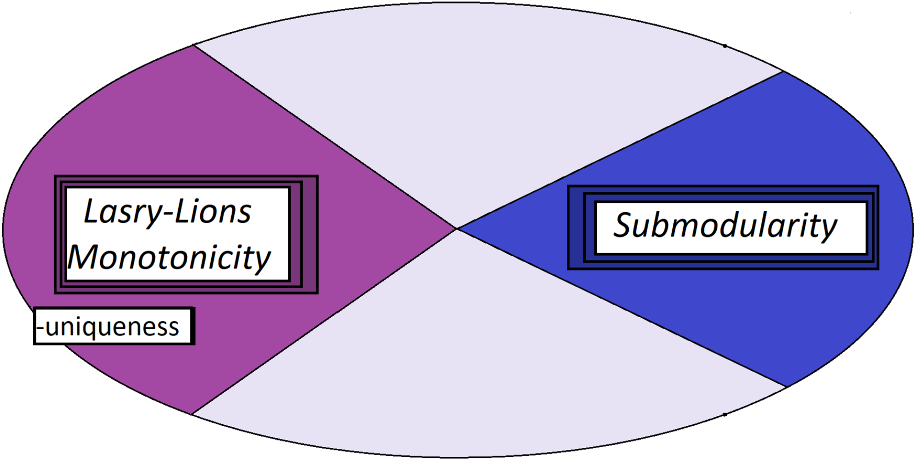
Concluding: structural conditions on MFG models



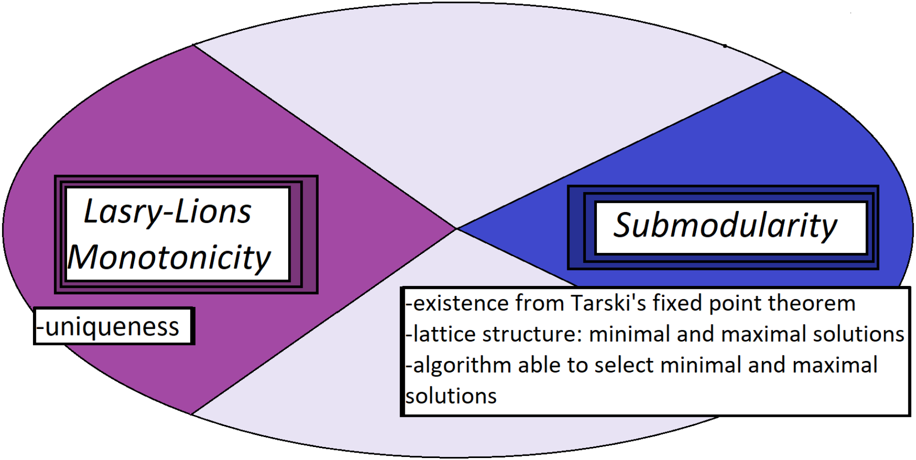
Concluding: structural conditions on MFG models



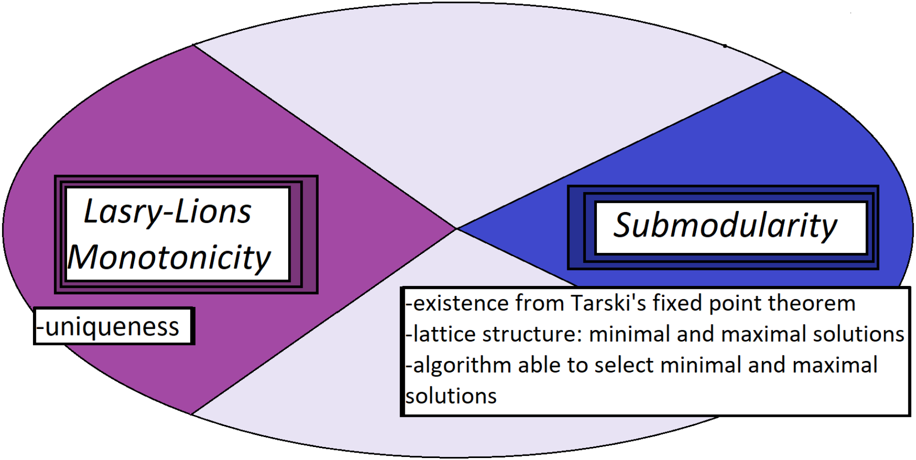
Concluding: structural conditions on MFG models



Concluding: structural conditions on MFG models



Concluding: structural conditions on MFG models



Thank you for your attention!
[D.-Ferrari-Fischer-Nendel (2021), AAP] and [D.-Ferrari-Fischer-Nendel (2022), arXiv preprint]