# Randomization method in optimal control and BSDEs with constrained jumps

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# Plan

1. A classical optimal control problem.

2. The randomization method.

3. Equivalence of an auxiliary control problem and the original problem.

4. Constrained BSDE representation for the value function.

5. Application: non-Markovian control problems.

6. Application: control with partial observation.

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## 1. A classical optimal control problem.

## Classical stochastic optimal control problem

Controlled SDE in $\mathbb{R}^n$:

$$\begin{cases} dX_s^\alpha &= b(X_s^\alpha, \alpha_s)\, ds + \sigma(X_s^\alpha, \alpha_s)\, dW_s, \qquad s \in [t,T] \subset [0,T], \\ X_t^\alpha &= x \in \mathbb{R}^n. \end{cases}$$

Reward functional and value function:

$$J(\alpha,t,x) = \mathbb{E}\left[\int_t^T f(X_s^\alpha, \alpha_s)\, ds + g(X_T^\alpha)\right], \quad v(t,x) = \sup_{\alpha \in \mathcal{A}_d} J(\alpha,t,x).$$

- $W$ is a Wiener process in $\mathbb{R}^d$, defined in $(\Omega, \mathcal{F}, \mathbb{P})$;
- $A$, the space of control actions, is a complete separable metric space (or a Borel subset of it);
- $\mathcal{A}_d = \{\alpha : \Omega \times [0,T] \to A,\ \mathbb{F}^W = (\mathcal{F}_t^W)\text{-progressive}\}$ is the space of admissible controls;
- $\sigma(x,a) \in \mathbb{R}^{n\times d}$, $b(x,a) \in \mathbb{R}^n$, $f(x,a) \in \mathbb{R}$, $g(x) \in \mathbb{R}$ are functions of $x \in \mathbb{R}^n$ and $a \in A$.

## Assumptions (A) on the coefficients

On the data $b(x,a), \sigma(x,a), f(x,a), g(x)$ of the control problem we assume:

- $b, \sigma, f, g$ are continuous.
- $b, \sigma$ are Lipschitz in $x$ uniformly in $a$: $\exists L \geq 0$ such that

$$|b(x,a) - b(y,a)| + |\sigma(x,a) - \sigma(y,a)| \leq L|x-y|, \quad x, y \in \mathbb{R}^n, a \in A.$$

- $b(0,a), \sigma(0,a)$ are bounded in $a$: $\exists M \geq 0$ such that

$$|b(0,a)| + |\sigma(0,a)| \leq M, \quad a \in A.$$

- $f, g$ have polynomial growth in $x$ uniformly in $a$: $\exists r \geq 0$ such that

$$|f(x,a)| + |g(x)| \leq M\,(1 + |x|^r), \quad x \in \mathbb{R}^n, a \in A.$$

Under these assumptions $X^\alpha$ is well defined and $v(t,x)$ is finite.

$$\begin{cases} dX_s^\alpha &= b(X_s^\alpha, \alpha_s)\, ds + \sigma(X_s^\alpha, \alpha_s)\, dW_s, \qquad s \in [t,T] \subset [0,T], \\ X_t^\alpha &= x \in \mathbb{R}^n. \end{cases}$$

Basic issue: find characterizations of the value function $v$.

Possible approaches:

• prove that $v$ is the unique solution to the Hamilton-Jacobi-Bellman equation (HJB), in general a fully non linear PDE;

• in some cases (e.g. $\sigma(x,a) = \sigma(x)$ and $\sigma^{-1}$ bounded) use the theory of classical backward stochastic differential equations (BSDEs); (E. Pardoux, S. Peng)

• use the theory of second order BSDEs; (M. Soner, N. Touzi)

• use the theory of $G$-expectations (S. Peng).

## 2. The randomization method.

# The randomization method in optimal control

Introduced in

B. Bouchard. A stochastic target formulation for optimal switching problems in finite horizon. Stochastics 81, no. 2 (2009), 171-197.

$$\begin{cases} dX_s^\alpha = b(X_s^\alpha, \alpha_s)\, ds + \sigma(X_s^\alpha, \alpha_s)\, dW_s, & s \in [t, T], \\ X_t^\alpha = x \in \mathbb{R}^n. \end{cases}$$

$$v(t,x) = \sup_{\alpha \in \mathcal{A}_d} \mathbb{E}\left[ g(X_T^\alpha) + \int_t^T f(X_s^\alpha, \alpha_s)\, ds \right].$$

Idea:

1) replace $(\alpha_s)$ by a random (uncontrolled) process $(I_s)$ with values in $A$;

2) formulate an auxiliary ("randomized") control problem, where "the law of $I$ is controlled", having value denoted $v^{\mathcal{R}}(t,x)$;

3) prove that $v(t,x) = v^{\mathcal{R}}(t,x)$;

4) represent $v^{\mathcal{R}}(t,x)$ by a BSDE.

## The randomized control problem

We replace the control $\alpha \in \mathcal{A}_d$ by an $A$-valued process $I$
- independent of $W$;
- with piecewise constant trajectories.

We consider the "randomized" state equation:

$$dX_s = b(X_s, I_s)\, ds + \sigma(X_s, I_s)\, dW_s, \; s \in [t, T]; \qquad X_t = x.$$

Then we construct, via a Girsanov theorem, a suitable family of probability measures $\mathbb{P}^\nu$, depending on $\nu \in \mathcal{V}$, such that
- $\mathbb{P}^\nu \sim \mathbb{P}$ (dominated model)
- $W$ remains a Wiener process under $\mathbb{P}^\nu$.

Then we optimize among $\mathbb{P}^\nu$: we formulate an auxiliary ("randomized") control problem with value function:

$$v^{\mathcal{R}}(t, x) = \sup_{\nu \in \mathcal{V}} \mathbb{E}^\nu \left[ \int_t^T f(X_s, I_s)\, ds + g(X_T) \right].$$

# The piecewise constant process $I$

$$I_t = a_0\, 1(0 \le t < T_1) + A_1\, 1(T_1 \le t < T_2) + A_2\, 1(T_2 \le t < T_3) + \ldots,$$

$T_n$: random times, $0 < T_n < T_{n+1} \uparrow \infty$
$A_n$: $A$-valued random variables ($a_0 \in A$).
We identify $I \equiv (T_n, A_n)_{n \ge 1} \equiv \mu$, a random measure on $(0, \infty) \times A$:

$$\mu(dt\, da) = \sum_{n \ge 1} \delta_{(T_n, A_n)}(dt\, da)\, 1_{\{T_n < \infty\}}.$$

We will use the filtration $\mathbb{F}^{W,\mu} = (\mathcal{F}_t^{W,\mu})$ generated by $W$ and $\mu$:
$\mathcal{F}_t^{W,\mu} = \sigma\{W_s,\, \mu((0, s] \times C) \,:\, s \in [0, t],\, C \in \mathcal{B}(A)\}.$

We take for $I \equiv \mu$ a Poisson process, independent of $W$, with arbitrary fixed intensity $\lambda$, a finite measure on $A$. If we set

$$\nu(dt\, da) = \lambda(da) dt$$

then $\mu - \nu$ is a martingale measure.

Then we perform a Girsanov change of measure. Choose

$$\nu_t(\omega, a) > 0$$

a bounded $\mathcal{P}(\mathbb{F}^{W,\mu}) \otimes \mathcal{B}(A)$-measurable random field. Set

$$\kappa_t^\nu = \exp\left(\int_0^t \int_A (1 - \nu_s(a))\lambda(da)ds\right) \prod_{T_n \le t} \nu_{T_n}(A_n), \qquad d\mathbb{P}^\nu = \kappa_T^\nu \, d\mathbb{P}$$

$\kappa^\nu > 0$ is a Doléans exponential martingale and under $\mathbb{P}^\nu$ the compensator (= dual predictable projection) of $\mu$ is

$$\nu(dt\, da) = \nu_t(\omega, a)\lambda(da)dt.$$

Formally, for $C \in \mathcal{B}(A)$ and $n \ge 1$ the processes

$$\mu((0, t \wedge T_n] \times C) - \nu((0, t \wedge T_n] \times C), \qquad t \ge 0,$$

are $\mathbb{P}^\nu$-martingales with respect to $\mathbb{F}^{W,\mu}$.

## The randomized control problem

Let $b, \sigma, f, g$ satisfy Assumption (A). Consider $(\Omega, \mathcal{F}, \mathbb{P}, W, \mu)$ where:

- $\mu \equiv (T_n, A_n)_{n \geq 1} \equiv I$ is a Poisson random measure with finite intensity $\lambda(da)$ with full topological support;
- $W$ is an $\mathbb{R}^d$-valued Brownian motion, independent of $\mu$.

Consider the "randomized" state equation:

$$dX_s = b(X_s, I_s)\,ds + \sigma(X_s, I_s)\,dW_s,\ s \in [t, T]; \qquad X_t = x.$$

The admissible controls are now random fields

$\mathcal{V} = \{\nu_t(\omega, a) : \mathcal{P}(\mathbb{F}^{W,\mu}) \otimes \mathcal{B}(A)\text{−measurable}, 0 < \nu \leq \sup \nu < \infty\}$

Given $\nu \in \mathcal{V}$, we construct $\mathbb{P}^\nu$ such that, on $[t, T]$,

- $\mu$ has compensator $\nu_t(a)\lambda(da)dt$ under $\mathbb{P}^\nu$;
- $W$ remains a Wiener process under $\mathbb{P}^\nu$.

We define an auxiliary ("randomized") value function:

$$v^{\mathcal{R}}(t, x) = \sup_{\nu \in \mathcal{V}} \mathbb{E}^\nu \left[ \int_t^T f(X_s, I_s)\,ds + g(X_T) \right].$$

$$dX_s = b(X_s, I_s)\,ds + \sigma(X_s, I_s)\,dW_s,\ s \in [t,T]; \qquad X_t = x.$$

$$v^{\mathcal{R}}(t,x) = \sup_{\nu \in \mathcal{V}} \mathbb{E}^{\nu}\left[\int_t^T f(X_s, I_s)\,ds + g(X_T)\right].$$

It is known that the compensator $\nu_t(a)\lambda(da)dt$ determines the law of $\mu \equiv I$: choosing $\nu \in \mathcal{V}$ is way to "control" the process $I$.

We take $\lambda(da)$ with full topological support: the process $I$ visits every open set in $A$.

It can be proved that $v^{\mathcal{R}}$ only depends on $b, \sigma, f, g$ (not on $\Omega, \mathcal{F}, \mathbb{P}$, $W$, $\mu$, $\lambda, a_0$).

Under these conditions we expect $v(t,x) \geq v^{\mathcal{R}}(t,x)$ but even

$$v(t,x) = v^{\mathcal{R}}(t,x).$$

## 3. Equivalence of an auxiliary control problem and the original problem.

## Equivalence with the randomized problem

Original problem: for $s \in [t, T] \subset [0, T]$, $x \in \mathbb{R}^n$,

$$\begin{array}{lcl} dX_s^\alpha & = & b(X_s^\alpha, \alpha_s)\, ds + \sigma(X_s^\alpha, \alpha_s)\, dW_s, \\ {X_t}^\alpha & = & x, \\ v(t,x) & = & \sup_{\alpha \in \mathcal{A}_d} \mathbb{E}\left[\int_t^T f(X_s^\alpha, \alpha_s)\, ds + g(X_T^\alpha)\right], \\ \mathcal{A}_d & = & \{\alpha_t(\omega) \,:\, \mathbb{F}^W\text{-progressive}\} \end{array}$$

Randomized problem:

$$\begin{array}{lcl} dX_s & = & b(X_s, I_s)\, ds + \sigma(X_s, I_s)\, dW_s, \\ X_t & = & x, \\ v^{\mathcal{R}}(t,x) & = & \sup_{\nu \in \mathcal{V}} \mathbb{E}^\nu \left[\int_t^T f(X_s, I_s)\, ds + g(X_T)\right], \\ \mathcal{V} & = & \{\nu_t(\omega, a) : \mathbb{F}^{W,\mu}\text{-predictable}, 0 < \nu \le \sup \nu < \infty\} \end{array}$$

**Theorem.** Assume (A). Then $v(t,x) = v^{\mathcal{R}}(t,x)$.

Kharroubi-Pham AOP 2015, F.-Pham AAP 2015, E. Bandini, A. Cosso, M. F., H. Pham. AAP 2018,

Method of proof: control-theoretic arguments and point process constructions.

## Proof of the inequality $v(t,x) \le v^{\mathcal{R}}(t,x)$

Define

$$J(\alpha) = \mathbb{E}\left[\int_t^T f(X_s^\alpha, \alpha_s)\, ds + g(X_T^\alpha)\right],$$
$$J(\nu) = \mathbb{E}^\nu\left[\int_t^T f(X_s, I_s)\, ds + g(X_T)\right].$$

For $\epsilon > 0$ let $\alpha \in \mathcal{A}_d$ be such that $J(\alpha) \ge v(t,x) - \epsilon$.
Find a piecewise-constant process $I' \in \mathcal{A}_d$ such that $J(I') \ge v(t,x) - 2\epsilon$. Identify $I' \equiv \mu' \equiv (T_n', A_n')$.

Take independent random elements $(T^\epsilon, A^\epsilon)$ in $(\Omega', \mathcal{F}', \mathbb{P}')$ and construct the product probability $\mathbb{Q} := \mathbb{P} \times \mathbb{P}'$ on $\Omega \times \Omega'$.
Perturb $\mu'$ using random elements

$$T_n = T_n' + T^\epsilon, \quad A_n = A_n' + A^\epsilon,$$

and construct $I \equiv \mu \equiv (T_n, A_n)$ in such a way that:

- $J(I) \geq v(t,x) - 3\epsilon$.

- the compensator of under $\mathbb{Q}$ is of the form $\nu_t(a)\lambda(da)dt$ with

$$0 < c \leq \nu_t(a) \leq C.$$

By Girsanov define $\mathbb{P} := \mathbb{Q}^{\nu^{-1}}$ and verify:

- $\mathbb{Q} = \mathbb{P}^{\nu}$.
- under $\mathbb{P}$, $\mu$ has compensator $\lambda(da)dt$, so it is a Poisson process;
- under $\mathbb{P}$, $\mu$ is independent of $W$.

It follows that $J(\nu) = J(I) \geq v(t,x) - 3\epsilon$, so that

$$v^{\mathcal{R}}(t,x) \geq v(t,x) - 3\epsilon.$$

## 4. Constrained BSDE representation for the value function.

## A class of constrained BSDEs to represent the value

Solve $dX_s = b(X_s, I_s)\,ds + \sigma(X_s, I_s)\,dW_s, \quad X_t = x$
and consider the BSDE on $[t, T]$: $\mathbb{P}$-a.s.

$$Y_s + \int_s^T Z_r\,dW_r + \int_s^T \int_A U_r(a)\,\mu(dr\,da)$$
$$= g(X_T) + \int_s^T f(X_r, I_r)\,dr + K_T - K_s,$$

$U_s(a) \le 0$, a constraint to hold $d\mathbb{P}\lambda(da)ds$-a.s.

A solution is a quadruple $(Y_s(\omega), Z_s(\omega), U_s(\omega, a), K_s(\omega))$
where $s \in [t, T]$, $a \in A$, such that
$Y$ is adapted; $Z, U, K$ are predictable (w.r.t. $\mathbb{F}^{W,\mu}$);
$Y$ is càdlàg, $K$ càdlàg increasing, $K_t = 0$;

$$\mathbb{E}\Big[\sup_{s\in[t,T]} |Y_s|^2 + \int_t^T \|Z_s\|^2\,ds + \int_t^T \int_A |U_s(a)|^2\,\lambda(da)\,ds + K_T^2\Big] < \infty.$$

**Theorem** Under assumptions (A) there exists a unique solution $(Y, Z, U, K) = (Y^{t,x}, Z^{t,x}, U^{t,x}, K^{t,x})$ to the constrained BSDE which is minimal, i.e. for any other solution $(Y', Z', U', K')$ as above we have $\mathbb{P}$-a.s.

$$Y_t \le Y'_t, \quad t \ge 0.$$

Moreover we have

$$Y_t^{t,x} = v^{\mathcal{R}}(t,x) = v(t,x)$$

and more generally

$$Y_s = \text{ess sup}_{\nu \in \mathcal{V}} \mathbb{E}^{\nu} \left[ \int_s^T f(X_r, I_r)\, dr + g(X_T) \middle| \mathcal{F}_s^{W,\mu} \right].$$

Kharroubi-Pham AOP 2015, Kharroubi, Ma, Pham, Zhang AOP 2010.

In these papers it is proved that $(t,x) \mapsto Y_t^{t,x}$ is a solution to HJB or to a QVI in optimal impulse problems.

## Proof by penalization and monotonic limit

Set $t = 0$. Solve $dX_t = b(X_t, I_t)\, dt + \sigma(X_t, I_t)\, dW_t$, $X_0 = x$.
Solve the penalized equations, for unknown $(Y_t^n, Z_t^n, U_t^n(a))$:

$$Y_t^n + \int_t^T Z_s^n\, dW_s + \int_t^T \int_A U_s^n(a)\, \mu(ds\, da) = g(X_T) + \int_t^T f(X_s, I_s)\, ds + K_T^n - K_t^n,$$

where $K_t^n := n \int_0^t \int_A [U_s^n(a)]^+ \lambda(da)\, ds$. One proves

$$Y_t^n = \operatorname*{ess\,sup}_{\nu \in \mathcal{V},\, \nu \le n} \mathbb{E}^\nu \left[ \int_t^T f(X_s, I_s)\, ds + g(X_T) \Big| \mathcal{F}_t^{W,\mu} \right]$$

and so $Y_t^n \le Y_t^{n+1}$ and then $(Y^n, Z^n, U^n, K^n) \to (Y, Z, U, K)$, the solution. In the limit, $[U_t(a)]^+ = 0$. For $t = 0$,

$$Y_0 = \uparrow \lim_n Y_0^n = \sup_{\nu \in \mathcal{V},\, \nu \le n} \mathbb{E}^\nu \left[ \int_0^T f(X_s, I_s)\, ds + g(X_T) \right] = v^{\mathcal{R}}(0, x).$$

## 5. Application: non-Markovian control problems.

# Non-Markovian optimal control

$$\begin{cases} dX_s^\alpha = b_s(X^\alpha, \alpha_s)\, ds + \sigma_s(X^\alpha, \alpha_s)\, dW_s, & s \in [t,T] \subset [0,T], \\ X_s^\alpha = x(s), \ s \in [0,t]. \end{cases}$$

Here $b_t(x,a) \in \mathbb{R}^n$, $\sigma_t(x,a) \in \mathbb{R}^{n \times d}$ depend on $a \in A$ and

$$x = x(\cdot) : [0,T] \to \mathbb{R}^n \text{ continuous,}$$

but they are non-anticipative functionals:

$$x(s) = x'(s), \ s \in [0,t] \quad \Rightarrow \quad \begin{cases} b_t(x,a) = b_t(x',a) \\ \sigma_t(x,a) = \sigma_t(x',a) \end{cases}$$

More precisely, we require $b_t(x,a)$ etc. to be a progressive process with respect to the canonical coordinate filtration on the space of continuous paths $x(\cdot)$.

Examples:

$$X_t^\alpha = x + \ldots + \int_0^t k(t-s)\, X_s^\alpha\, ds + b(X_{t-\delta}^\alpha) + \ldots$$

for a memory kernel $k$ and a delay $\delta > 0$, or more general path-dependent coefficients.

Path-dependence might be included in the control as well:

$$dX_s^\alpha = b_s(X^\alpha, \alpha)\, ds + \sigma_s(X^\alpha, \alpha)\, dW_s.$$

Maximize: $$\sup_{\alpha \in \mathcal{A}_d} \mathbb{E}\left[\int_t^T f_s(X^\alpha, \alpha_s)\, ds + g(X^\alpha)\right].$$

## Assumptions (A1)

- Continuity: $b_t(x,a)$, $\sigma_t(x,a)$, $f_t(x,a)$, $g(x,a)$ are continuous functions of $t \in [0,T]$, $a \in A$, $x \in C([0,T];\mathbb{R}^n)$.
[$A$ is given its metric, $C([0,T];\mathbb{R}^n)$ the supremum norm.]

- $b, \sigma$ are Lipschitz continuous in $x$ uniformly in $a$: $\exists L \geq 0$ such that

$$|b_t(x,a) - b_t(x',a)| + |\sigma_t(x,a) - \sigma_t(x',a)| \leq L \sup_{s\in[0,t]} |x(s) - x'(s)|.$$

- $b_t(0,a), \sigma_t(0,a)$ are bounded: $\exists M \geq 0$ such that

$$|b_t(0,a)| + |\sigma_t(0,a)| \leq M.$$

- $f, g$ have polynomial growth in $x$ uniformly in $a$: $\exists r \geq 0$ such that

$$|f_t(x,a)| + |g(x,a)| + |c_t(x,a,a')| \leq M(1 + \sup_{s\in[0,t]} |x(s)|^r).$$

$$\begin{cases} dX_s^\alpha = b_s(X^\alpha, \alpha_s)\, ds + \sigma_s(X^\alpha, \alpha_s)\, dW_s, & s \in [t,T] \subset [0,T], \\ X_s^\alpha = x(s), \ s \in [0,t]. \end{cases}$$

The value

$$v(t, x(\cdot)) = \sup_{\alpha \in \mathcal{A}_d} \mathbb{E}\left[\int_t^T f_s(X^\alpha, \alpha_s)\, ds + g(X^\alpha)\right]$$

is a function of $(t, (x(s))_{s \in [0,t]})$.

Possible characterizations for $v$:

• prove that $v$ is the unique solution to the path-dependent HJB equation, a PPDE.

• constrained BSDEs and the randomization method.

We take $I \equiv \mu$ Poisson, independent of $W$, with intensity $\lambda$ (finite measure with full support in $A$). Randomized state equation:

$$\begin{cases} dX_s = b_s(X, I_s)\, ds + \sigma_s(X, I_s)\, dW_s, & s \in [t, T] \subset [0, T], \\ X_s = x(s), \ s \in [0, t]. \end{cases}$$

We construct probabilities $\mathbb{P}^\nu$ as before and define

$$v^{\mathcal{R}}(t, x(\cdot)) = \sup_{\nu \in \mathcal{V}} \mathbb{E}^\nu \left[ \int_t^T f_s(X, I_s)\, ds + g(X) \right].$$

**Theorem.** Assume (A1). Then

$$v(t, x(\cdot)) = v^{\mathcal{R}}(t, x(\cdot)) = Y_t,$$

where $(Y, Z, U, K)$ is the unique minimal solution to the constrained BSDE on $[t, T]$:

$$\begin{cases} Y_s + \int_s^T Z_r\, dW_r + \int_s^T \int_A U_r(a)\, \mu(dr\, da) \\ \qquad = g(X) + \int_s^T f_r(X, I_r)\, dr + K_T - K_s, \\ U_s(a) \leq 0. \end{cases}$$

## 6. Application: control with partial observation.

## Randomization method for partially observed optimal control problems

We start from some reminders on the filtering problem.
Equation for the state $X$ in $\mathbb{R}^n$: on $[0,T]$,

$$dX_t = b(X_t)\,dt + \sigma^1(X_t)\,dV_t^1 + \sigma^2(X_t)\,dV_t^2, \qquad X_0 = x_0$$

$(V^1, V^2)$ standard Wiener process in $\mathbb{R}^{m+d}$ defined in $(\Omega, \mathcal{F}, \bar{\mathbb{P}})$
$\bar{\mathbb{P}}$ = the "physical" probability.
Equation for the observation $W$ in $\mathbb{R}^d$:

$$dW_t = h(X_t)\,dt + dV_t^2, \qquad W_0 = 0.$$

$b, \sigma^1, \sigma^2, h$ "nice" (e.g. Lipschitz bounded).

$\mathbb{F}^W = (\mathcal{F}_t^W)_{t\geq 0}$ = the filtration generated by $W$.

Filtering problem: characterize the filter process $(\pi_t)$ with values in $\mathbb{P}(\mathbb{R}^n)$ such that

$$\pi_t(\phi) = \bar{\mathbb{E}}\,[\phi(X_t) \mid \mathcal{F}_t^W],$$

(optional projection) for every $\phi$ test (e.g. bounded smooth).

# The reference probability method

"Reference" probability $d\mathbb{P} = Z_T^{-1}\, d\bar{\mathbb{P}}$ where

$$Z_t^{-1} = \exp\left(-\int_0^t h(X_s)\, dV_s^2 - \frac{1}{2}\int_0^t |h(X_s)|^2\, ds\right)$$

$(V^1, W)$ standard Wiener in $(\Omega, \mathcal{F}, \mathbb{P})$ on $[0, T]$.

Define the unnormalized filter process $(\rho_t)$: for every $\phi$ test,

$$\rho_t(\phi) = \mathbb{E}\left[\phi(X_t) Z_t \mid \mathcal{F}_t^W\right].$$

Then

$$\pi_t(\phi) = \rho_t(\phi)/\rho_t(1)$$

and $(\rho_t)$ solves the Zakai equation

$$d\rho_t(\phi) = \rho_t(\mathcal{L}\phi)\, dt + \rho_t(h\, \phi + \mathcal{M}\phi)\, dW_t$$

where $\mathcal{L}\phi = \frac{1}{2} Tr(\sigma\sigma^T \nabla^2 \phi) + \nabla\phi\, b$, $\mathcal{M}\phi = \sigma^2 \nabla\phi$, $\sigma = (\sigma^1, \sigma^2)$.

In $(\Omega, \mathcal{F}, \mathbb{P})$ the process $(V^1, W)$ is Wiener,

$$dX_t = (b - \sigma^2 h)(X_t)\, dt + \sigma^1(X_t)\, dV_t^1 + \sigma^2(X_t)\, dW_t, \qquad X_0 = x_0,$$

$$dZ_t = Z_t\, h(X_t)\, dW_t, \qquad Z_0 = 1.$$

Given a functional

$$J = \bar{\mathbb{E}}\left[\int_0^T f(X_t)\, dt + g(X_T)\right]$$

we have

$$J = \mathbb{E}\left[\int_0^T Z_t\, f(X_t)\, dt + Z_T\, g(X_T)\right].$$

## Weak formulation of the partially observed problem

See e.g. Bensoussan (1993). In the reference probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we take a Wiener process $(V^1, W)$ and call $W$ the observation.
A control is a process $(\alpha_t)$ progressive for $\mathbb{F}^W$ with values in $A$.

Controlled state equation and equation for the density process:

$$\begin{aligned} dX_t &= (b - \sigma^2 h)(X_t, \alpha_t)\, dt + \sigma^1(X_t, \alpha_t)\, dV_t^1 + \sigma^2(X_t, \alpha_t)\, dW_t, \\ dZ_t &= Z_t\, h(X_t, \alpha_t)\, dW_t. \end{aligned}$$

Maximize $J(\alpha) = \mathbb{E}\left[\int_0^T Z_t\, f(X_t, \alpha_t)\, dt + Z_T\, g(X_T)\right]$.

Let $V^2$ be defined by $dV_t^2 = dW_t - h(X_t, \alpha_t)\, dt$. Then, under the "physical" probability $d\bar{\mathbb{P}} = Z_T\, d\mathbb{P}$, $(V^1, V^2)$ is Wiener and

$$\begin{aligned} dX_t &= b(X_t, \alpha_t)\, dt + \sigma^1(X_t, \alpha_t)\, dV_t^1 + \sigma^2(X_t, \alpha_t)\, dV_t^2, \ X_0 = x_0, \\ J(\alpha) &= \bar{\mathbb{E}}\left[\int_0^T f(X_t, \alpha_t)\, dt + g(X_T)\right]. \end{aligned}$$

Note: $X, Z, V^2, \bar{\mathbb{P}}$ depend on $\alpha$.

$(V^1, W)$ Wiener in $(\Omega, \mathcal{F}, \mathbb{P})$, controls $(\alpha_t)$ progressive for $\mathbb{F}^W$.

$$\begin{cases} dX_t &= (b - \sigma^2 h)(X_t, \alpha_t)\, dt + \sigma^1(X_t, \alpha_t)\, dV_t^1 + \sigma^2(X_t, \alpha_t)\, dW_t, \\ X_0 &= x_0, \\ dZ_t &= Z_t\, h(X_t, \alpha_t)\, dW_t, \\ Z_0 &= 1 \end{cases}$$

Maximize $J(\alpha) = \mathbb{E}\left[\int_0^T Z_t\, f(X_t, \alpha_t)\, dt + Z_T\, g(X_T)\right]$.

Classical approach: controlled Zakai equation for $\rho_t(\phi) = \mathbb{E}\left[\phi(X_t) Z_t \mid \mathcal{F}_t^W\right]$ :

$$d\rho_t(\phi) = \rho_t(\mathcal{L}^{\alpha_t}\phi)\, dt + \rho_t(h(\cdot, \alpha_t)\phi + \mathcal{M}^{\alpha_t}\phi)\, dW_t$$

where $\mathcal{L}^a\phi = \frac{1}{2} Tr(\sigma\sigma^T(\cdot, a) D^2\phi) + \nabla\phi b(\cdot, a)$, $\mathcal{M}^a\phi = \nabla\phi\sigma^2(\cdot, a)$
$\sigma = (\sigma^1, \sigma^2)$, $\phi$ test.

Maximize $J(\alpha) = \mathbb{E}\left[\int_0^T \rho_t\,(f(\cdot, \alpha_t))\, dt + \rho_T\,(g(\cdot))\right]$.
A full observation infinite-dimensional optimal control problem. Its Hamilton-Jacobi-Bellman equation is fully nonlinear and very degenerate.

$(V^1, W)$ Wiener in $(\Omega, \mathcal{F}, \mathbb{P})$, controls $(\alpha_t)$ progressive for $\mathbb{F}^W$.

$$\begin{cases} dX_t &= (b - \sigma^2 h)(X_t, \alpha_t)\, dt + \sigma^1(X_t, \alpha_t)\, dV_t^1 + \sigma^2(X_t, \alpha_t)\, dW_t, \\ X_0 &= x_0, \\ dZ_t &= Z_t\, h(X_t, \alpha_t)\, dW_t, \\ Z_0 &= 1 \end{cases}$$

Maximize $J(\alpha) = \mathbb{E}\left[\int_0^T Z_t\, f(X_t, \alpha_t)\, dt + Z_T\, g(X_T)\right]$.

Setting $X_t^\alpha = (X_t, Z_t)$ we write the above problem in the form

$$\begin{cases} dX_t^\alpha &= \tilde{b}(X_t^\alpha, \alpha_t)\, dt + \tilde{\sigma}^1(X_t^\alpha, \alpha_t)\, dV_t^1 + \tilde{\sigma}^2(X_t^\alpha, \alpha_t)\, dW_t, \\ X_0^\alpha &= (x_0, 1) \end{cases}$$

Maximize

$$J(\alpha) = \mathbb{E}\left[\int_0^T \tilde{f}(X_t^\alpha, \alpha_t)\, dt + \tilde{g}(X_T^\alpha)\right]$$

over all $(\alpha_t)$ progressive for $\mathbb{F}^W$.

Note: in this reformulation we have $h = 0$ and so $\pi_t = \rho_t$.

## The addressed control problem with partial observation

Let $b, \sigma, f, g$ satify assumptions (A).
Given $(\Omega, \mathcal{F}, \mathbb{P})$ and a Wiener process $(V, W)$ consider

$$\begin{aligned} dX_s^\alpha &= b(X_s^\alpha, \alpha_s)\, ds + \sigma^1(X_s^\alpha, \alpha_s)\, dV_s + \sigma^2(X_s^\alpha, \alpha_s)\, dW_s, \qquad s \in [t, T], \\ X_t{}^\alpha &= x_0 \text{ with law } \xi, \end{aligned}$$

Controls: $(\alpha_t)$ progressive for $\mathbb{F}^W$ with values in $A$.

Define the reward (depending only on $\xi$)

$$J(t, \xi, \alpha) = \mathbb{E}\left[\int_t^T f(X_s^\alpha, \alpha_s)\, ds + g(X_T^\alpha)\right].$$

and the value function

$$v(t, \xi) = \sup_{\alpha(\cdot)} J(t, \xi, \alpha).$$

## The equivalent randomized control problem

We take $I \equiv \mu$ Poisson, independent of $(V, W)$, with intensity $\lambda$ (finite with full support in $A$). Randomized state equation:

$$dX_s = b(X_s, I_s)\, ds + \sigma^1(X_s, I_s)\, dV_s + \sigma^2(X_s, I_s)\, dW_s, \quad X_t = x_0.$$

Let $\mathbb{F}^{W,\mu} = (\mathcal{F}_t^{W,\mu})$ be generated by $W, \mu$ alone.

$\mathcal{V} = \{\nu_t(\omega, a) : \mathcal{P}(\mathbb{F}^{W,\mu}) \otimes \mathcal{B}(A)$−measurable, $0 < \nu \leq \sup \nu < \infty\}$

Under $d\mathbb{P}^\nu = \kappa_T^\nu \, d\mathbb{P}$, on $[t, T]$:

- $(V, W)$ remains Wiener, and
- $\mu$ has compensator $\nu_t(a)\lambda(da)dt$.

$$v^{\mathcal{R}}(t, \xi) = \sup_{\nu \in \mathcal{V}} \mathbb{E}^\nu \left[ \int_t^T f(X_s, I_s)\, ds + g(X_T) \right],$$

**Theorem.** We have $v(t, \xi) = v^{\mathcal{R}}(t, \xi)$.
It only depends on $b, \sigma, f, g$ (not on $\Omega, \mathcal{F}, \mathbb{P}, V, W, \mu, \lambda, x_0, a$).

E. Bandini, A. Cosso, M. F., H. Pham. AAP 2018, where non Markovian case is also treated.

## Filter processes in the randomized framework.

For $\phi$ test, set

$$\rho_s(\phi) = \mathbb{E}\,[\phi(X_s) \mid \mathcal{F}_s^{W,\mu}].$$

The pair $(\rho, I)$ is Markovian in $\mathbb{P}(\mathbb{R}^n) \times A$ where

- $(I_s) = (I_s^{t,a_0})_{s\in[t,T]}$ is Poisson $\lambda$ starting at $a_0 \in A$ at time $t$;
- $(\rho_s) = (\rho_s^{t,\xi,a_0})_{s\in[t,T]}$ satisfies the randomized Zakai equation:

for $s \in [t,T]$,

$$d\rho_s(\phi) = \rho_s(\mathcal{L}^{I_s}\phi)\,ds + \rho_s(\mathcal{M}^{I_s}\phi)\,dW_s, \qquad \rho_t(\phi) = \xi(\phi).$$

where $\mathcal{L}^a\phi = \frac{1}{2}Tr(\sigma\sigma^T(\cdot,a)\nabla^2\phi) + \nabla\phi b(\cdot,a)$, $\mathcal{M}^a\phi = \nabla\phi\sigma^2(\cdot,a)$, $\sigma = (\sigma^1,\sigma^2)$.

Compare with the controlled Zakai equation (for a different $\rho_s$)

$$d\rho_s(\phi) = \rho_s(\mathcal{L}^{\alpha_s}\phi)\,ds + \rho_s(h(\cdot,\alpha_s)\phi + \mathcal{M}^{\alpha_s}\phi)\,dW_s.$$

# The constrained BSDE representing the value function

**Theorem.** We have

$$v(t,\xi) = v^{\mathcal{R}}(t,\xi) = Y_t^{t,\xi,a_0},$$

where $(Y,Z,U,K) = (Y^{t,\xi,a_0}, Z^{t,\xi,a_0}, U^{t,\xi,a_0}, K^{t,\xi,a_0})$ is the unique minimal solution to the constrained BSDE on $[t,T]$:

$$\begin{cases} Y_s + \int_s^T Z_r\, dW_r + \int_s^T \int_A U_r(a)\, \mu(dr\, da) \\ \qquad = \rho_T(g) + \int_s^T \rho_r(f(\cdot, I_r))\, dr + K_T - K_s, \\ U_s(a) \leq 0. \end{cases}$$

Moreover,

$$Y_s^{t,\xi,a_0} = v(s, \rho_s^{t,\xi,a_0}), \qquad s \in [t,T].$$

Based on this results one can prove that $v(t,\xi)$ is a viscosity solution to a HJB equation on $[0,T] \times \mathbb{P}(\mathbb{R})$.

Bandini, Cosso, F., Pham, SPA 2019.

## 7. Related results and final comments.

## Some general comments

- No nondegeneracy condition on $\sigma$.
- Markovian and non-Markovian case treated similarly.
- No result on existence of an optimal control.
- Numerical methods have been developed for constrained BS-DEs of this form.

## Other applications of the control randomization method

- Optimal switching
(Bouchard 09, Elie-Kharroubi 10, 14, 14, F.-Morlais 19)
- Impulse control (Kharroubi-Pham-Ma-Zhang 10).
- Jump-diffusion (Kharroubi-Pham 14)
- Optimal stopping (F.-Pham-Zeni 15).
- Control of pure jump processes (Bandini-F. 17)
- Control of piecewise-deterministic Markov processes
(Bandini 19, 21, Bandini-Thieullen 21)
- Infinite horizon (Confortola-Cosso-Fuhman 19)
- Ergodic control (Cosso-F.-Pham 16)
- Markovian jump-diffusion with controlled intensity
(Choukroun-Cosso 16).
- Control of McKean-Vlasov systems
(Bayraktar-Cosso-Pham 18)
- Control of infinite-dimensional jump-diffusions
(Bandini-Confortola-Cosso 19)
- Numerical methods (Kharroubi-Langrené-Pham 14, 15)
- Weak formulation (F.-Pham 15)

**Thank you for your attention!**