Control randomization method and BSDEs with constrained jumps

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Plan

1. Reminders on classical stochastic optimal control problems.

2. The value function and its characterizations: Hamilton-Jacobi-Bellman equations (HJB), BSDEs.


4. Digression on marked point processes and associated control problems.

5. Back to the control randomization method.

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1. Reminders on classical stochastic optimal control problems.
Classical stochastic optimal control

Controlled SDE in $\mathbb{R}^n$:
\[
\begin{aligned}
\text{d}X_s &= b(X_s, \alpha_s) \text{d}s + \sigma(X_s, \alpha_s) \text{d}W_s, \\
X_t &= x \in \mathbb{R}^n,
\end{aligned}
\quad s \in [t,T] \subset [0,T],
\]

- $W$ is a Wiener process in $\mathbb{R}^d$, defined in $(\Omega, \mathcal{F}, \mathbb{P})$.

We will use the Brownian filtration $\mathbb{F}^W = (\mathcal{F}_t^W)_{t \geq 0}$, namely we assume $\mathcal{F}$ to be $\mathbb{P}$-complete and we set
\[
\mathcal{F}_t^W := \sigma(W_s, s \in [0,t]) \lor \mathcal{N}
\]
where $\mathcal{N}$ are $\mathbb{P}$-null sets of $\mathcal{F}$.

The coefficients of an SDE are controlled:
\[
b(x, a) \in \mathbb{R}^n, \quad \sigma(x, a) \in \mathbb{R}^{n \times d},
\]
i.e. they depend on a parameter $a \in A$:
- $A$, the space of control actions, is a complete separable metric space (or a Borel subset of it).
\[
\begin{align*}
\left\{
\begin{array}{ll}
\mathrm{d}X_s &= b(X_s, \alpha_s) \, \mathrm{d}s + \sigma(X_s, \alpha_s) \, \mathrm{d}W_s, \\
X_t &= x \in \mathbb{R}^n, \\
\end{array}
\right. \\
&\quad s \in [t, T],
\end{align*}
\]

A controller dynamically selects her actions by choosing a control process

\[\alpha_s(\omega) \in A.\]

• \(A_d = \{\alpha : \Omega \times [0, T] \to A, \mathbb{F}^W\text{-progressive}\}\) is the space of admissible controls.

The corresponding solution \(X_s = X_s^\alpha = X_{s, t, x}^\alpha\) is called the trajectory corresponding to the control \(\alpha\).
Controlled equation:
\[
\begin{align*}
\left\{ 
\begin{array}{l}
\frac{dX^\alpha_s}{ds} = b(X^\alpha_s, \alpha_s) ds + \sigma(X^\alpha_s, \alpha_s) dW_s, \\
X^\alpha_t = x \in \mathbb{R}^n.
\end{array}
\right. 
\end{align*}
\]

The controller tries to maximize the reward functional
\[
J(\alpha, t, x) = \mathbb{E} \left[ g(X^\alpha_{t}, t, x) + \int_{t}^{T} f(X^\alpha_{s}, t, x, \alpha_s) ds \right].
\]

Here
\[
g(x) \in \mathbb{R}, \quad f(x, a) \in \mathbb{R}
\]
are: 
\(i\) the reward corresponding to the final position \(x \in \mathbb{R}^n\) of the state;
\(ii\) the running reward rate when the current state is \(x \in \mathbb{R}^n\) and the control action is \(a \in A\).

Value function:
\[
v(t, x) = \sup_{\alpha \in A_d} J(\alpha, t, x).
\]
Assumptions (A) on the coefficients

On the data of the control problem \( b(x, a), \sigma(x, a), f(x, a), g(x) \) we assume:

• \( b : \mathbb{R}^n \times A \to \mathbb{R}^n, \sigma : \mathbb{R}^n \times A \to \mathbb{R}^{n \times d}, f : \mathbb{R}^n \times A \to \mathbb{R}, g : \mathbb{R}^n \to \mathbb{R} \) are continuous.

• \( b, \sigma \) are Lipschitz in \( x \) uniformly in \( a \): \( \exists L \geq 0 \) such that
  \[ |b(x, a) - b(y, a)| + |\sigma(x, a) - \sigma(y, a)| \leq L|x - y|, \quad x, y \in \mathbb{R}^n, a \in A. \]

• \( b(0, a), \sigma(0, a) \) are bounded in \( a \): \( \exists M \geq 0 \) such that
  \[ |b(0, a)| + |\sigma(0, a)| \leq M, \quad a \in A. \]

• \( f, g \) have polynomial growth in \( x \) uniformly in \( a \): \( \exists r \geq 0 \) such that
  \[ |f(x, a)| + |g(x)| \leq M (1 + |x|^r), \quad x \in \mathbb{R}^n, a \in A. \]

Under these assumptions \( X^\alpha \) is well defined and \( v(t, x) \) is finite.
2. The value function and its characterizations: Hamilton-Jacobi-Bellman equations (HJB), BSDEs.
Hamilton-Jacobi-Bellman equation (HJB)

\[
-\partial v(t,x) = \sup_{a \in A} \left[ \mathcal{L}^a v(t,x) + f(x,a) \right], \quad t \in [0,T], \; x \in \mathbb{R}^n,
\]

\[
v(T,x) = g(x),
\]

Controlled Kolmogorov operator:

\[
\mathcal{L}^a \phi(x) = \frac{1}{2} \text{tr} \left[ \sigma(x,a) \sigma^T(x,a) D^2 \phi(x) \right] + D \phi(x) b(x,a)
\]

Available results are two-fold:

- Any classical solution to HJB coincides with the value function:

\[
v(t,x) = \sup_{\alpha \in \mathcal{A}_d} \mathbb{E} \left[ g(X^\alpha_T) + \int_t^T f(X^\alpha_s, \alpha_s) \, ds \right].
\]

Results of this kind are called verification theorems and often an optimal control is also found.

- The value function, defined by the formula above, is a solution to HJB, possibly in viscosity sense. Often, uniqueness of the viscosity solution is also proved.
The search for an associated BSDE

Aim: find a BSDE giving a representation of the value function:

\[ v(t, x) = \sup_{\alpha \in A_d} \mathbb{E} \left[ g(X_T^\alpha) + \int_t^T f(X_s^\alpha, \alpha_s) \, ds \right]. \]

Motivations:
- Efficient numerical methods are often available for BSDEs.
- Immediate extensions to non-Markovian systems (e.g. with memory).
- Associated BSDEs are known in special cases.

Available approaches:
- The theory of \textit{G}-expectation (Peng). It is based on a different foundation of stochastic calculus and even probability theory. It replaces the expectation operator by a (non-linear) analogue satisfying appropriate axioms.
- The theory of \textit{second order BSDEs} (2BSDEs, Soner-Touzi). One formulates a BSDE that can be solved under a family of (mutually singular) probability measures. It often requires non-degenerate diffusion (i.e. invertible \( \sigma(x, a) \)).
The randomization method in optimal control

Introduced in

\[
\begin{align*}
\begin{cases}
    \ dX^\alpha_s &= b(X^\alpha_s, \alpha_s) \, ds + \sigma(X^\alpha_s, \alpha_s) \, dW_s, & s \in [t, T], \\
    \ X^\alpha_t &= x \in \mathbb{R}^n.
\end{cases}
\end{align*}
\]

\[
v(t, x) = \sup_{\alpha \in A_d} \mathbb{E}\left[ g(X^\alpha_T) + \int_t^T f(X^\alpha_s, \alpha_s) \, ds \right],
\]

Idea:
1) replace \((\alpha_s)\) by a random (uncontrolled) process \((I_s)\) with piecewise constant trajectories and values in \(A\);
2) formulate an auxiliary (“randomized”) control problem, where “the law of \(I\) is controlled”, having value denoted \(v^R(t, x)\);
3) prove that \(v = v^R\);
4) represent \(v^R(t, x)\) by a BSDE.
The randomized control problem

We replace the control $\alpha \in A_d$ by an $A$-valued process $I$, independent of $W$. We consider the “randomized” state equation:

$$dX_s = b(X_s, I_s) \, ds + \sigma(X_s, I_s) \, dW_s, \quad s \in [t, T]; \quad X_t = x.$$ 

Then we construct a suitable family of probability measures $P^\nu$, depending on a parameter $\nu \in \mathcal{V}$, such that

- $P^\nu \sim P$ (dominated model)
- $W$ remains a Wiener process under $P^\nu$.

Then we optimize among $P^\nu$: we formulate an auxiliary (“randomized”) control problem with value function:

$$v^R(t, x) = \sup_{\nu \in \mathcal{V}} \mathbb{E}^\nu \left[ \int_t^T f(X_s, I_s) \, ds + g(X_T) \right].$$
We wish to prove equivalence with the randomized problem namely that

\[ v(t, x) = v^R(t, x). \]

Original problem: for \( s \in [t, T] \subset [0, T], \ x \in \mathbb{R}^n, \)

\[
\begin{align*}
    dX^\alpha_s &= b(X^\alpha_s, \alpha_s) \, ds + \sigma(X^\alpha_s, \alpha_s) \, dW_s, \\
    X^\alpha_t &= x, \\
    v(t, x) &= \sup_{\alpha \in A} \mathbb{E} \left[ \int_t^T f(X^\alpha_s, \alpha_s) \, ds + g(X^\alpha_T) \right],
\end{align*}
\]

Randomized problem:

\[
\begin{align*}
    dX_s &= b(X_s, I_s) \, ds + \sigma(X_s, I_s) \, dW_s, \\
    X_t &= x, \\
    v^R(t, x) &= \sup_{\nu \in \mathcal{V}} \mathbb{E}^\nu \left[ \int_t^T f(X_s, I_s) \, ds + g(X_T) \right].
\end{align*}
\]
The choice of the process $I$

When $A = \mathbb{R}^k$ one can choose $(I_s)$ as another Brownian motion, independent of $W$. The probabilities $\mathbb{P}^\nu$ are then defined by a classical Girsanov theorem. See for instance S. Choukroun, A. Cosso. Backward SDE representation for stochastic control problems with nondominated controlled intensity. The Annals of Applied Probability 26, no. 2 (2016), 1208-1259.

For the general case we first note that

$$v(t, x) = \sup_{\alpha \in \mathcal{A}_d} \mathbb{E} \left[ \int_t^T f(X_\alpha^s, \alpha_s) \, ds + g(X_\alpha^T) \right]$$

remains unchanged if we restrict to $(\alpha_s)$ being a piecewise constant process: see for instance Lemma 3.2.6 in N.V. Krylov. Controlled diffusion processes. Springer, 2009.

We will choose $(I_s)$ to be a (pure jump) Poisson process in $A$. 
4. Digression on marked point processes and associated control problems.
Reminders on pure jump and marked point processes

We use notions from


See also


On a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with a right-continuous filtration \(\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}\) we define a marked point process \((T_n, A_n)_{n \geq 1}\) where

- \(T_n\): a (non-explosive) point process, i.e. \(T_n\) are \(\mathbb{F}\)-stopping times, \(0 < T_n \leq T_{n+1} \uparrow \infty, T_n < T_{n+1}\) if \(T_n < \infty\);

- \(A_n\): \(A\)-valued random variables, \(A_n\) is \(\mathcal{F}_{T_n}\)-measurable.
To \((T_n, A_n)_{n \geq 1}\) we associate:

- a pure jump \(A\)-valued process

\[
I_t = a_0 \mathbb{1}_{\{0 \leq t < T_1\}} + A_1 \mathbb{1}_{\{T_1 \leq t < T_2\}} + A_2 \mathbb{1}_{\{T_2 \leq t < T_3\}} + \ldots ,
\]

where \(a_0 \in A\) is deterministic and fixed (in fact, \(I_s = I_{s_0}^a\)).

- A random measure on \((0, \infty) \times A\):

\[
\mu(dt \, da) = \sum_{n \geq 1} \delta_{(T_n, A_n)}(dt \, da) \mathbb{1}_{\{T_n < \infty\}}.
\]

In the sequel we identify \(I \equiv (T_n, A_n)_{n \geq 1} \equiv \mu\).

The natural filtration \(\mathbb{F}^\mu = (\mathcal{F}^\mu_t)\) is defined as

\[
\mathcal{F}^\mu_t = \sigma\{\mu((0, s] \times C) : s \in [0, t], C \in \mathcal{B}(A)\} \lor \mathcal{N}.
\]
Reminders on compensators

The compensator, or dual predictable projection, of $\mu$ is a predictable random measure $\nu(dt\,da)$ such that $\mu - \nu$ is a martingale measure. Formally,

- for $C \in \mathcal{B}(A)$ the process $\nu((0, t] \times C)$ is $\mathbb{F}$-predictable;
- the following equivalent conditions are satisfied:
  
  i) for $C \in \mathcal{B}(A)$ and $n \geq 1$ the processes

  \[ \mu((0, t \land T_n] \times C) - \nu((0, t \land T_n] \times C), \quad t \geq 0, \]

  are $\mathbb{P}$-martingales with respect to $\mathbb{F}$.

  ii) denote $\mathcal{P}(\mathbb{F})$ the predictable $\sigma$-algebra of $\mathbb{F}$; then for any $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(A)$-measurable random field $H(\omega, t, a) \geq 0$,

  \[ \mathbb{E} \int_{(0,t]} H(t, a) \mu(dt\,da) = \mathbb{E} \int_{(0,t]} H(t, a) \nu(dt\,da). \]
One proves that $\nu$ exists and is unique (up to null sets). Moreover

- $\nu$ depends on $\mathbb{P}$.
- $\nu$ determines the law of $I \equiv (T_n, A_n)_{n \geq 1} \equiv \mu$ under $\mathbb{P}$.

For instance, if the compensator is $\nu(dt \, da) = \lambda(da) \, dt$, for some (finite) measure $\lambda$ on $A$, then $\mu$ is a (non-explosive) Poisson process, i.e.

- $(T_n)$ is a standard counting Poisson process with intensity $\lambda(A)$;
- $(A_n)$ is an i.i.d. sequence with law $\frac{\lambda(da)}{\lambda(A)}$ (independent of $(T_n)$).

If $\lambda$ has full topological support then $I$ visits every open set of $A$

One way to “control” the process $I$ is to fix a desired compensator $\nu$ (corresponding to some desired law for $I$) and find a new probability $\mathbb{P}^\nu$ under which $\nu$ is the compensator of $\mu$. 
Instead of choosing an arbitrary compensator we may start from a probability \( \mathbb{P} \) under which \( \mu \) a Poisson process with compensator

\[
\lambda(da)dt.
\]

We look for a probability under which \( I \) has a compensator of the form

\[
\nu_t(\omega, a) \lambda(da)dt
\]

where \( \nu_t(\omega, a) \geq 0 \) is a chosen \( \mathbb{F} \)-predictable random field.

**Theorem** (of Girsanov type). Let \( \nu_t(\omega, a) \) be bounded. The required probability is given by \( d\mathbb{P}^\nu = \kappa^\nu_T d\mathbb{P} \) where \( \kappa^\nu_t \) is the Doléans exponential martingale

\[
\kappa^\nu_t = \exp \left( \int_0^t \int_A \left( 1 - \nu_s(a) \right) \lambda(da)ds \right) \prod_{T_n \leq t} \nu_{T_n}(A_n).
\]
5. Back to the control randomization method.
Return to the original control problem

Let $b, \sigma, f, g$ satisfy Assumption (A).

Given $(\Omega, \mathcal{F}, \mathbb{P}, W)$ consider

$$\begin{cases} 
    dX^\alpha_s &= b(X^\alpha_s, \alpha_s) \, ds + \sigma(X^\alpha_s, \alpha_s) \, dW_s, \quad s \in [t, T], \\
    X^\alpha_t &= x \in \mathbb{R}^n. 
\end{cases}$$

$$v(t, x) = \sup_{\alpha \in \mathcal{A}_d} \mathbb{E} \left[ g(X^\alpha_T) + \int_t^T f(X^\alpha_s, \alpha_s) \, ds \right],$$

Admissible controls:

$$\mathcal{A}_d = \{ \alpha : \Omega \times [0, T] \to A, \mathbb{F}^W - \text{progressive} \}.$$
The randomized control problem

Let $b, \sigma, f, g$ satisfy Assumption (A). Consider $(\Omega, \mathcal{F}, \mathbb{P}, W, \mu)$ where:

- $\mu \equiv (T_n, A_n)_{n \geq 1} \equiv I$ is a Poisson random measure with finite intensity $\lambda(da)$ with full topological support;
- $W$ is an $\mathbb{R}^d$-valued Brownian motion, independent of $\mu$.

We will use the filtration $\mathbb{F}^{W, \mu} = (\mathcal{F}^{W, \mu}_t)$ generated by $W$ and $\mu$:

$\mathcal{F}^{W, \mu}_t = \sigma\{W_s, \mu((0, s] \times C) : s \in [0, t], C \in \mathcal{B}(A)\} \vee \mathcal{N}$.

We consider the “randomized” state equation:

$$dX_s = b(X_s, I_s) \, ds + \sigma(X_s, I_s) \, dW_s, \quad s \in [t, T]; \quad X_t = x.$$ 

The admissible controls are now random fields

$\mathcal{V} = \{\nu_t(\omega, a) : \Omega \times [0, \infty) \times A \to (0, \infty), \mathbb{F}^{W, \mu}-\text{predictable bounded}\}$
Given $\nu_t(\omega, a) \in \mathcal{V}$, we define $d\mathbb{P}^\nu = \kappa^\nu_T d\mathbb{P} \sim d\mathbb{P}$ where

$$\kappa^\nu_t = \exp \left( \int_0^t \int_A (1 - \nu_s(a)) \lambda(da) ds \right) \prod_{T_n \leq t} \nu_{T_n}(A_n).$$

Then, on $[0, T]$,

- $\mu$ has compensator $\nu_t(\omega, a) \lambda(da) dt$ under $\mathbb{P}^\nu$;
- $W$ remains a Wiener process under $\mathbb{P}^\nu$.

We define an auxiliary ("randomized") value function:

$$v^R(t, x) = \sup_{\nu \in \mathcal{V}} \mathbb{E}^\nu \left[ \int_t^T f(X_s, I_s) ds + g(X_T) \right].$$

It can be proved that $v^R$ only depends on $b, \sigma, f, g$ (not on $\Omega, \mathcal{F}, \mathbb{P}, W, \mu, \lambda, a_0$). $\nu$ changes the intensity of the Poisson component $I$ and therefore has an influence on $X$ as well.
Equivalent with the randomized problem

Original problem: for $s \in [t, T] \subset [0, T]$, $x \in \mathbb{R}^n$,

$$dX^\alpha_s = b(X^\alpha_s, \alpha_s) \, ds + \sigma(X^\alpha_s, \alpha_s) \, dW_s,$$
$$X^\alpha_t = x,$$
$$v(t, x) = \sup_{\alpha \in A_d} \mathbb{E} \left[ \int_t^T f(X^\alpha_s, \alpha_s) \, ds + g(X^\alpha_T) \right],$$
$$A_d = \{ \alpha_t(\omega) : \mathbb{F}^W \text{–progressive} \}$$

Randomized problem:

$$dX_s = b(X_s, I_s) \, ds + \sigma(X_s, I_s) \, dW_s,$$
$$X_t = x,$$
$$v^R(t, x) = \sup_{\nu \in \mathcal{V}} \mathbb{E}^\nu \left[ \int_t^T f(X_s, I_s) \, ds + g(X_T) \right],$$
$$\mathcal{V} = \{ \nu_t(\omega, a) : \mathbb{F}^W, \mu \text{–predictable}, 0 < \nu \leq \sup \nu < \infty \}$$

**Theorem.** Assume (A). Then $v(t, x) = v^R(t, x)$.

6. The search for an associated BSDE.
A BSDE for the randomized control problem

Let $b, \sigma, f, g$ satisfy Assumption (A).
Consider $(\Omega, \mathcal{F}, \mathbb{P}, W, \mu)$ where:
- $\mu \equiv (T_n, A_n)_{n \geq 1} \equiv I$ is a Poisson random measure with finite intensity $\lambda(da)$ with full topological support;
- $W$ is an $\mathbb{R}^d$-valued Brownian motion, independent of $\mu$.

We consider the randomized problem starting at $t = 0$:

$$dX_s = b(X_s, I_s) \, ds + \sigma(X_s, I_s) \, dW_s,$$

$$X_0 = x \text{ fixed},$$

$$v^R = \sup_{\nu \in \mathcal{V}} \mathbb{E}^\nu \left[ \int_0^T f(X_s, I_s) \, ds + g(X_T) \right],$$

$$\mathcal{V} = \{ \nu_t(\omega, a) : \mathbb{F}_{W, \mu}-\text{predictable}, 0 < \nu \leq \sup \nu < \infty \}.$$

We approximate $v^R$ by

$$v_n^R = \sup_{\nu \in \mathcal{V}_n} \mathbb{E}^\nu \left[ \int_0^T f(X_s, I_s) \, ds + g(X_T) \right],$$

where $\mathcal{V}_n := \{ \nu \in \mathcal{V}, \nu \leq n \}$. We look for a BSDE representation for the penalized value $v_n^R$ and then for $v^R$.
Clearly $v_n^R \leq v_{n+1}^R \leq v^R$ and even $v_n^R \uparrow v^R$. 

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Consider again \((\Omega, \mathcal{F}, \mathbb{P}, W, \mu)\) and \(\mathbb{F}^W,\mu = (\mathcal{F}^W_t,\mu)\). If \(M\) is an \(\mathbb{F}^W,\mu\)-martingale in \(L^2\) then

\[
M_t = M_0 + \int_0^t Z_s \, ds + \int_0^t \int_A U_s(a) [\mu(ds \, da) - \lambda(da) \, ds], \quad t \in [0,T],
\]

where \(Z\) and \(U\) are in the spaces

\[
L^2_W := \{Z_s(\omega) \in \mathbb{R}^d : \mathbb{F}^W,\mu - \text{predictable, } \mathbb{E} \int_0^T |Z_s|^2 ds < \infty\},
\]

\[
L^2_\mu := \{U_s(\omega, a) \in \mathbb{R} : \mathcal{P}(\mathbb{F}^W,\mu) \otimes \mathcal{B}(A) - \text{measurable, } \mathbb{E} \int_0^T \int_A |U_s(a)|^2 \lambda(da) \, ds < \infty\}.
\]

It follows that \(M\) belongs to

\[
S^2 := \{Y_s(\omega) \in \mathbb{R} : \mathbb{F}^W,\mu - \text{adapted and càdlàg, } \mathbb{E} \sup_{s \in [0,T]} |Y_s|^2 < \infty\}
\]

Note that \(\int_0^T \int_A U_s(a) \mu(ds \, da) = \sum_{n \geq 1} U_{T_n}(A_n) \mathbb{1}_{\{T_n \leq T\}}\).
Now consider the BSDE: for \( s \in [0, T] \),
\[
Y_s + \int_s^T Z_r \, dW_r + \int_s^T \int_A U_r(a) \left[ \mu(dr \, da) - \lambda(da) \, dr \right] = g(X_T) + \int_s^T f(X_r, I_r) \, dr
\]
for the unknown \( (Y, Z, U) \in S^2 \times L^2_W \times L^2_\mu \). Define
\[
Y_s = \mathbb{E} \left[ g(X_T) + \int_s^T f(X_r, I_r) \, dr \mid \mathcal{F}^W_{s, \mu} \right] = M_s + \int_0^s f(X_r, I_r) \, dr
\]
where
\[
M_s = \mathbb{E} \left[ g(X_T) + \int_0^T f(X_r, I_r) \, dr \mid \mathcal{F}^W_{s, \mu} \right]
\]
Represent \( M \) and before by \( Z, U \) and check that \( (Y, Z, U) \) is the unique required solution.
As usual, similar results hold if we add a Lipschitz nonlinearity to the BSDE.
A BSDE for the penalized randomized control problem

Now recall

\begin{align*}
    dX_s &= b(X_s, I_s) \, ds + \sigma(X_s, I_s) \, dW_s, \quad X_0 = x, \\
    J(\nu) &= \mathbb{E}^\nu \left[ \int_0^T f(X_s, I_s) \, ds + g(X_T) \right], \\
    \nu_n^{\mathcal{R}} &= \sup_{\nu \in \mathcal{V}_n} \mathbb{E}^\nu \left[ \int_0^T f(X_s, I_s) \, ds + g(X_T) \right], \\
    \mathcal{V}_n &= \{ \nu_t(\omega, a) : \mathbb{F}W,\mu-\text{predictable}, 0 < \nu \leq n < \infty \}
\end{align*}

Consider the penalized equation

\begin{align*}
    Y_s + \int_s^T Z_r \, dW_r + \int_s^T \int_A U_r(a) \mu(dr \, da) \\
    &= g(X_T) + \int_s^T f(X_r, I_r) \, dr + n \int_s^T \int_A U_r(a)^+ \lambda(da) \, dr
\end{align*}

and find a unique solution \((Y^n, Z^n, U^n) \in S^2 \times L^2_W \times L^2_\mu\).

We claim that \(\nu_n^{\mathcal{R}} = Y^n_0\).
Write \((Y, Z, U)\) instead of \((Y^n, Z^n, U^n)\). Fix \(\nu \in \mathcal{V}\) and take \(\mathbb{E}^\nu\) in

\[
Y_0 + \int_0^T Z_r \, dW_r + \int_0^T \int_A U_r(a) \, \mu(dr \, da) \\
= g(X_T) + \int_0^T f(X_r, I_r) \, dr + n \int_0^T \int_A U_r(a)^+ \, \lambda(da) \, dr.
\]

Then

\[
Y_0 + \mathbb{E}^\nu \int_0^T \int_A U_r(a) \, \mu(dr \, da) = J(\nu) + n \mathbb{E}^\nu \int_0^T \int_A U_r(a)^+ \, \lambda(da) \, dr.
\]

Since \(U\) is a predictable random field,

\[
\mathbb{E}^\nu \int_0^T \int_A U_r(a) \, \mu(dr \, da) = \mathbb{E}^\nu \int_0^T \int_A U_r(a) \, \nu_r(a) \, \lambda(da) \, dr.
\]

Substituting and rearranging,

\[
Y_0 = J(\nu) + \mathbb{E}^\nu \int_0^T \int_A [nU_r(a)^+ - \nu_r(a)U_r(a)] \, \lambda(da) \, dr.
\]

Since \(nu^+ - \nu u \geq 0\) for \(\nu \in [0, n]\) with equality when \(\nu = n \mathbb{1}_{u \geq 0}\), we have \(Y_0 \geq J(\nu)\) with equality when \(\nu_s(a) = n \mathbb{1}_{U_s \geq 0}\).
Convergence of the penalized BSDE

The penalized equation

\[ Y^n_s + \int_s^T Z^n_r \, dW_r + \int_s^T \int_A U^n_r(a) \, \mu(dr \, da) \]
\[ = g(X_T) + \int_s^T f(X_r, I_r) \, dr + n \int_s^T \int_A U^n_r(a)^+ \, \lambda(da) \, dr \]

can be written

\[ Y^n_s + \int_s^T Z^n_r \, dW_r + \int_s^T \int_A U^n_r(a) \, \mu(dr \, da) \]
\[ = g(X_T) + \int_s^T f(X_r, I_r) \, dr + K^n_T - K^n_s \]

where

\[ K^n_s = n \int_0^s \int_A U^n_r(a)^+ + \lambda(da) \, dr \]

satisfy \( K^n_0 = 0 \), are increasing and adapted continuous (hence predictable).

Since \( v^n_R \leq v^{R}_{n+1} \leq v^R \) we have \( Y^n_0 \leq Y^{n+1}_0 \leq v^R \).
Convergence of $Y^n$.

We have proved

$$Y_0^n = \nu_n^R = \sup_{\nu \in \mathcal{N}_n} \mathbb{E}^\nu \left[ \int_0^T f(X_s, I_s) \, ds + g(X_T) \right].$$

By similar arguments

$$Y_t^n = \text{ess sup}_{\nu \in \mathcal{N}_n} \mathbb{E}^\nu \left[ \int_t^T f(X_s, I_s) \, ds + g(X_T) \right| \mathcal{F}_t^W,\mu]$$

so that

$$Y_t^n \leq Y_t^{n+1} \uparrow \text{ess sup}_{\nu \in \mathcal{N}} \mathbb{E}^\nu \left[ \int_t^T f(X_s, I_s) \, ds + g(X_T) \right| \mathcal{F}_t^W,\mu] =: Y_t.$$
Convergence of \( Z^n, U^n, K^n \).

By standard estimates on the BSDE
\[
Y^n_s + \int_s^T Z^n_r \, dW_r + \int_s^T \int_A U^n_r(a) \, \mu(dr \, da) = g(X_T) + \int_s^T f(X_r, I_r) \, dr + K^n_T - K^n_s
\]
we have
- \( Z^n \) bounded in \( L^2_W \),
- \( U^n \) bounded in \( L^2_\mu \),
- \( K^n_T \) bounded in \( L^2 \) (and \( 0 \leq K^n_s \leq K^n_T \)).

So we can extract weakly convergent subsequences and pass to the limit in the (linear) BSDE, obtaining:
\[
Y_s + \int_s^T Z_r \, dW_r + \int_s^T \int_A U_r(a) \, \mu(dr \, da) = g(X_T) + \int_s^T f(X_r, I_r) \, dr + K_T - K_s
\]
It follows that \( Y \in S^2 \).
A sign constraint for $U$.

The functional $U \mapsto \mathbb{E} \int_0^T \int_A U^n_r(a)^+ \lambda(da) \, dr$ is convex in the space

$$L^2_{\mu} := \{ U_s(\omega, a) \in \mathbb{R} : \mathcal{P}(\mathbb{F}^W, \mu) \otimes \mathcal{B}(A) -\text{measurable}, \quad ||U||^2 := \mathbb{E} \int_0^T \int_A |U_s(a)|^2 \lambda(da) \, ds < \infty \}.$$  

So it is weakly l.s.c. so that

$$\mathbb{E} \int_0^T \int_A U_r(a)^+ \lambda(da) \, dr \leq \liminf_n \mathbb{E} \int_0^T \int_A U^n_r(a)^+ \lambda(da) \, dr$$

Since

$$n \int_0^T \int_A U^n_r(a)^+ \lambda(da) \, dr = K^n_T$$

is bounded in $L^2$, we conclude that $\mathbb{E} \int_0^T \int_A U_r(a)^+ \lambda(da) \, dr = 0$. The limit BSDE satisfies the following jump constraint:

$$U_t(\omega, a) \leq 0, \quad dt \otimes \lambda(da) \otimes \mathbb{P}(d\omega) - \text{a.s.}$$
7. BSDEs with constrained jumps.
A class of constrained BSDEs


We have proved the first part of the following result.

**Theorem i)** There exists a solution \((Y, Z, U, K)\) to the BSDE with constrained jumps: \(\mathbb{P}\)-a.s.

\[
\begin{cases}
Y_t + \int_t^T Z_s \, dW_s + \int_t^T \int_A U_s(a) \, \mu(ds \, da) \\
= g(X_T) + \int_t^T f(X_s, I_s) \, ds + K_T - K_t, \quad t \in [0, T], \\
U_t(a) \leq 0 \quad d\mathbb{P}d\lambda dt - a.s.
\end{cases}
\]

A solution is a quadruple \((Y_t(\omega), Z_t(\omega), U_t(\omega, a), K_t(\omega))\) where \(t \in [0, T], a \in A\) such that

- \(Y\) is adapted; \(Z, U, K\) are predictable \((w.r.t. \mathbb{F}_W^W, \mu)\);
- \(Y\) is càdlàg, \(K\) càdlàg increasing, \(K_0 = 0\),

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |Y_t|^2 + \int_0^T |Z_t|^2 \, dt + \int_0^T \int_A |U_t(a)|^2 \lambda(da) \, dt + K_T^2 \right] < \infty.
\]
ii) The solution \((Y, Z, U, K)\) is minimal, namely for any other solution \((Y', Z', U', K')\) we have \(\mathbb{P}\)-a.s.

\[
Y_t \leq Y'_t, \quad t \in [0, T].
\]

iii) The minimal solution is unique.

**Proof of ii).**

\[
Y'_t + \int_t^T Z'_r \, dW_r + \int_t^T \int_A U'_r(a) \, \mu(dr \, da)
= g(X_T) + \int_t^T f(X_r, I_r) \, dr + K'_T - K'_t
\]

since \(U' \leq 0\) and \(K' \uparrow\),

\[
Y'_t + \int_t^T Z'_r \, dW_r \geq g(X_T) + \int_t^T f(X_r, I_r) \, dr
\]

Taking expectation under \(\mathbb{P}^\nu\),

\[
Y'_t \geq \mathbb{E}^\nu \left[ \int_t^T f(X_s, I_s) \, ds + g(X_T) \big| \mathcal{F}_t^{W, \mu} \right].
\]

Taking \(\text{ess sup}_{\nu \in \mathcal{V}}\) we get \(Y'_t \geq Y_t\).
Proof of iii). If \((Y, Z, U, K)\) and \((Y', Z', U', K')\) are minimal then

\[
Y_t \leq Y'_t, \quad Y'_t \leq Y_t, \quad t \in [0, T],
\]

and so \(Y = Y'\). Rearranging the BSDE,

\[
\int_0^t Z'_r \, dr + \int_0^t \int_A U'_r(a) \, \mu(dr \, da) + K'_t = \int_0^t Z_r \, dr + \int_0^t \int_A U_r(a) \, \mu(dr \, da) + K_t
\]

Taking joint variation with \(W\) we get

\[
\int_0^t Z'_r \, dr = \int_0^t Z_r \, dr
\]

and so \(Z'_t = Z_t, dt \otimes d\mathbb{P}\)-a.s. Then

\[
\int_0^t \int_A U'_r(a) \, \mu(dr \, da) + K'_t = \int_0^t \int_A U_r(a) \, \mu(dr \, da) + K_t
\]
\[
\int_0^t \int_A U_{t}^r(a) \, \mu(da) \, dr + K'_t = \int_0^t \int_A U_{t}(a) \, \mu(da) \, dr + K_t
\]

Next recall that \( \int_0^t \int_A U_{s}(a) \, \mu(ds) \, da = \sum_{n \geq 1} U_{T_n}(A_n) \, 1\{T_n \leq t\} \),

Possible jump times for these stochastic integrals are \( T_n \), which are totally inaccessible, hence disjoint from jump times of \( K \) or \( K' \) which are predictable. Identifying jumps at \( T_n \) we obtain

\[
U_{T_n}'(A_n) = U_{T_n}(A_n)
\]

which implies

\[
0 = \mathbb{E} \int_0^T \int_A |U_{t}'(a) - U_{t}(a)| \, \mu(da) \, dt = \mathbb{E} \int_0^T \int_A |U_{t}'(a) - U_{t}(a)| \, \lambda(da) \, dt
\]

and so

\[
U_{t}'(a) = U_{t}(a), \quad d\mathbb{P}d\lambda dt - a.s.
\]

From the equality above we finally have \( K' = K \).
The BSDE representing the value function

Let $b, \sigma, f, g$ satisfy Assumption (A). Consider $(\Omega, \mathcal{F}, \mathbb{P}, W, \mu)$ and the associated $\mu \equiv (I_s) = (I_s^{a_0})$ starting at $a_0 \in A$. Solve

$$dX_s = b(X_s, I_s) \, ds + \sigma(X_s, I_s) \, dW_s, \quad X_t = x.$$ 

Denote $X_s = X^{t,x,a_0}_s$ its solution and define

$$v^R(t, x) = \sup_{\nu \in \mathcal{V}} \mathbb{E}^{\nu} \left[ \int_t^T f(X_s, I_s) \, ds + g(X_T) \right].$$

**Theorem.** Let $(Y, Z, U, K) = (Y^{t,x,a_0}, Z^{t,x,a_0}, U^{t,x,a_0}, K^{t,x,a_0})$ be the unique minimal solution to the constrained BSDE on $[t, T]$: 

$$\begin{cases}
Y_s + \int_s^T Z_r \, dW_r + \int_s^T \int_A U_r(a) \, \mu(\, dr \, da) = \\
U_s(a) \leq 0
\end{cases}$$

Then $v^R(t, x) = Y^{t,x,a_0}_t$, and moreover $Y^{t,x,a_0}_s = v(s, X^{t,x,a_0}_s)$, $s \in [t, T]$. 
Finally

\[ v(t, x) = v^R(t, x) = Y_{t}^{t,x,a_0}, \]

where

\[
 v(t, x) = \sup_{\alpha \in A_d} \mathbb{E} \left[ \int_{t}^{T} f(X_{\alpha}^{\alpha_s}, \alpha_s) \, ds + g(X_{T}^{\alpha}) \right],
\]

and

\[
 dX_{s}^{\alpha} = b(X_{s}^{\alpha}, \alpha_s) \, ds + \sigma(X_{s}^{\alpha}, \alpha_s) \, dW_s, \\
 X_{t}^{\alpha} = x,
\]

and \( A_d = \{ \alpha_t(\omega) : \mathbb{F}^W \text{–progressive} \} \).
Constrained BSDEs and fully non linear PDEs

Recall

\[ v(t, x) = Y^t_{t,x,a_0}, \]

where \( Y^t_{t,x,a_0} \) is the first component of the unique solution to the constrained BSDE.

This suggests that the function

\[ (t, x) \mapsto Y^t_{t,x,a_0} \]

is a solution to the HJB equation. Several authors have used this fact to construct solutions to HJB or other PDEs.


The starting point is the following: denoting \( Y_s = Y_{s}^{t,x,a_0} \) we have the following functional equality: for \( t \leq s \leq \theta \leq T \),

\[
Y_s = \text{ess sup}_{\nu \in \mathcal{V}} \mathbb{E}^{\nu} \left[ \int_{s}^{\theta} f(X_r, I_r) \, dr + Y_\theta \bigg| \mathcal{F}_s^{W,\mu} \right],
\]

which can be seen as a "randomized dynamic programming principle".

It is much easier to prove than the usual DPP.
Some comments

- $\sigma$ may be degenerate (or even null: deterministic control problem).
- Existence of an optimal control is not proved.
- Numerical methods have been developed for constrained BS-DEs of this form:


Thank you for your attention!
• Optimal switching.


• Impulse control.

• Constrained BSDEs to represent solution to integral PDEs.

• A weaker formulation of the control problem.

• Optimal stopping (non-Markovian diffusion).

• Optimal control of pure jump Markovian processes.
- **Optimal control of piecewise-deterministic Markov processes.**

- **Ergodic control.**

- **Optimal control with partial observation.**

- **Infinite horizon optimal control and elliptic HJB.**

- **Control of infinite-dimensional jump-diffusions.**

- **Markovian jump-diffusion, controlled intensity.**
• **Control of McKean-Vlasov systems.**

• **Numerical methods.**