Control randomization method and BSDEs with constrained jumps

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1. Reminders on classical stochastic optimal control problems.

Classical stochastic optimal control

Controlled SDE in $\mathbb{R}^n$:

$$\begin{cases} dX_s = b(X_s, \alpha_s)\, ds + \sigma(X_s, \alpha_s)\, dW_s, & s \in [t, T] \subset [0, T], \\ X_t = x \in \mathbb{R}^n, \end{cases}$$

• W is a Wiener process in $\mathbb{R}^d$, defined in $(\Omega, \mathcal{F}, \mathbb{P})$.

We will use the Brownian filtration $\mathbb{F}^W = (\mathcal{F}_t^W)_{t \geq 0}$, namely we assume $\mathcal{F}$ to be $\mathbb{P}$-complete and we set

$$\mathcal{F}_t^W := \sigma(W_s,\ s \in [0, t]) \vee \mathcal{N}$$

where $\mathcal{N}$ are $\mathbb{P}$-null sets of $\mathcal{F}$.
The coefficients of an SDE are controlled:

$$b(x, a) \in \mathbb{R}^n, \quad \sigma(x, a) \in \mathbb{R}^{n \times d},$$

i.e. they depend on a parameter $a \in A$:
• A, the space of control actions, is a complete separable metric space (or a Borel subset of it).

$$\begin{cases} dX_s &= b(X_s, \alpha_s)\, ds + \sigma(X_s, \alpha_s)\, dW_s, \qquad s \in [t, T], \\ X_t &= x \in \mathbb{R}^n. \end{cases}$$

A controller dynamically selects her actions by choosing a control process

$$\alpha_s(\omega) \in A.$$

- $\mathcal{A}_d = \{\alpha : \Omega \times [0, T] \to A,\ \mathbb{F}^W\text{-progressive}\}$ is the space of admissible controls.

The corresponding solution $X_s = X_s^\alpha = X_s^{\alpha,t,x}$ is called the trajectory corresponding to the control α.

Controlled equation:

$$\begin{cases} dX_s^\alpha &= b(X_s^\alpha, \alpha_s)\, ds + \sigma(X_s^\alpha, \alpha_s)\, dW_s, \qquad s \in [t,T], \\ X_t^\alpha &= x \in \mathbb{R}^n. \end{cases}$$

The controller tries to maximize the reward functional

$$J(\alpha, t, x) = \mathbb{E}\left[g(X_T^{\alpha,t,x}) + \int_t^T f(X_s^{\alpha,t,x}, \alpha_s)\, ds \right].$$

Here

$$g(x) \in \mathbb{R}, \qquad f(x,a) \in \mathbb{R}$$

are: i) the reward corresponding to the final position $x \in \mathbb{R}^n$ of the state;
ii) the running reward rate when the current state is $x \in \mathbb{R}^n$ and the control action is $a \in A$.

Value function:

$$v(t,x) = \sup_{\alpha \in \mathcal{A}_d} J(\alpha, t, x).$$

Assumptions (A) on the coefficients

On the data of the control problem $b(x,a), \sigma(x,a), f(x,a), g(x)$ we assume:

- $b : \mathbb{R}^n \times A \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \times A \to \mathbb{R}^{n\times d}$, $f : \mathbb{R}^n \times A \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}$ are continuous.
- b, σ are Lipschitz in x uniformly in a: $\exists L \geq 0$ such that

$$|b(x,a) - b(y,a)| + |\sigma(x,a) - \sigma(y,a)| \leq L|x-y|, \quad x, y \in \mathbb{R}^n, a \in A.$$

- $b(0,a), \sigma(0,a)$ are bounded in a: $\exists M \geq 0$ such that

$$|b(0,a)| + |\sigma(0,a)| \leq M, \quad a \in A.$$

- f, g have polynomial growth in x uniformly in a: $\exists r \geq 0$ such that

$$|f(x,a)| + |g(x)| \leq M\,(1 + |x|^r), \quad x \in \mathbb{R}^n, a \in A.$$

Under these assumptions X^α is well defined and $v(t,x)$ is finite.

2. The value function and its characterizations: Hamilton-Jacobi-Bellman equations (HJB), BSDEs.

Hamilton-Jacobi-Bellman equation (HJB)

$$\begin{cases} -\partial v(t,x) & = \sup_{a \in A} [\mathcal{L}^a v(t,x) + f(x,a)], \qquad t \in [0,T],\ x \in \mathbb{R}^n, \\ v(T,x) & = g(x), \end{cases}$$

Controlled Kolmogorov operator:

$$\mathcal{L}^a \phi(x) = \frac{1}{2} \mathrm{tr}\Big[\sigma(x,a)\sigma^T(x,a) D^2 \phi(x)\Big] + D\phi(x)\, b(x,a)$$

Available results are two-fold:

• Any classical solution to HJB coincides with the value function:

$$v(t,x) = \sup_{\alpha \in \mathcal{A}_d} \mathbb{E}\left[g(X_T^\alpha) + \int_t^T f(X_s^\alpha, \alpha_s)\, ds \right].$$

Results of this kind are called verification theorems and often an optimal control is also found.

• The value function, defined by the formula above, is a solution to HJB, possibly in viscosity sense. Often, uniqueness of the viscosity solution is also proved.

The search for an associated BSDE

Aim: find a BSDE giving a representation of the value function:

$$v(t,x) = \sup_{\alpha \in \mathcal{A}_d} \mathbb{E}\left[g(X_T^\alpha) + \int_t^T f(X_s^\alpha, \alpha_s)\, ds \right].$$

Motivations:

• Efficient numerical methods are often available for BSDEs.

• Immediate extensions to non-Markovian systems (e.g. with memory).

• Associated BSDEs are known in special cases.

Available approaches:

• The theory of G-expectation (Peng). It is based on a different foundation of stochastic calculus and even probability theory. It replaces the expectation operator by a (non-linear) analogue satisfying appropriate axioms.

• The theory of second order BSDEs (2BSDEs, Soner-Touzi). One formulates a BSDE that can be solved under a family of (mutually singular) probability measures. It often requires non-degenerate diffusion (i.e. invertible $\sigma(x,a)$).

3. Control randomization method: introduction.

The randomization method in optimal control

Introduced in

B. Bouchard. A stochastic target formulation for optimal switching problems in finite horizon. Stochastics 81, no. 2 (2009), 171-197.

$$\begin{cases} dX_s^\alpha &= b(X_s^\alpha, \alpha_s)\, ds + \sigma(X_s^\alpha, \alpha_s)\, dW_s, \qquad s \in [t,T], \\ X_t^\alpha &= x \in \mathbb{R}^n. \end{cases}$$

$$v(t,x) = \sup_{\alpha \in \mathcal{A}_d} \mathbb{E}\left[g(X_T^\alpha) + \int_t^T f(X_s^\alpha, \alpha_s)\, ds \right],$$

Idea:

1) replace (α_s) by a random (uncontrolled) process (I_s) with piecewise constant trajectories and values in A;

2) formulate an auxiliary ("randomized") control problem, where "the law of I is controlled", having value denoted $v^{\mathcal{R}}(t,x)$;

3) prove that $v = v^{\mathcal{R}}$;

4) represent $v^{\mathcal{R}}(t,x)$ by a BSDE.

The randomized control problem

We replace the control $\alpha \in \mathcal{A}_d$ by an A-valued process I, independent of W. We consider the "randomized" state equation:

$$dX_s = b(X_s, I_s)\, ds + \sigma(X_s, I_s)\, dW_s, \; s \in [t,T]; \qquad X_t = x.$$

Then we construct a suitable family of probability measures $\mathbb{P}^\nu$, depending on a parameter $\nu \in \mathcal{V}$, such that

- $\mathbb{P}^\nu \sim \mathbb{P}$ (dominated model)
- W remains a Wiener process under $\mathbb{P}^\nu$.

Then we optimize among $\mathbb{P}^\nu$: we formulate an auxiliary ("randomized") control problem with value function:

$$v^{\mathcal{R}}(t,x) = \sup_{\nu \in \mathcal{V}} \mathbb{E}^\nu \left[\int_t^T f(X_s, I_s)\, ds + g(X_T) \right].$$

We wish to prove equivalence with the randomized problem namely that

$$v(t,x) = v^{\mathcal{R}}(t,x).$$

Original problem: for $s \in [t,T] \subset [0,T]$, $x \in \mathbb{R}^n$,

$$\begin{array}{rcl} dX_s^{\alpha} & = & b(X_s^{\alpha}, \alpha_s)\, ds + \sigma(X_s^{\alpha}, \alpha_s)\, dW_s, \\ {X_t}^{\alpha} & = & x, \\ v(t,x) & = & \sup_{\alpha \in \mathcal{A}_d} \mathbb{E}\left[\int_t^T f(X_s^{\alpha}, \alpha_s)\, ds + g(X_T^{\alpha})\right], \end{array}$$

Randomized problem:

$$\begin{array}{rcl} dX_s & = & b(X_s, I_s)\, ds + \sigma(X_s, I_s)\, dW_s, \\ X_t & = & x, \\ v^{\mathcal{R}}(t,x) & = & \sup_{\nu \in \mathcal{V}} \mathbb{E}^{\nu}\left[\int_t^T f(X_s, I_s)\, ds + g(X_T)\right] \end{array}$$

The choice of the process I

When $A = \mathbb{R}^k$ one can choose (I_s) as another Brownian motion, independent of W. The probabilities $\mathbb{P}^\nu$ are then defined by a classical Girsanov theorem. See for instance

S. Choukroun, A. Cosso. Backward SDE representation for stochastic control problems with nondominated controlled intensity. The Annals of Applied Probability 26, no. 2 (2016), 1208-1259.

For the general case we first note that

$$v(t,x) = \sup_{\alpha \in \mathcal{A}_d} \mathbb{E}\left[\int_t^T f(X_s^\alpha, \alpha_s)\, ds + g(X_T^\alpha)\right]$$

remains unchanged if we restrict to (α_s) being a piecewise constant process: see for instance Lemma 3.2.6 in

N.V. Krylov. Controlled diffusion processes. Springer, 2009.

We will choose (I_s) to be a (pure jump) Poisson process in A.

4. Digression on marked point processes and associated control problems.

Reminders on pure jump and marked point processes

We use notions from

J. Jacod. Multivariate point processes: predictable projection, Radon-Nikodym derivatives, representation of martingales. Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete 31, no. 3 (1975), 235-253.

See also

P. Brémaud. Point processes and queues: martingale dynamics. Springer, 1981.

G. Last, A. Brandt. Marked Point Processes on the real line: the dynamical approach. Springer, 1995.

On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a right-continuous filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ we define a marked point process $(T_n, A_n)_{n \geq 1}$ where

T_n: a (non-explosive) point process, i.e. T_n are $\mathbb{F}$-stopping times, $0 < T_n \leq T_{n+1} \uparrow \infty$, $T_n < T_{n+1}$ if $T_n < \infty$;

A_n: A-valued random variables, A_n is $\mathcal{F}_{T_n}$-measurable.

To $(T_n, A_n)_{n\geq 1}$ we associate:

- a pure jump A-valued process

$$I_t = a_0\, 1_{\{0\leq t<T_1\}} + A_1\, 1_{\{T_1\leq t<T_2\}} + A_2\, 1_{\{T_2\leq t<T_3\}} + \dots,$$

where $a_0 \in A$ is deterministic and fixed (in fact, $I_s = I_s^{a_0}$).

- A random measure on $(0,\infty) \times A$:

$$\mu(dt\, da) = \sum_{n\geq 1} \delta_{(T_n,A_n)}(dt\, da)\, 1_{\{T_n<\infty\}}.$$

In the sequel we identify $I \equiv (T_n, A_n)_{n\geq 1} \equiv \mu$.

The natural filtration $\mathbb{F}^\mu = (\mathcal{F}_t^\mu)$ is defined as
$\mathcal{F}_t^\mu = \sigma\{\mu((0,s] \times C) \,:\, s \in [0,t],\, C \in \mathcal{B}(A)\} \vee \mathcal{N}$.

Reminders on compensators

The compensator, or dual predictable projection, of μ is a predictable random measure $\nu(dt\, da)$ such that $\mu - \nu$ is a martingale measure. Formally,

• for $C \in \mathcal{B}(A)$ the process $\nu((0,t] \times C)$ is $\mathbb{F}$-predictable;

• the following equivalent conditions are satisfied:

i) for $C \in \mathcal{B}(A)$ and $n \geq 1$ the processes

$$\mu((0, t \wedge T_n] \times C) - \nu((0, t \wedge T_n] \times C), \qquad t \geq 0,$$

are $\mathbb{P}$-martingales with respect to $\mathbb{F}$.

ii) denote $\mathcal{P}(\mathbb{F})$ the predictable σ-algebra of $\mathbb{F}$; then for any $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(A)$-measurable random field $H(\omega, t, a) \geq 0$,

$$\mathbb{E} \int_{(0,t]} H(t,a)\, \mu(dtda) = \mathbb{E} \int_{(0,t]} H(t,a)\, \nu(dtda).$$

One proves that ν exists and is unique (up to null sets). Moreover

- ν depends on $\mathbb{P}$.
- ν determines the law of $I \equiv (T_n, A_n)_{n\geq 1} \equiv \mu$ under $\mathbb{P}$.

For instance, if the compensator is $\nu(dt\, da) = \lambda(da)dt$, for some (finite) measure λ on A, then μ is a (non-explosive) Poisson process, i.e.

- (T_n) is a standard counting Poisson process with intensity $\lambda(A)$;
- (A_n) is an i.i.d. sequence with law $\frac{\lambda(da)}{\lambda(A)}$ (independent of (T_n)).

If λ has full topological support then I visits every open set of A

One way to "control" the process I is to fix a desired compensator ν (corresponding to some desired law for I) and find a new probability $\mathbb{P}^\nu$ under which ν is the compensator of μ.

Instead of choosing an arbitrary compensator we may start from a probability $\mathbb{P}$ under which μ a Poisson process with compensator

$$\lambda(da)dt.$$

We look for a probability under which I has a compensator of the form

$$\nu_t(\omega, a)\lambda(da)dt$$

where $\nu_t(\omega, a) \geq 0$ is a chosen $\mathbb{F}$-predictable random field.

Theorem (of Girsanov type). Let $\nu_t(\omega, a)$ be bounded. The required probability is given by $d\mathbb{P}^\nu = \kappa_T^\nu \, d\mathbb{P}$ where κ^ν is the Doléans exponential martingale

$$\kappa_t^\nu = \exp\left(\int_0^t \int_A (1 - \nu_s(a))\lambda(da)ds\right) \prod_{T_n \leq t} \nu_{T_n}(A_n).$$

5. Back to the control randomization method.

Return to the original control problem

Let b, σ, f, g satisfy Assumption (A).
Given $(\Omega, \mathcal{F}, \mathbb{P}, W)$ consider

$$\begin{cases} dX_s^\alpha &= b(X_s^\alpha, \alpha_s)\, ds + \sigma(X_s^\alpha, \alpha_s)\, dW_s, \qquad s \in [t, T], \\ X_t^\alpha &= x \in \mathbb{R}^n. \end{cases}$$

$$v(t,x) = \sup_{\alpha \in \mathcal{A}_d} \mathbb{E}\left[g(X_T^\alpha) + \int_t^T f(X_s^\alpha, \alpha_s)\, ds \right],$$

Admissibile controls:

$$\mathcal{A}_d = \{\alpha : \Omega \times [0, T] \to A,\ \mathbb{F}^W\text{–progressive}\}.$$

The randomized control problem

Let b, σ, f, g satisfy Assumption (A).
Consider $(\Omega, \mathcal{F}, \mathbb{P}, W, \mu)$ where:

- $\mu \equiv (T_n, A_n)_{n\geq 1} \equiv I$ is a Poisson random measure with finite intensity $\lambda(da)$ with full topological support;
- W is an $\mathbb{R}^d$-valued Brownian motion, independent of μ.

We will use the filtration $\mathbb{F}^{W,\mu} = (\mathcal{F}_t^{W,\mu})$ generated by W and μ:
$\mathcal{F}_t^{W,\mu} = \sigma\{W_s,\, \mu((0,s] \times C) \,:\, s \in [0,t],\, C \in \mathcal{B}(A)\} \vee \mathcal{N}$.

We consider the "randomized" state equation:

$$dX_s = b(X_s, I_s)\, ds + \sigma(X_s, I_s)\, dW_s,\ s \in [t,T]; \qquad X_t = x.$$

The admissible controls are now random fields

$\mathcal{V} = \{\nu_t(\omega, a) : \Omega \times [0,\infty) \times A \to (0,\infty), \mathbb{F}^{W,\mu}-\text{predictable bounded}\}$

Given $\nu_t(\omega, a) \in \mathcal{V}$, we define $d\mathbb{P}^\nu = \kappa_T^\nu \, d\mathbb{P} \sim d\mathbb{P}$ where

$$\kappa_t^\nu = \exp\left(\int_0^t \int_A (1 - \nu_s(a))\lambda(da)ds\right) \prod_{T_n \leq t} \nu_{T_n}(A_n).$$

Then, on $[0, T]$,

- μ has compensator $\nu_t(\omega, a)\lambda(da)dt$ under $\mathbb{P}^\nu$;
- W remains a Wiener process under $\mathbb{P}^\nu$.

We define an auxiliary ("randomized") value function:

$$v^{\mathcal{R}}(t, x) = \sup_{\nu \in \mathcal{V}} \mathbb{E}^\nu \left[\int_t^T f(X_s, I_s)\, ds + g(X_T)\right].$$

It can be proved that $v^{\mathcal{R}}$ only depends on b, σ, f, g (not on $\Omega, \mathcal{F}, \mathbb{P}, W, \mu, \lambda, a_0$). ν changes the intensity of the Poisson component I and therefore has an influence on X as well.

Equivalence with the randomized problem

Original problem: for $s \in [t, T] \subset [0, T]$, $x \in \mathbb{R}^n$,

$$\begin{aligned} dX_s^\alpha &= b(X_s^\alpha, \alpha_s)\, ds + \sigma(X_s^\alpha, \alpha_s)\, dW_s, \\ X_t{}^\alpha &= x, \\ v(t,x) &= \mathrm{sup}_{\alpha \in \mathcal{A}_d} \mathbb{E}\left[\int_t^T f(X_s^\alpha, \alpha_s)\, ds + g(X_T^\alpha)\right], \\ \mathcal{A}_d &= \{\alpha_t(\omega) : \mathbb{F}^W\text{--progressive}\} \end{aligned}$$

Randomized problem:

$$\begin{aligned} dX_s &= b(X_s, I_s)\, ds + \sigma(X_s, I_s)\, dW_s, \\ X_t &= x, \\ v^{\mathcal{R}}(t,x) &= \mathrm{sup}_{\nu \in \mathcal{V}} \mathbb{E}^\nu \left[\int_t^T f(X_s, I_s)\, ds + g(X_T)\right], \\ \mathcal{V} &= \{\nu_t(\omega, a) : \mathbb{F}^{W,\mu}\text{--predictable}, 0 < \nu \leq \mathrm{sup}\, \nu < \infty\} \end{aligned}$$

Theorem. Assume (A). Then $v(t,x) = v^{\mathcal{R}}(t,x)$.

E. Bandini, A. Cosso, M. F., H. Pham. Backward SDEs for optimal control of partially observed path-dependent stochastic systems: a control randomization approach. The Annals of Applied Probability 28, no. 3 (2018), 1634-1678.

6. The search for an associated BSDE.

A BSDE for the randomized control problem

Let b, σ, f, g satisfy Assumption (A).
Consider $(\Omega, \mathcal{F}, \mathbb{P}, W, \mu)$ where:

- $\mu \equiv (T_n, A_n)_{n \geq 1} \equiv I$ is a Poisson random measure with finite intensity $\lambda(da)$ with full topological support;
- W is an $\mathbb{R}^d$-valued Brownian motion, independent of μ.

We consider the randomized problem starting at $t = 0$:

$$\begin{array}{lcl} dX_s & = & b(X_s, I_s)\, ds + \sigma(X_s, I_s)\, dW_s, \\ X_0 & = & x \text{ fixed}, \\ v^{\mathcal{R}} & = & \sup_{\nu \in \mathcal{V}} \mathbb{E}^{\nu} \left[\int_0^T f(X_s, I_s)\, ds + g(X_T) \right], \\ \mathcal{V} & = & \{\nu_t(\omega, a) : \mathbb{F}^{W,\mu}\text{–predictable}, 0 < \nu \leq \sup \nu < \infty\} \end{array}$$

We approximate $v^{\mathcal{R}}$ by

$$v_n^{\mathcal{R}} = \sup_{\nu \in \mathcal{V}_n} \mathbb{E}^{\nu} \left[\int_0^T f(X_s, I_s)\, ds + g(X_T) \right],$$

where $\mathcal{V}_n := \{\nu \in \mathcal{V},\ \nu \leq n\}$. We look for a BSDE representation for the penalized value $v_n^{\mathcal{R}}$ and then for $v^{\mathcal{R}}$.
Clearly $v_n^{\mathcal{R}} \leq v_{n+1}^{\mathcal{R}} \leq v^{\mathcal{R}}$ and even $v_n^{\mathcal{R}} \uparrow v^{\mathcal{R}}$.

BSDEs with respect to Wiener + Poisson

Consider again $(\Omega, \mathcal{F}, \mathbb{P}, W, \mu)$ and $\mathbb{F}^{W,\mu} = (\mathcal{F}_t^{W,\mu})$.
If M is an $\mathbb{F}^{W,\mu}$-martingale in L^2 then

$$M_t = M_0 + \int_0^t Z_s\, ds + \int_0^t \int_A U_s(a)\, [\mu(ds\, da) - \lambda(da)\, ds], \quad t \in [0, T],$$

where Z and U are in the spaces

$$L^2_W := \{Z_s(\omega) \in \mathbb{R}^d \,:\, \mathbb{F}^{W,\mu}-\text{predictable}, \mathbb{E} \int_0^T |Z_s|^2 ds < \infty\},$$

$$L^2_\mu := \{U_s(\omega, a) \in \mathbb{R} \,:\, \mathcal{P}(\mathbb{F}^{W,\mu}) \otimes \mathcal{B}(A)-\text{measurable},$$
$$\mathbb{E} \textstyle\int_0^T \int_A |U_s(a)|^2 \lambda(da)\, ds < \infty\}.$$

It follows that M belongs to

$$\mathcal{S}^2 := \{Y_s(\omega) \in \mathbb{R} \,:\, \mathbb{F}^{W,\mu} \text{ -adapted and càdlàg}, \mathbb{E} \sup_{s \in [0,T]} |Y_s|^2 < \infty\}$$

Note that $\int_0^T \int_A U_s(a)\, \mu(ds\, da) = \sum_{n \geq 1} U_{T_n}(A_n)\, 1_{\{T_n \leq T\}}$.

Now consider the BSDE: for $s \in [0, T]$,

$$
\begin{aligned}
Y_s + \int_s^T Z_r \, dW_r + \int_s^T \int_A U_r(a) \, [\mu(dr \, da) - \lambda(da) \, dr] \\
= g(X_T) + \int_s^T f(X_r, I_r) \, dr
\end{aligned}
$$

for the unknown $(Y, Z, U) \in \mathcal{S}^2 \times L^2_W \times L^2_\mu$. Define

$$
Y_s = \mathbb{E}\left[g(X_T) + \int_s^T f(X_r, I_r) \, dr \,\middle|\, \mathcal{F}^{W,\mu}_s \right] = M_s + \int_0^s f(X_r, I_r) \, dr
$$

where

$$
M_s = \mathbb{E}\left[g(X_T) + \int_0^T f(X_r, I_r) \, dr \,\middle|\, \mathcal{F}^{W,\mu}_s \right]
$$

Represent M and before by Z, U and check that (Y, Z, U) is the unique required solution.
As usual, similar results hold if we add a Lipschitz nonlinearity to the BSDE.

A BSDE for the penalized randomized control problem

Now recall

$$\begin{array}{rcl} dX_s & = & b(X_s, I_s)\, ds + \sigma(X_s, I_s)\, dW_s, \qquad X_0 = x, \\ J(\nu) & = & \mathbb{E}^\nu \left[\int_0^T f(X_s, I_s)\, ds + g(X_T) \right], \\ v_n^{\mathcal{R}} & = & \sup_{\nu \in \mathcal{V}_n} \mathbb{E}^\nu \left[\int_0^T f(X_s, I_s)\, ds + g(X_T) \right], \\ \mathcal{V}_n & = & \{\nu_t(\omega, a) : \mathbb{F}^{W,\mu}\text{–predictable}, 0 < \nu \le n < \infty\} \end{array}$$

Consider the penalized equation

$$\begin{aligned} Y_s + \int_s^T Z_r\, dW_r + \int_s^T \int_A U_r(a)\, \mu(dr\, da) \\ = g(X_T) + \int_s^T f(X_r, I_r)\, dr + n \int_s^T \int_A U_r(a)^+ \lambda(da)\, dr \end{aligned}$$

and find a unique solution $(Y^n, Z^n, U^n) \in \mathcal{S}^2 \times L^2_W \times L^2_\mu$.

We claim that $v_n^{\mathcal{R}} = Y_0^n$.

Write (Y, Z, U) instead of (Y^n, Z^n, U^n). Fix $\nu \in \mathcal{V}$ and take $\mathbb{E}^\nu$ in

$$Y_0 + \int_0^T Z_r\, dW_r + \int_0^T \int_A U_r(a)\, \mu(dr\, da)$$
$$= g(X_T) + \int_0^T f(X_r, I_r)\, dr + n \int_0^T \int_A U_r(a)^+ \lambda(da)\, dr.$$

Then

$$Y_0 + \mathbb{E}^\nu \int_0^T \int_A U_r(a)\, \mu(dr\, da) = J(\nu) + n\mathbb{E}^\nu \int_0^T \int_A U_r(a)^+ \lambda(da)\, dr.$$

Since U is a predictable random field,

$$\mathbb{E}^\nu \int_0^T \int_A U_r(a)\, \mu(dr\, da) = \mathbb{E}^\nu \int_0^T \int_A U_r(a)\, \nu_r(a)\, \lambda(da)\, dr.$$

Substituting and rearranging,

$$Y_0 = J(\nu) + \mathbb{E}^\nu \int_0^T \int_A [nU_r(a)^+ - \nu_r(a) U_r(a)]\, \lambda(da)\, dr.$$

Since $nu^+ - \nu u \geq 0$ for $\nu \in [0, n]$ with equality when $\nu = n1_{u \geq 0}$, we have $Y_0 \geq J(\nu)$ with equality when $\nu_s(a) = n1_{U_s \geq 0}$.

Convergence of the penalized BSDE

The penalized equation

$$
\begin{aligned}
Y_s^n + \int_s^T Z_r^n \, dW_r + \int_s^T \int_A U_r^n(a)\, \mu(dr\, da) \\
= g(X_T) + \int_s^T f(X_r, I_r)\, dr + n \int_s^T \int_A U_r^n(a)^+ \, \lambda(da)\, dr
\end{aligned}
$$

can be written

$$
\begin{aligned}
Y_s^n + \int_s^T Z_r^n \, dW_r + \int_s^T \int_A U_r^n(a)\, \mu(dr\, da) \\
= g(X_T) + \int_s^T f(X_r, I_r)\, dr + K_T^n - K_s^n
\end{aligned}
$$

where

$$
K_s^n = n \int_0^s \int_A U_r^n(a)^+ \, \lambda(da)\, dr
$$

satisfy $K_0^n = 0$, are increasing and adapted continuous (hence predictable).

Since $v_n^{\mathcal{R}} \le v_{n+1}^{\mathcal{R}} \le v^{\mathcal{R}}$ we have $Y_0^n \le Y_0^{n+1} \le v^{\mathcal{R}}$.

Convergence of Y^n.

We have proved

$$Y_0^n = v_n^{\mathcal{R}} = \sup_{\nu \in \mathcal{V}_n} \mathbb{E}^\nu \left[\int_0^T f(X_s, I_s)\, ds + g(X_T) \right].$$

By similar arguments

$$Y_t^n = \operatorname*{ess\,sup}_{\nu \in \mathcal{V}_n} \mathbb{E}^\nu \left[\int_t^T f(X_s, I_s)\, ds + g(X_T) \middle| \mathcal{F}_t^{W,\mu} \right]$$

so that

$$Y_t^n \le Y_t^{n+1} \uparrow \operatorname*{ess\,sup}_{\nu \in \mathcal{V}} \mathbb{E}^\nu \left[\int_t^T f(X_s, I_s)\, ds + g(X_T) \middle| \mathcal{F}_t^{W,\mu} \right] =: Y_t.$$

Convergence of Z^n, U^n, K^n.

By standard estimates on the BSDE

$$\begin{aligned} Y_s^n + \textstyle\int_s^T Z_r^n\, dW_r + \int_s^T \int_A U_r^n(a)\, \mu(dr\, da) \\ = g(X_T) + \textstyle\int_s^T f(X_r, I_r)\, dr + K_T^n - K_s^n \end{aligned}$$

we have

- Z^n bounded in L^2_W,
- U^n bounded in L^2_μ,
- K_T^n bounded in L^2 (and $0 \leq K_s^n \leq K_T^n$).

So we can extract weakly convergent subsequences and pass to the limit in the (linear) BSDE, obtaining:

$$\begin{aligned} Y_s + \int_s^T Z_r\, dW_r + \int_s^T \int_A U_r(a)\, \mu(dr\, da) \\ = g(X_T) + \int_s^T f(X_r, I_r)\, dr + K_T - K_s \end{aligned}$$

It follows that $Y \in \mathcal{S}^2$.

A sign constraint for U.

The functional $U \mapsto \mathbb{E}\int_0^T \int_A U_r^n(a)^+ \,\lambda(da)\,dr$ is convex in the space

$$L^2_\mu := \{U_s(\omega,a) \in \mathbb{R} \,:\, \mathcal{P}(\mathbb{F}^{W,\mu}) \otimes \mathcal{B}(A)-\text{measurable},$$
$$\|U\|^2 := \mathbb{E}\int_0^T \int_A |U_s(a)|^2 \lambda(da)\,ds < \infty\}.$$

So it is weakly l.s.c. so that

$$\mathbb{E}\int_0^T \int_A U_r(a)^+ \,\lambda(da)\,dr \le \liminf_n \mathbb{E}\int_0^T \int_A U_r^n(a)^+ \,\lambda(da)\,dr$$

Since

$$n\int_0^T \int_A U_r^n(a)^+ \,\lambda(da)\,dr = K_T^n$$

is bounded in L^2, we conclude that $\mathbb{E}\int_0^T \int_A U_r(a)^+ \,\lambda(da)\,dr = 0$.
The limit BSDE satisfies the following jump constraint:

$$U_t(\omega,a) \le 0, \qquad dt \otimes \lambda(da) \otimes \mathbb{P}(d\omega) - a.s.$$

7. BSDEs with constrained jumps.

A class of constrained BSDEs

I. Kharroubi and H. Pham. Feynman–Kac representation for Hamilton–Jacobi–Bellman IPDE. Ann. Probab. 43, no. 4 (2015) 1823–1865.

We have proved the first part of the following result.

Theorem i) There exists a solution (Y, Z, U, K) to the BSDE with constrained jumps: $\mathbb{P}$-a.s.

$$\begin{cases} Y_t + \int_t^T Z_s\, dW_s + \int_t^T \int_A U_s(a)\, \mu(ds\, da) \\ \qquad = g(X_T) + \int_t^T f(X_s, I_s)\, ds + K_T - K_t, \qquad t \in [0, T], \\ U_t(a) \leq 0 \quad d\mathbb{P} d\lambda dt - a.s. \end{cases}$$

A solution is a quadruple $(Y_t(\omega), Z_t(\omega), U_t(\omega, a), K_t(\omega))$ where $t \in [0, T]$, $a \in A$ such that

Y is adapted; Z, U, K are predictable (w.r.t. $\mathbb{F}^{W,\mu}$);

Y is càdlàg, K càdlàg increasing, $K_0 = 0$,

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |Y_t|^2 + \int_0^T |Z_t|^2\, dt + \int_0^T \int_A |U_t(a)|^2\, \lambda(da)\, dt + K_T^2\Big] < \infty.$$

ii) The solution (Y, Z, U, K) is minimal, namely for any other solution (Y', Z', U', K') we have $\mathbb{P}$-a.s.

$$Y_t \leq Y'_t, \quad t \in [0, T].$$

iii) The minimal solution is unique.

Proof of ii).

$$\begin{aligned} Y'_t + \textstyle\int_t^T Z'_r\, dW_r + \int_t^T \int_A U'_r(a)\, \mu(dr\, da) \\ = g(X_T) + \textstyle\int_t^T f(X_r, I_r)\, dr + K'_T - K'_t \end{aligned}$$

since $U' \leq 0$ and $K' \uparrow$,

$$Y'_t + \int_t^T Z'_r\, dW_r \geq g(X_T) + \int_t^T f(X_r, I_r)\, dr$$

Taking expectation under $\mathbb{P}^\nu$,

$$Y'_t \geq \mathbb{E}^\nu \left[\int_t^T f(X_s, I_s)\, ds + g(X_T) \Big| \mathcal{F}_t^{W,\mu} \right].$$

Taking $\operatorname{ess\,sup}_{\nu \in \mathcal{V}}$ we get $Y'_t \geq Y_t$.

Proof of iii). If (Y, Z, U, K) and (Y', Z', U', K') are minimal then

$$Y_t \leq Y'_t, \quad Y'_t \leq Y_t, \quad t \in [0, T],$$

and so $Y = Y'$. Rearranging the BSDE,

$$\begin{aligned}\int_0^t Z'_r\, dW_r + \int_0^t \int_A U'_r(a)\, \mu(dr\, da) + K'_t \\ = \int_0^t Z_r\, dW_r + \int_0^t \int_A U_r(a)\, \mu(dr\, da) + K_t\end{aligned}$$

Taking joint variation with W we get

$$\int_0^t Z'_r\, dr = \int_0^t Z_r\, dr$$

and so $Z'_t = Z_t$, $dt \otimes d\mathbb{P}$-a.s. Then

$$\int_0^t \int_A U'_r(a)\, \mu(dr\, da) + K'_t = \int_0^t \int_A U_r(a)\, \mu(dr\, da) + K_t$$

$$\int_0^t \int_A U'_r(a)\, \mu(dr\, da) + K'_t = \int_0^t \int_A U_r(a)\, \mu(dr\, da) + K_t$$

Next recall that $\int_0^t \int_A U_s(a)\, \mu(ds\, da) = \sum_{n\geq 1} U_{T_n}(A_n)\, 1_{\{T_n \leq t\}}$,

Possible jump times for these stochastic integrals are T_n, which are totally inaccessible, hence disjoint from jump times of K or K' which are predictable. Identifying jumps at T_n we obtain

$$U'_{T_n}(A_n) = U_{T_n}(A_n)$$

which implies

$$0 = \mathbb{E} \int_0^T \int_A |U'_t(a) - U_t(a)| \mu(ds\, da) = \mathbb{E} \int_0^T \int_A |U'_t(a) - U_t(a)| \lambda(da)\, dt$$

and so

$$U'_t(a) = U_t(a), \qquad d\mathbb{P} d\lambda dt - a.s.$$

From the equality above we finally have $K' = K$.

The BSDE representing the value function

Let b, σ, f, g satisfy Assumption (A). Consider $(\Omega, \mathcal{F}, \mathbb{P}, W, \mu)$ and the associated $\mu \equiv (I_s) = (I_s^{a_0})$ starting at $a_0 \in A$. Solve
$dX_s = b(X_s, I_s)\, ds + \sigma(X_s, I_s)\, dW_s$, $X_t = x$.
Denote $X_s = X_s^{t,x,a_0}$ its solution and define

$$v^{\mathcal{R}}(t,x) = \sup_{\nu \in \mathcal{V}} \mathbb{E}^{\nu} \left[\int_t^T f(X_s, I_s)\, ds + g(X_T) \right].$$

Theorem. Let $(Y, Z, U, K) = (Y^{t,x,a_0}, Z^{t,x,a_0}, U^{t,x,a_0}, K^{t,x,a_0})$ be the unique minimal solution to the constrained BSDE on $[t, T]$:

$$\begin{cases} Y_s + \int_s^T Z_r\, dW_r \;+\; \int_s^T \int_A U_r(a)\, \mu(dr\, da) = \\ \qquad\qquad g(X_T) + \int_s^T f(X_r, I_r)\, dr + K_T - K_s, \\ U_s(a) \le 0 \end{cases}$$

Then $v^{\mathcal{R}}(t,x) = Y_t^{t,x,a_0}$, and moreover $Y_s^{t,x,a_0} = v(s, X_s^{t,x,a_0})$, $s \in [t, T]$.

Finally

$$v(t,x) = v^{\mathcal{R}}(t,x) = Y_t^{t,x,a_0},$$

where

$$v(t,x) = \sup_{\alpha \in \mathcal{A}_d} \mathbb{E}\left[\int_t^T f(X_s^\alpha, \alpha_s)\, ds + g(X_T^\alpha)\right],$$

and

$$\begin{aligned} dX_s^\alpha &= b(X_s^\alpha, \alpha_s)\, ds + \sigma(X_s^\alpha, \alpha_s)\, dW_s, \\ X_t{}^\alpha &= x, \end{aligned}$$

and $\mathcal{A}_d = \{\alpha_t(\omega) : \mathbb{F}^W-\text{progressive}\}$.

Constrained BSDEs and fully non linear PDEs

Recall

$$v(t,x) = Y_t^{t,x,a_0},$$

where Y^{t,x,a_0} is the first component of the unique solution to the constrained BSDE.

This suggests that the function

$$(t,x) \mapsto Y_t^{t,x,a_0}$$

is a solution to the HJB equation. Several authors have used this fact to construct solutions to HJB or other PDEs.

I. Kharroubi, J. Ma, H. Pham, J. Zhang. Backward SDEs with constrained jumps and quasi-variational inequalities. Ann. Probab. 38 (2010), no. 2, 794-840.

I. Kharroubi, H. Pham. Feynman-Kac representation for Hamilton-Jacobi-Bellman IPDE. Ann. Probab. 43 (2015), no. 4, 1823-1865.

The starting point is the following: denoting $Y_s = Y_s^{t,x,a_0}$ we have the following functional equality: for $t \leq s \leq \theta \leq T$,

$$Y_s = \operatorname{ess\,sup}_{\nu \in \mathcal{V}} \mathbb{E}^{\nu} \left[\int_s^{\theta} f(X_r, I_r)\, dr + Y_{\theta} \middle| \mathcal{F}_s^{W,\mu} \right],$$

which can be seen as a *"randomized dynamic programming principle"*.

It is much easier to prove than the usual DPP.

Some comments

• σ may be degenerate (or even null: deterministic control problem).

• Existence of an optimal control is not proved.

• Numerical methods have been developed for constrained BSDEs of this form:

I. Kharroubi, N. Langrené, H. Pham. A numerical algorithm for fully nonlinear HJB equations: an approach by control randomization. Monte Carlo Methods Appl. 20, no. 2 (2014), 145–165.

I. Kharroubi, N. Langrené, H. Pham. Discrete time approximation of fully nonlinear HJB equations via BSDEs with nonpositive jumps. Ann. Appl. Probab. 25, no. 4 (2015),2301–2338.

Thank you for your attention!

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