Control of state-constrained McKean-Vlasov equations: application to portfolio selection

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Mean-field control

The joint law of (X_t^α, α_t) appears in the cost and dynamics

$$\min_\alpha J(\alpha) := \mathbb{E}\Big[\int_0^T f(s, X_s^\alpha, \mathbb{P}_{(X_s^\alpha, \alpha_s)}, \alpha_s)\ \mathrm{d}s + g(X_T^\alpha, \mathbb{P}_{X_T^\alpha}) \Big]$$

with McKean-Vlasov dynamics (in $\mathbb{R}^d$)

$$\mathrm{d}X_t^\alpha = b(t, X_t^\alpha, \mathbb{P}_{(X_t^\alpha, \alpha_t)}, \alpha_t)\ \mathrm{d}t + \sigma(t, X_t^\alpha, \mathbb{P}_{(X_t^\alpha, \alpha_t)}, \alpha_t)\ \mathrm{d}W_t.$$

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Example: Markowitz mean-variance problem in continuous time

$$\min_\alpha \; \lambda \mathrm{Var}(X_T^\alpha) - \mathbb{E}[X_T^\alpha]$$
$$\mathrm{d}X_t^\alpha = \alpha_t b \, \mathrm{d}t + \alpha_t \sigma \, \mathrm{d}W_t.$$

Contents

Constraints

What about adding **state constraints** on X_t^α? Examples: bounded state, positive state...

In the stochastic control case (no mean-field interaction), this problem is treated by Bokanowski, Picarelli, and Zidani 2015; Bokanowski, Picarelli, and Zidani 2016 with constraints

$$X_t^\alpha \in \mathcal{K} \ \forall t \in [0, T], \text{a.s.},$$

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$$X_t^\alpha \in \mathcal{K} \; \forall t \in [0,T], \text{a.s.},$$

for a non empty closed set $\mathcal{K}$. It turns out that we can go further and enforce **probabilistic constraints**:

$$\Psi(t, \mathbb{P}_{X_t^\alpha}) \leq 0 \; \forall t \in [0,T].$$

Related literature

- For MFG: state constrained to stay in a **compact** $\rightarrow$ Cannarsa, Capuani, and Cardaliaguet 2018; Graber and Mayorga 2021...
- Deterministic MFC by control of Fokker-Planck equations $\rightarrow$ Bonnet 2019
- MFC with **smooth terminal expectation** constraint $\rightarrow$ Chen and Wang 2019
- MFC cost with probabilistic constraints for a standard diffusion $\rightarrow$ Daudin 2021

Examples of possible constraints in the form

$$\Psi(t, \mathbb{P}_{X_t^\alpha}) \leq 0 \ \forall t \in [0, T]$$

- $\mathbb{P}(X_t^\alpha \in \mathcal{K}_t) \geq p_t$
- $\mathcal{W}_2(\mathbb{P}_{X_t^\alpha}, \eta_t) \leq \delta_t$
- $\varphi(\mathbb{P}_{X_T^\alpha}) \leq 0$
- $\phi(t_i, \mathbb{P}_{X_{t_i}^\alpha}) \leq 0$ for $t_1 < \cdots < t_k$

Primal Markowitz mean-variance problem

$$\begin{aligned} \min_\alpha \ & -\mathbb{E}[X_T^\alpha] \\ & \mathrm{d}X_t^\alpha = \alpha_t b \ \mathrm{d}t + \alpha_t \sigma \ \mathrm{d}W_t \\ & \mathrm{Var}(X_T) \leq \vartheta. \end{aligned}$$

Markowitz mean-variance problem with conditional expectation constraint

$$
\begin{aligned}
&\inf_{\alpha} \lambda \mathrm{Var}(X_T^\alpha) - \mathbb{E}[X_T^\alpha] \\
&\quad \mathrm{d}X_t^\alpha = \alpha_t b \, \mathrm{d}t + \alpha_t \sigma \, \mathrm{d}W_t \\
&\quad E[X_t^\alpha \mid X_t^\alpha \leq \theta] \geq \delta.
\end{aligned}
$$

Contents

An auxiliary problem with exact penalization

$$J(\alpha) = \mathbb{E}\Big[\int_0^T f(s, X_s^\alpha, \alpha_s, \mathbb{P}_{(X_s^\alpha, \alpha_s)})\, \mathrm{d}s + g(X_T^\alpha, \mathbb{P}_{X_T^\alpha})\Big]$$

$$V^\Psi := \inf_{\alpha \in \mathcal{A}} \big\{ J(\alpha) : \Psi(t, \mathbb{P}_{X_t^\alpha}) \leq 0,\ \forall\, t \in [0, T] \big\}.$$

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Define a new state variable

$$Z_z^\alpha(\cdot) := z - \mathbb{E}\Big[\int_0^\cdot f\big(s, X_s^\alpha, \alpha_s, \mathbb{P}_{(X_s^\alpha, \alpha_s)}\big) \, \mathrm{d}s \Big] = z - \int_0^\cdot \widehat{f}\big(s, \mathbb{P}_{(X_s^\alpha, \alpha_s)}\big) \, \mathrm{d}s,$$

with $\widehat{f}(t, \nu) = \int_{\mathbb{R}^d \times A} f\big(t, x, a, \nu\big) \, \nu(\mathrm{d}x, \mathrm{d}a)$ and $\widehat{g}(\mu) = \int_{\mathbb{R}^d} g(x, \mu) \mu(\mathrm{d}x)$.

An auxiliary problem with exact penalization

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$$J(\alpha) = z - Z_z^\alpha(T) + \widehat{g}(\mathbb{P}_{X_T^\alpha})$$

An auxiliary problem with exact penalization

Then

$$
\begin{aligned}
& J(\alpha) \leq z,\ \Psi(s, \mathbb{P}_{X_s^\alpha}) \leq 0,\ \forall\ s \in [0, T] \\
\iff\ & \widehat{g}(\mathbb{P}_{X_T^\alpha}) \leq Z_z^\alpha(T),\ \Psi(s, \mathbb{P}_{X_s^\alpha}) \leq 0,\ \forall\ s \in [0, T] \\
\iff\ & \{\widehat{g}(\mathbb{P}_{X_T^\alpha}) - Z_z^\alpha(T)\}_+ + \{\Psi(s, \mathbb{P}_{X_s^\alpha})\}_+ = 0,\ \forall\ s \in [0, T],
\end{aligned}
$$

where $x_+ = \max(x, 0)$ is the positive part.

An auxiliary problem with exact penalization

Then

$$\begin{aligned} & J(\alpha) \leq z,\ \Psi(s, \mathbb{P}_{X_s^\alpha}) \leq 0,\ \forall\ s \in [0,T] \\ \iff & \widehat{g}(\mathbb{P}_{X_T^\alpha}) \leq Z_z^\alpha(T),\ \Psi(s, \mathbb{P}_{X_s^\alpha}) \leq 0,\ \forall\ s \in [0,T] \\ \iff & \{\widehat{g}(\mathbb{P}_{X_T^\alpha}) - Z_z^\alpha(T)\}_+ + \{\Psi(s, \mathbb{P}_{X_s^\alpha})\}_+ = 0,\ \forall\ s \in [0,T], \end{aligned}$$

where $x_+ = \max(x, 0)$ is the positive part. Hence, rewriting the value of the control problem we obtain

$$\begin{aligned} & V^\Psi \\ & = \inf\{z \in \mathbb{R} \mid \exists\ \alpha \in \mathcal{A} \text{ s.t. } J(\alpha) \leq z,\ \Psi(s, \mathbb{P}_{X_s^\alpha}) \leq 0,\ \forall\ s \in [0,T]\} \\ & = \inf\{z \in \mathbb{R} \mid \exists\ \alpha \in \mathcal{A} \text{ s.t. } \{\widehat{g}(\mathbb{P}_{X_T^\alpha}) - Z_z^\alpha(T)\}_+ + \{\Psi(s, \mathbb{P}_{X_s^\alpha})\}_+ = 0, \forall\ s \in [0,T] \end{aligned}$$

An auxiliary problem with exact penalization

Unconstrained mean-field control problem

$$\mathcal{Y}^{\Psi} : z \in \mathbb{R} \mapsto \inf_{\alpha \in \mathcal{A}} \Big[\{\widehat{g}(\mathbb{P}_{X_T^\alpha}) - Z_z^\alpha(T)\}_+ + \sup_{s \in [0,T]} \{\Psi(s, \mathbb{P}_{X_s^\alpha})\}_+ \Big],$$

with the notation $\{x\}_+ = \max(x, 0)$. We see that $\mathcal{Y}^{\Psi}(z) \geq 0$.

An auxiliary problem with exact penalization

Unconstrained mean-field control problem

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with the notation $\{x\}_+ = \max(x, 0)$. We see that $\mathcal{Y}^{\Psi}(z) \geq 0$. We consider the infimum of the **zero level-set**

$$\mathcal{Z}^{\Psi} := \inf\{z \in \mathbb{R} \mid \mathcal{Y}^{\Psi}(z) = 0\}.$$

$\mathcal{Y}^{\Psi}$ being convex, positive and non-increasing, if $\mathcal{Z}^{\Psi} < \infty$ then $\mathcal{Y}^{\Psi}$ is decreasing on $(-\infty, \mathcal{Z}^{\Psi}]$ then $\mathcal{Y}^{\Psi}(z) = 0$ on $[\mathcal{Z}^{\Psi}, \infty)$.

Representation results

Theorem 1 (make the constraint function vary)

If $V^\Psi < +\infty$, it verifies the bounds

$$\mathcal{Z}^\Psi \leq V^\Psi \leq \inf_{\varepsilon > 0} \mathcal{Z}^{\Psi+\varepsilon}.$$

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$$\mathcal{Z}^\Psi \le V^\Psi \le \inf_{\varepsilon>0} \mathcal{Z}^{\Psi+\varepsilon}.$$

Theorem 2

If $V^\Psi < +\infty$ then $\varepsilon \mapsto \mathcal{Z}^{\Psi+\varepsilon}$ is continuous at zero. Hence

$$\mathcal{Z}^\Psi = V^\Psi.$$

Summary

Solving

$$\mathcal{Y}^{\Psi} : z \in \mathbb{R} \mapsto \inf_{\alpha \in \mathcal{A}} \Big[\{\widehat{g}(\mathbb{P}_{X_T^\alpha}) - Z_z^\alpha(T)\}_+ + \sup_{s \in [0,T]} \{\Psi(s, \mathbb{P}_{X_s^\alpha})\}_+ \Big],$$

and computing the infimum of the zero level set

$$\mathcal{Z}^{\Psi} := \inf\{z \in \mathbb{R} \mid \mathcal{Y}^{\Psi}(z) = 0\},$$

gives us the value of the control problem $V^{\Psi} = \mathcal{Z}^{\Psi}$.

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gives us the value of the control problem $V^{\Psi} = \mathcal{Z}^{\Psi}$.

Moreover it can be seen that ε-optimal controls α^{ε} for the auxiliary problem $\mathcal{Y}^{\Psi}(V^{\Psi})$ are ε-admissible ε-optimal controls for the original problem in the sense that

$$J(\alpha^{\varepsilon}) \leq V + \varepsilon, \ \sup_{0 \leq s \leq T} \Psi(s, \mathbb{P}_{X_s^{\alpha^{\varepsilon}}}) \leq \varepsilon.$$

An alternative standard auxiliary problem

Solving

$$\bar{\mathcal{Y}}^{\Psi} : z \in \mathbb{R} \mapsto \inf_{\alpha \in \mathcal{A}} \left[\{ \widehat{g}(\mathbb{P}_{X_T^\alpha}) - Z_z^\alpha(T) \}_+ + \int_0^T \{ \Psi(s, \mathbb{P}_{X_s^\alpha}) \}_+ \mathrm{d}s \right],$$

instead of

$$\mathcal{Y}^{\Psi} : z \in \mathbb{R} \mapsto \inf_{\alpha \in \mathcal{A}} \left[\{ \widehat{g}(\mathbb{P}_{X_T^\alpha}) - Z_z^\alpha(T) \}_+ + \sup_{s \in [0,T]} \{ \Psi(s, \mathbb{P}_{X_s^\alpha}) \}_+ \right],$$

and computing the infimum of the zero level set

$$\bar{\mathcal{Z}}^{\Psi} := \inf\{ z \in \mathbb{R} \mid \bar{\mathcal{Y}}^{\Psi}(z) = 0 \},$$

also allows us to obtain the value and optimal control of the problem, if we assume existence of optimal controls for the auxiliary problem.

On open loop and closed loop controls

We focus here on the problem with integral penalization. In general (see Cosso, Gozzi, Kharroubi, Pham, and Rosestolato 2020) the infimum over open-loop controls α in $\mathcal{A}$ can be taken equivalently over randomized feedback policies, i.e. controls α in the form

$$\alpha_t = \alpha(t, X_t^\alpha, \mathbb{P}_{X_t^\alpha}, Z_t^{z,\alpha}, U),$$

for some deterministic function $\alpha : [0,T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \times \mathbb{R} \times [0,1] \mapsto A$, where U is an $\mathcal{F}_0$-measurable uniform random variable on $[0,1]$.

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$$\alpha_t = \alpha(t, X_t^\alpha, \mathbb{P}_{X_t^\alpha}, Z_t^{z,\alpha}) \ ?$$

We assume that f, b, σ do not depend on the law of the control process. We also assume that the running cost $f = f(t,x,\mu)$ does not depend on the control argument.

On open loop and closed loop controls

The Bellman equation for the auxiliary problem (in dynamic form) is:

$$\begin{cases} \partial_t w(t,\mu,z) + \int_{\mathbb{R}^d} \{\inf_{a\in A}\{b\big(t,x,a,\mu)\partial_\mu w(t,\mu,z)(x) - \bar{f}(t,\mu)\partial_z w(t,\mu,z) \\ +\frac{1}{2}\mathrm{Tr}(\sigma\sigma^\top\big(t,x,a,\mu)\partial_x\partial_\mu w(t,\mu,z)(x))\}\}\mu(\mathrm{d}x) + \{\Psi(t,\mu)\}_+ = 0 \text{ for } (t,\mu,z) \in \\ w(T,\mu,z) = \{\widehat{g}(\mu) - z\}_+ \text{ for } (\mu,z) \in \mathcal{P}_2(\mathbb{R}^d)\times\mathbb{R}. \end{cases}$$

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By assuming that w is a smooth solution to this Bellman equation, and when the infimum in

$$\inf_{a\in A}\{b\big(t,x,a,\mu)\partial_\mu w(t,\mu,z)(x) + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^\top\big(t,x,a,\mu)\partial_x\partial_\mu w(t,\mu,z)(x))\}$$

is attained for some measurable function $\hat{\alpha}(t,x,\mu,z)$ on $[0,T]\times\mathbb{R}^d\times\mathcal{P}(\mathbb{R}^d)\times\mathbb{R}$,

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is attained for some measurable function $\hat{\alpha}(t,x,\mu,z)$ on $[0,T]\times\mathbb{R}^d\times\mathcal{P}(\mathbb{R}^d)\times\mathbb{R}$, we get an optimal control given in feedback form by $\alpha_t^* = \hat{\alpha}(t, X_t^{\alpha^*}, \mathbb{P}_{X_t^{\alpha^*}}, Z_t^{z,\alpha^*})$, $0\le t\le T$, which shows that one can restrict to deterministic feedback policies.

Contents

Numerical scheme

We discretize the problem in time with $t_k = k\frac{T}{N}$ and use the algorithm from Carmona and Laurière 2019 by taking a sequence of neural network $(\alpha_i^{\theta_i})_{i=1,\cdots,N} : \mathbb{R}^d \times \mathbb{R} \mapsto \mathbb{R}^d$ with parameters θ_i to approximate the (Markovian feedback) control.

We solve the auxiliary problem in a segment K in which we assume $\mathcal{Z}^\Psi$ lies, discretized as $z_1 < \cdots < z_M$. For $i = 0, \cdots, N_T - 1$, $j = 1, \cdots, N$

$$
\begin{aligned}
X_{i+1}^j &= X_i^j + b\big(t_i, X_i^j, \alpha_i^{\theta_i}(X_i^j, z), \overline{\mu}_i\big)\Delta t_i + \sigma\big(t_i, X_i^j, \alpha_i^{\theta_i}(X_i^j, z), \overline{\mu}_i\big)\Delta W_i^j \\
Z_z^\alpha &= z - \frac{1}{N}\sum_{i=0}^{N_T-1}\sum_{l=1}^{N} f\big(t_i, X_i^l, \alpha_i^{\theta_i}(X_i^l, z), \overline{\mu}_i\big)\ \Delta t_i \\
\overline{\mu}_i &= \frac{1}{N}\sum_{j=1}^{N}\delta_{(X_i^j, \alpha_i^{\theta_i}(X_i^j, z))} \\
X_0^j &\sim \mu_0
\end{aligned}
$$

Numerical scheme

We solve by stochastic gradient descent $\inf_\theta \sum_{m=1}^{M} w_\Lambda(z_m)$ with w defined by

$$w_\Lambda(z)$$

$$:= \mathbb{E}\Big[\{\frac{1}{N}\sum_{l=1}^{N} g\Big(X^l_{N_T}, \frac{1}{N}\sum_{j=1}^{N}\delta_{X^j_{N_T}}\Big) - Z^\alpha_z\}_+ + \Lambda\sum_{i=1}^{N_T}\{\Psi\Big(t_i, \frac{1}{N}\sum_{j=1}^{N}\delta_{X^j_i}\Big)\}_+\Delta t_i\Big].$$

and for $i = 0, \cdots, N_T - 1$, $j = 1, \cdots, N$

$$X^j_{i+1} = X^j_i + b\big(t_i, X^j_i, \alpha^\theta_i(X^j_i, z), \overline{\mu}_i\big)\Delta t_i + \sigma\big(t_i, X^j_i, \alpha^\theta_i(X^j_i, z), \overline{\mu}_i\big)\Delta W^j_i$$

$$Z^\alpha_z = z - \frac{1}{N}\sum_{i=0}^{N_T-1}\sum_{l=1}^{N} f\big(t_i, X^l_i, \alpha^\theta_i(X^l_i, z), \overline{\mu}_i\big)\ \Delta t_i$$

$$\overline{\mu}_i = \frac{1}{N}\sum_{j=1}^{N}\delta_{(X^j_i, \alpha^\theta_i(X^j_i, z))}$$

$$X^j_0 \sim \mu_0$$

Numerical scheme

- Define $\alpha^* = \alpha^{\theta^*}$ with θ^* the solution to the previous minimization problem.
- Compute $V_0 = \inf\{z_i,\ i \in [\![1, M]\!] \mid w_\Lambda(z_i) \leq \varepsilon\}$ with $\alpha = \alpha^*$ in the dynamics for some threshold ε.
- Return the value V_0 and the optimal controls $\hat{\alpha}_i : x \mapsto \alpha_i^*(x, V_0)$ for $i = 0, \cdots, N_T - 1$.

Numerical results

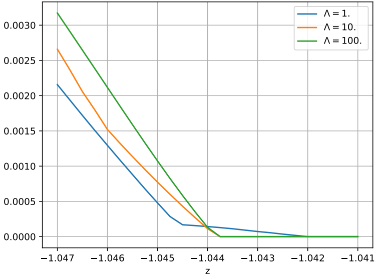
Markowitz mean-variance problem with conditional expectation constraint

$$\begin{aligned}
&\min_{\alpha} \ \lambda \mathrm{Var}(X_T^\alpha) - \mathbb{E}[X_T^\alpha] \\
&\qquad \mathrm{d}X_t^\alpha = \alpha_t b \ \mathrm{d}t + \alpha_t \sigma \ \mathrm{d}W_t \\
&\qquad E[X_t^\alpha \mid X_t^\alpha \leq \theta] \geq \delta.
\end{aligned}$$

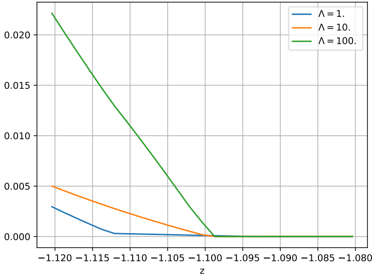
Primal Markowitz mean-variance problem

$$\begin{aligned}
&\min_{\alpha} \ -\mathbb{E}[X_T^\alpha] \\
&\qquad \mathrm{d}X_t^\alpha = \alpha_t b \ \mathrm{d}t + \alpha_t \sigma \ \mathrm{d}W_t \\
&\qquad \mathrm{Var}(X_T) \leq \vartheta.
\end{aligned}$$

Auxiliary value functions



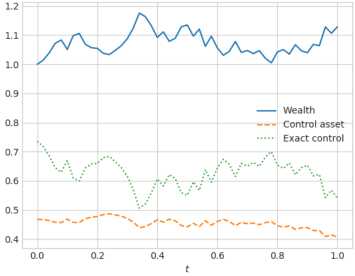
Conditional expectation constraint



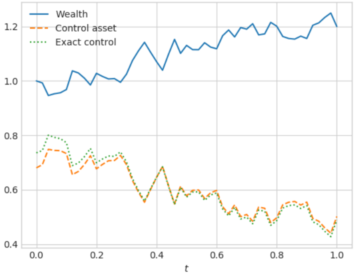
Terminal variance constraint

Auxiliary value function $\mathcal{Y}_\Lambda(z)$ for several values of Λ

Trajectories



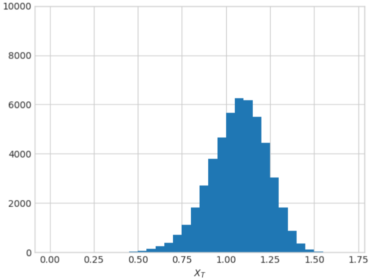
Conditional expectation constraint



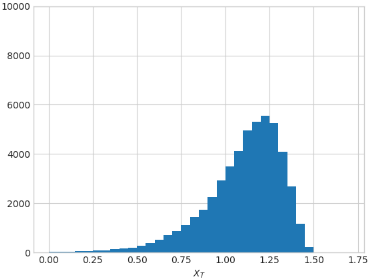
Terminal variance constraint

Sample path of the controlled process X_t^α, with the analytical optimal control (for the unconstrained case) and the computed control. On the left figure we don't have the true control but plot the unconstrained one for comparison

Histogram



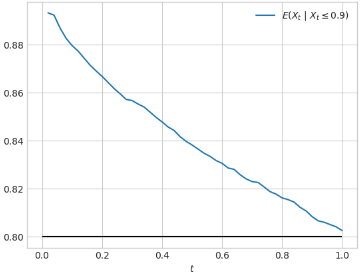
Conditional expectation constraint



Terminal variance constraint

Histogram of X_T for 50000 samples

Constraints



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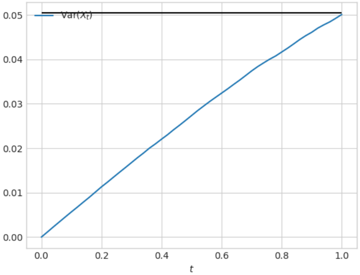

Terminal variance constraint

On the left: conditional expectation $E[X_t^\alpha \mid X_t^\alpha \leq 0.9]$ estimated with 50000 samples. On the right: variance $\mathrm{Var}(X_t^\alpha)$ estimated with 50000 samples.

Conclusion

We have been able to:

- **Extend the level-set approach** to the mean-field control problem.
- Prove **representation results** of the constrained problem by an unconstrained one.
- Design a machine learning numerical scheme to compute the **optimal value and control**.

Potential future research

- Carefully assess the optimality of the computed control.
- Solve more difficult cases with an explicit solution.

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