State-constrained controlled mean-field flows

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Outline

We consider:

+ $(\Omega, \mathcal{F}, \mathbb{P})$ - complete probability space;
+ B - Brownian motion (for simplicity here: all in dimension 1);
+ $\mathcal{F}_0 \subset \mathcal{F}$ - sub-σ-field independent of B, "rich enough":

$$\mathcal{P}(\mathbb{R}^d) = \{\mathbb{P}_\xi,\ \xi \in \mathbb{L}^0(\mathcal{F}_0; \mathbb{R}^d)\}; d \geq 1;$$

+ $\mathcal{P}(\mathbb{R}^d)$ the space of all probability measures over $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$;
+ $\mathbb{F} = \mathbb{F}^B \vee \mathcal{F}_0$ - our filtration;
+ U $(\subset \mathbb{R}^{d'}$, for some integer $d' \geq 1)$: some compact metric space.

+ $\mathcal{P}_p(\mathbb{R}^d)$ the space of probability measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ with finite p-th moment, $p \geq 1$, endowed with the Wasserstein metric:

$$W_p(\mu, \nu) := \inf \Big\{ \Big(\int_{\mathbb{R}^d \times \mathbb{R}^d} |z - z'|^p \rho(dzdz') \Big)^{\frac{1}{p}} \Big| \ \rho \in \mathcal{P}_p(\mathbb{R}^d \times \mathbb{R}^d) \text{ with } \rho(\cdot \times \mathbb{R}^d) = \mu,\ \rho(\mathbb{R}^d \times \cdot) = \nu \Big\}.$$

Note: $(\mathcal{P}_p(\mathbb{R}^d), W_p(\cdot, \cdot))$ is a complete metric space.

Brief state of the art

Viability problems:

1) Given a controlled stochastic flow, satisfy some state restrictions for a suitably-chosen admissible control. This property is called (near-)viability;

2) Viability or near-viability goes back to the pioneering work by

Nagumo (1942);

3) Viability properties for stochastic differential equations:

Aubin, Da Prato (1990);
Buckdahn, Peng, Quincampoix, and Rainer (1998);

4) Viability properties for backward stochastic differential equations:

Buckdahn, Quincampoix, Răşcanu (2000, PTRF);

5) Viability of moving sets for a nonlinear Neumann problem

Maticiuc, Răşcanu (2007, Nonlinear Analysis);

....

Brief state of the art

6) The long-established methods to deal with the viability concept can be categorised as follows:

- To characterize the distance function to the set of constraints in relation with (viscosity) solutions of an associated HJB system: e.g.,

  (forward) Buckdahn, Peng, Quincampoix, and Rainer (1998);

  (bck) Buckdahn, Quincampoix, Rășcanu (2002);

  (SPDE) Buckdahn, Quincampoix, Tessitore (2009);

  (open sets) Buckdahn, Frankowska, Quincampoix (2019);

  G. (2019);

- Tangency related concepts:

  Aubin, Da Prato (1990);

  Gautier, Thibault (1993);

  Carja, Necula, Vrabie (2007)

  ....

1. Objectives of the talk

<u>**Characterize near-viability**</u> for (controlled) mean-field flows. Extends:

- Buckdahn, Li, Peng and Rainer (2017, AOP (2014, Arxiv))

<u>**The novelties in our work**</u>:

• We deal with law constraints by asking the law of the solution to belong to some given closed subset $\mathcal{K} \subset \mathcal{P}_2(\mathbb{R}^d)$.

• We study a more general state and law restriction, roughly speaking, by simultaneously controlling the statistical law in a set of constraints $\mathcal{K} \subset \mathcal{P}_2(\mathbb{R}^d)$ and the evolution of a particular particle in a set of state constraints $K \subset \mathbb{R}^d$.

• We give some structure considerations for mean-field controlled dynamics whose importance exceeds their application to viability.

• We give a Nagumo-type characterization of invariance. In the case of regular-bound domains, this characterization, together with Itô's formula for mean-field flows lead to explicit (necessary) conditions on the coefficient functions.

2.1. The Control System

We consider the following split controlled SDE of mean-field type:

$$\begin{cases} dX_s^{t,\xi,u} = b(X_s^{t,\xi,u}, \mathbb{P}_{X_s^{t,\xi,u}}, u_s)ds + \sigma(X_s^{t,\xi,u}, \mathbb{P}_{X_s^{t,\xi,u}}, u_s)dB_s, \ t \le s \le T, \\ dX_s^{t,x,\xi,u} = b(X_s^{t,x,\xi,u}, \mathbb{P}_{X_s^{t,\xi,u}}, u_s)ds + \sigma(X_s^{t,x,\xi,u}, \mathbb{P}_{X_s^{t,\xi,u}}, u_s)dB_s, \ t \le s \le T, \\ X_t^{t,\xi,u} = \xi \in \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d), \ X_t^{t,x,\xi,u} = x \in \mathbb{R}^d. \end{cases} \tag{2.1}$$

$\rightsquigarrow$ Buckdahn, Li, Peng, Rainer (2017)

+**Assumption**: $b : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times U \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times U \to \mathbb{R}^{d\times d}$: bounded, uniformly conti. and Lipschitz conti. in (x, μ), uniformly w.r.t. $u \in U$, i.e., $\exists\ c > 0$

$$|b(x, \mu, u) - b(y, \nu, u)| + |\sigma(x, \mu, u) - \sigma(y, \nu, u)| \le c(|x - y| + W_2(\mu, \nu)). \tag{2.2}$$

$\rightsquigarrow$ Under our assumptions (2.1) has a unique solution $(X^{t,x,\xi,u}, X^{t,\xi,u})$, $\forall (t, x, \xi) \in [0, T] \times \mathbb{R}^d \times \mathbb{L}^2_{\mathcal{F}_t}(\mathbb{R}^d)$, $u \in \mathcal{U}_{t,T}$.

2.2. Elementary Controls. The First Main Result

In this subsection: Let the initial time be $t = 0$. Denote by:

$$+\ \mathcal{U}^0 := \{v : U^{\mathbb{N}} \to \Omega \times U :\ \exists (\Omega_i)_{i\geq 1} \text{ an } \mathcal{F}_0\text{-partition of } \Omega,\ v(u) = \sum_{i\geq 1} \mathbf{1}_{\Omega_i} u^i,\ \forall u = (u^i)_{i\geq 1} \subset U\}.$$

$$+\ \mathcal{U}^e := \{v := v(u) : v \in \mathcal{U}^0,\ u = (u^i)_{i\geq 1},\ u^i \in \mathbb{L}^0_{\mathbb{F}^B}([0,T];U),\ i \geq 1\}.$$

For every initial datum $\bar{\xi} := (\xi, \xi') \in \mathbb{L}^2_{\mathcal{F}_0}(\mathbb{R}^{2d})$, we set

$$\bar{X}^{\bar{\xi},u} = \bar{X}^{0,\bar{\xi},u} := \left(X^{0,\xi,u}, X^{0,\xi',\xi,u}\right)(= (X^{0,\xi,u}, X^{0,x,\xi,u}|_{x=\xi'})).$$

With the above notations, for $t = 0$, our split SDE writes:

$$\begin{cases} d\bar{X}^{t,\bar{\xi},u}_s = \bar{b}(\bar{X}^{t,\bar{\xi},u}_s, \mathbb{P}_{\bar{X}^{t,\bar{\xi},u}_s}, u_s)ds + \bar{\sigma}(\bar{X}^{t,\bar{\xi},u}_s, \mathbb{P}_{\bar{X}^{t,\bar{\xi},u}_s}, u_s)dB_s, \\ \bar{X}^{t,\bar{\xi},u}_t = \bar{\xi}. \qquad s \in [t,T]. \end{cases} \tag{2.3}$$

2.2. Elementary Controls. The First Main Result

Given $u = v(u) \in \mathcal{U}^e$, we have the following result stating that, for initial data $\bar{\xi} \in \mathbb{L}^2_{\mathcal{F}_0}(\mathbb{R}^{2d})$, the law of the solution $\bar{X}^{\bar{\xi},u}$ depends on $(\bar{\xi}, v) \in \mathbb{L}^2_{\mathcal{F}_0}(\mathbb{R}^{2d}) \times \mathcal{U}^0$ only through the law of these data.

Theorem 2.1

If $(\bar{\xi}, v), (\bar{\xi}', v') \in \mathbb{L}^2_{\mathcal{F}_0}(\mathbb{R}^{2d}) \times \mathcal{U}^0$ have the same law, then, for every $u = (u^i)_{i\geq 1} \subset \mathbb{L}^0_{\mathbb{F}^B}([0,T]; U)$,

$$\mathbb{P}_{\bar{X}_t^{\bar{\xi},v(u)}} = \mathbb{P}_{\bar{X}_t^{\bar{\xi}',v'(u)}}, \ t \in [0,T]. \tag{2.4}$$

Remark 2.2. The above theorem shows that the density (in $\mathbb{L}^2$-sense) of the elementary controls $\mathcal{U}^e$ in the family of admissible ones $\mathcal{U}_{0,T}$ implies that the value functions in mean-field dynamics control problems depends on the initial argument $\xi \in \mathbb{L}^2_{\mathcal{F}_0}$ only through its law $\mathbb{P}_\xi$.

2.3. Constrained Problem From now on, we let

- K denote a closed subset of $\mathbb{R}^d$;
- $\mathcal{K}$ be some closed subset of $\mathcal{P}_2(\mathbb{R}^d)$.

Definition 2.3

$\mathcal{K} \subset \mathcal{P}_2(\mathbb{R}^d)$ is **near-viable** w.r.t. (2.1) if,
$\forall t \in [0,T], \ \forall \mu \in \mathcal{K}, \ \forall \varepsilon > 0, \ \exists (\xi, u) \in \mathbb{L}^2_{\mathcal{F}_t}(\mathbb{R}^d) \times \mathcal{U}_{t,T}$ such that
$\mathbb{P}_\xi = \mu$ and $d_{\mathcal{K}}(\mathbb{P}_{X_s^{t,\xi,u}}) \le \varepsilon, \forall s \in [t,T]$.

Remark 2.4. Theorem 2.1 allows to replace in the above definition "$\exists \ \xi \in \mathbb{L}^2_{\mathcal{F}_t}(\mathbb{R}^d)$ s.t. $\mathbb{P}_\xi = \mu$" with the stronger assertion "for all $\xi \in \mathbb{L}^2_{\mathcal{F}_t}(\mathbb{R}^d)$ s.t. $\mathbb{P}_\xi = \mu$."

Lemma 2.5

Let ξ, $\xi' \in \mathbb{L}^2_{\mathcal{F}_0}(\mathbb{R}^d)$ be of the same law and $v \in \mathcal{U}^0$. Then there exists $v' \in \mathcal{U}^0$ such that (ξ, v) and (ξ', v') obey the same law.

2.3. Constrained Problem (continued)

In general, both state and law restrictions, $K \subset \mathbb{R}^d, \mathcal{K} \subset \mathcal{P}_2(\mathbb{R}^d)$,

$$\mathbb{L}^2_{\mathcal{F}_s}(K) \otimes \mathcal{K} := \left\{ \theta = (\theta_1, \theta_2) \in \mathbb{L}^2_{\mathcal{F}_s}(\mathbb{R}^d \times \mathbb{R}^d) : \theta_1 \in K, \ \mathbb{P}\text{-a.s. and } \mathbb{P}_{\theta_2} \in \mathcal{K} \right\}. \quad (2.5)$$

Definition 2.6

$K \times \mathcal{K}$ is **near-viable** w.r.t (2.1) if
$\forall t \in [0, T], \ \forall \xi' \in \mathbb{L}^2_{\mathcal{F}_t}(K), \ \forall \xi \in \mathbb{L}^2_{\mathcal{F}_t}(\mathbb{R}^d)$ s.t. $\mathbb{P}_\xi \in \mathcal{K}$ and $\forall \varepsilon > 0$,
$\exists u \in \mathcal{U}_{t,T}$ s.t.

$$d_{\mathbb{L}^2_{\mathcal{F}_s}(K) \otimes \mathcal{K}}((X_s^{t,\xi',\xi,u}, X_s^{t,\xi,u})) \leq \varepsilon, \ s \in [t, T]. \quad (2.6)$$

Proposition 2.7

$\mathbb{R}^d \times \mathcal{K}$ is near-viable with respect to the controlled flow (2.1) (in the sense of Definition 2.6) if and only if $\mathcal{K}$ is near-viable in the sense of Definition 2.3.

3.1. Law Constrained Flows

The aim of this first subsection is to focus on controlled processes for which the constraints concern (only) the law.

For every initial time $t \in [0, T)$, every initial measure $\mu \in \mathcal{K}$ and every $T - t > \varepsilon > 0$, we define the family of processes

$$Y_s^{t,\xi,u} := \xi + \int_t^s b(\xi, \mathbb{P}_\xi, u_r)dr + \int_t^s \sigma(\xi, \mathbb{P}_\xi, u_r)dB_r, \ \ s \in [t, T], \quad (3.1)$$

indexed by $\xi \in \mathbb{L}^2_{\mathcal{F}_t}(\mathbb{R}^d)$ such that $\mathbb{P}_\xi = \mu$. Moreover, to the triplet (t, μ, ε), we associate the reachable set

$$\mathcal{S}_{t,\varepsilon}(\mu) := \left\{ Y_{t+\varepsilon}^{t,\xi,u} : \ (\xi, u) \in \mathbb{L}^2_{\mathcal{F}_t}(\mathbb{R}^d) \times \mathcal{U}_{t,t+\varepsilon} \text{ and } \mathbb{P}_\xi = \mu \right\}. \quad (3.2)$$

3.1. Law Constrained Flows (continued)

We also introduce the following notation

$$d_{t,\varepsilon}^2(\mu \to \mathcal{K}) := \inf \Big\{ \mathbb{E}[|\vartheta - \theta|^2] + \frac{1}{\varepsilon}\mathbb{E}[|\mathbb{E}^{\mathcal{F}_t}[\vartheta - \theta]|^2] : \vartheta \in \mathcal{S}_{t,\varepsilon}(\mu),\ \theta \in \mathbb{L}^2_{\mathcal{F}_{t+\varepsilon}}(\mathbb{R}^d) \text{ with } \mathbb{P}_\theta \in \mathcal{K} \Big\}. \tag{3.3}$$

Definition 3.1 (Mean-field quasi-tangency condition (MFQT$_1$))

The set $\mathcal{K} \subset \mathcal{P}_2(\mathbb{R}^d)$ is said to satisfy the **mean-field quasi-tangency** condition if

$$\liminf_{\varepsilon \to 0+} \frac{1}{\varepsilon} d_{t,\varepsilon}^2(\mu \to \mathcal{K}) = 0, \text{ for all } t \in [0,T] \text{ and all } \mu \in \mathcal{K}.$$

3.1. Law Constrained Flows (continued)

We are now able to state the main result in this framework.

Theorem 3.2

A closed set $\mathcal{K} \subset \mathcal{P}_2(\mathbb{R}^d)$ is near-viable (in the sense of Definition 2.3) if and only if $\mathcal{K}$ satisfies the mean-field quasi-tangency condition $\mathbf{MFQT_1}$.

This result will be a consequence of the more general one (with further state constraints) and the result on the equivalence of mean-field quasi-tangency conditions without state constraints (i.e., when $K = \mathbb{R}^d$, cf., Proposition 3.4).

3.2. Law and State Constrained Flows

For every initial time $t \in [0,T)$, every initial data $\xi, \xi' \in \mathbb{L}^2_{\mathcal{F}_t}(\mathbb{R}^d)$ such that $\xi' \in K$, $\mathbb{P}$-a.s. and $\mathbb{P}_\xi \in \mathcal{K}$ and every $T-t > \varepsilon > 0$, we define the processes

$$\begin{cases} Y_s^{t,\xi',\xi,u} := \xi' + \int_t^s b(\xi', \mathbb{P}_\xi, u_r)dr + \int_t^s \sigma(\xi', \mathbb{P}_\xi, u_r)dB_r, \\ Y_s^{t,\xi,u} := \xi + \int_t^s b(\xi, \mathbb{P}_\xi, u_r)dr + \int_t^s \sigma(\xi, \mathbb{P}_\xi, u_r)dB_r, \ \ s \in [t,T]. \end{cases} \tag{3.4}$$

Note that $Y_s^{t,\xi,\xi,u} = Y_s^{t,\xi,u}$, $s \in [t,T]$. Moreover, to $(t, \xi', \xi, \varepsilon)$ we associate the reachable set

$$\mathcal{S}_{t,\varepsilon}(\xi', \xi) := \left\{ (Y_{t+\varepsilon}^{t,\xi',\xi,u}, Y_{t+\varepsilon}^{t,\xi,u}) : u \in \mathcal{U}_{t,t+\varepsilon} \right\} \subset \mathbb{L}^2_{\mathcal{F}_{t+\varepsilon}}(\mathbb{R}^d) \times \mathbb{L}^2_{\mathcal{F}_{t+\varepsilon}}(\mathbb{R}^d). \tag{3.5}$$

3.2. Law and State Constrained Flows (continued)

We also introduce the following notation

$$d^2_{t,\varepsilon}((\xi',\xi) \to K \times \mathcal{K}) = \inf \Big\{ \mathbb{E}[|\vartheta - \theta|^2] + \frac{1}{\varepsilon}\mathbb{E}[|\mathbb{E}^{\mathcal{F}_t}[\vartheta - \theta]|^2] : \vartheta \in \mathcal{S}_{t,\varepsilon}(\xi',\xi),\ \theta \in \mathbb{L}^2_{\mathcal{F}_{t+\varepsilon}}(K) \otimes \mathcal{K} \Big\}. \tag{3.6}$$

Definition 3.3 $(\mathbf{MFQT_2})$

The set $K \times \mathcal{K} \subset \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$ is said to satisfy the **mean-field quasi-tangency** condition if

$$\liminf_{\varepsilon \to 0+} \frac{1}{\varepsilon} d^2_{t,\varepsilon}((\xi',\xi) \to K \times \mathcal{K}) = 0, \text{ for all } t \in [0,T], \text{ and all } (\xi',\xi) \in \mathbb{L}^2_{\mathcal{F}_t}(K) \otimes \mathcal{K}.$$

3.2. Law and State Constrained Flows (continued)

A priori, when $K = \mathbb{R}^d$ the quasi-tangency condition $\mathbf{MFQT_2}$ is more restrictive than $\mathbf{MFQT_1}$.

Indeed, the definitions imply that, given $t \in [0,T]$, $\varepsilon > 0$ and $\mu \in \mathcal{K}$,

$$\mathcal{S}_{t,\varepsilon}(\mu) = \bigcup_{\xi \in \mathbb{L}^2_{\mathcal{F}_t}(\mathbb{R}^d):\ \mathbb{P}_\xi = \mu} \Big\{ \vartheta_2 :\ (\vartheta_1, \vartheta_2) \in \bigcup_{\xi' \in \mathbb{L}^2_{\mathcal{F}_t}(\mathbb{R}^d)} \mathcal{S}_{t,\varepsilon}(\xi', \xi) \Big\}.$$

As consequence,

$$d_{t,\varepsilon}(\mu \to \mathcal{K}) \leq \inf_{\xi' \in \mathbb{L}^2_{\mathcal{F}_t}(\mathbb{R}^d)} d_{t,\varepsilon}((\xi', \xi) \to \mathbb{R}^d \times \mathcal{K}).$$

3.2. Law and State Constrained Flows (continued)

In fact, we can prove the equivalence of these both concepts.

Proposition 3.4

A closed set $\mathcal{K} \subset \mathcal{P}_2(\mathbb{R}^d)$ satisfies the mean-field quasi-tangency condition $\mathbf{MFQT_1}$ if and only if the set $\mathbb{R}^d \times \mathcal{K}$ satisfies the mean-field quasi-tangency condition $\mathbf{MFQT_2}$.

3.2. Law and State Constrained Flows (continued)

The main result of the section states that the notions of near-viability and mean-field quasi-tangency are equivalent.

Theorem 3.5

Let us consider two closed sets $K \subset \mathbb{R}^d$ and $\mathcal{K} \subset \mathcal{P}_2(\mathbb{R}^d)$. The set $K \times \mathcal{K}$ is near-viable (in the sense of Definition 2.6) if and only if the condition $\mathbf{MFQT_2}$ holds true.

Let us just hint at the main steps. The necessity of the condition $\mathbf{MFQT_2}$ will follow from standard estimates for solutions of (2.1). Basically, on small intervals $[t, t+\varepsilon]$, one needs to replace the actual process with the Euler-type scheme (3.4) and show that the error obeys the limiting condition in Definition 3.3. The same intuition for Euler schemes will be useful for the sufficiency part.

3.2.2. Sufficiency of $\mathbf{MFQT_2}$

Definition 3.6

ε-appr. constrained solutions $\big(\tilde{T}, \delta, u, (\varphi', \varphi), (\psi', \psi), (Y', Y)\big)$

i) The terminal time satisfies $t \le \tilde{T} \le T$;

ii) The measurable delay $\delta : [t, \tilde{T}] \longrightarrow [t, \tilde{T}]$ is non-decreasing and satisfies

$$s - \varepsilon \le \delta(s) \le s, \text{ for all } s \in [t, \tilde{T}];$$

iii) The control process u is admissible;

iv) The error-estimating processes $\phi \in \{\varphi, \varphi', \psi, \psi'\}$ are s.t. $\phi : [t, \tilde{T}] \times \Omega \to \mathbb{R}^d$ is predictable and satisfies

$$\mathbb{E}\Big[\int_t^{\tilde{T}} |\phi(s)|^2 ds\Big] \le \varepsilon\big(\tilde{T} - t\big);$$

3.2.2. Sufficiency of $\mathrm{MFQT_2}$ (continued)

Definition 3.6 (continued)

v) The process (Y', Y) satisfies

$$Y_s = \xi + \int_t^s b(Y_{\delta(r)}, \mathbb{P}_{Y_{\delta(r)}}, u_r)dr + \int_t^s \sigma(Y_{\delta(r)}, \mathbb{P}_{Y_{\delta(r)}}, u_r)dB_r + \int_t^s \varphi_r dr + \int_t^s \psi_r dB_r,$$
$$Y'_s = \xi' + \int_t^s b(Y'_{\delta(r)}, \mathbb{P}_{Y_{\delta(r)}}, u_r)dr + \int_t^s \sigma(Y'_{\delta(r)}, \mathbb{P}_{Y_{\delta(r)}}, u_r)dB_r + \int_t^s \varphi'_r dr + \int_t^s \psi'_r dB_r,$$

for all $s \in [t, \tilde{T}]$;

vi) Moreover,

(a) the delayed process belongs to the set of constraints, i.e., for every $s \in [t, \tilde{T}]$, one has $Y'_{\delta(s)} \in K$, $\mathbb{P}$-a.s. and $\mathbb{P}_{Y_{\delta(s)}} \in \mathcal{K}$;

(b) at time $\tilde{T}$, the constraint is satisfied, i.e., $Y'_{\tilde{T}} \in K$, $\mathbb{P}$-a.s. and $\mathbb{P}_{Y_{\tilde{T}}} \in \mathcal{K}$;

(c) the process is reasonably close to the delayed process, i.e.,

$$\sup_{s\in[t,\tilde{T}]} \mathbb{E}[|Y_s - Y_{\delta(s)}|^2] \leq \varepsilon \text{ and } \sup_{s\in[t,\tilde{T}]} \mathbb{E}\left[\left|Y'_s - Y'_{\delta(s)}\right|^2\right] \leq \varepsilon.$$

4. Nagumo-like Results. Application to Smooth Sets

Whenever the system (2.1) is no longer controlled, i.e.,

$$b : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d, \text{ respectively, } \sigma : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^{d\times d},$$

we get the following criterion (see, Nagumo (1942) for deterministic systems).

Corollary 4.1

$K \times \mathcal{K} \subset \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$ is invariant with respect to (2.1) if and only if, for every $t \in [0, T)$, every $\xi \in \mathbb{L}^2_{\mathcal{F}_t}(\mathbb{R}^d)$ with $\mathbb{P}_\xi \in \mathcal{K}$, for every $\xi' \in \mathbb{L}^2_{\mathcal{F}_t}(K)$ and for all $n \in \mathbb{N}^*$, there exists $\varepsilon_n > 0$, $\mathbb{R}^{2d}$-valued $\mathcal{F}_t$-measurable random variables $\bar{\varphi}^n = ((\varphi')^n, \varphi^n)$ and predictable processes $\bar{\psi}^n = ((\psi')^n, \psi^n)$ such that the following conditions hold true:

i) $\lim\limits_{n\to\infty} \varepsilon_n = 0$; ii) $\lim\limits_{n\to\infty} \mathbb{E}[|\bar{\varphi}^n|^2] = 0$; iii) $\lim\limits_{n\to\infty} \mathbb{E}[\int_t^{t+\varepsilon_n} |\bar{\psi}^n_s|^2 ds] = 0$;

Corollary 4.1 (continued)

iv) $\mathbb{P}_{\xi+\varepsilon_n b(\xi,\mathbb{P}_\xi)+\sigma(\xi,\mathbb{P}_\xi)(B_{t+\varepsilon_n}-B_t)+\varepsilon_n\varphi^n+\sqrt{\varepsilon_n}\int_t^{t+\varepsilon_n}\psi_s^n dB_s} \in \mathcal{K}$;

v) $\xi'+\varepsilon_n b(\xi',\mathbb{P}_\xi)+\sigma(\xi',\mathbb{P}_\xi)(B_{t+\varepsilon_n}-B_t)+\varepsilon_n(\varphi')^n+\sqrt{\varepsilon_n}\int_t^{t+\varepsilon_n}(\psi')_s^n dB_s \in K,\ \mathbb{P}$-a.s.

The preceding assertions i)-v) are a restatement of the quasi-tangency condition in the control-independent setting.

Let us now assume that a set of constraints Γ is described by the existence of a globally smooth function $\Phi : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ with $\Phi \in C_b^{2,1}(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$:

$$\Gamma := \{(x,\mu) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) :\ \Phi(x,\mu) \le 0\}.$$

Proposition 4.2

If the closed set $\Gamma = \{(x,\mu) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) : \ \Phi(x,\mu) \leq 0\}$ is near-viable with respect to the uncontrolled dynamics (2.1) then, for every $(\xi', \xi) \in \mathbb{L}^2_{\mathcal{F}_t}(\mathbb{R}^d \times \mathbb{R}^d)$ with $(\xi', \mathbb{P}_\xi) \in \Gamma, \ \Phi(\xi', \mathbb{P}_\xi) = 0, \ \mathbb{P}\text{-}a.s.$, the following two assertions hold simultaneously:

$$\begin{aligned}
&\sum_{1\leq i\leq d} \partial_{x_i}\Phi(\xi',\mathbb{P}_\xi) b_i(\xi',\mathbb{P}_\xi) + \frac{1}{2}\sum_{1\leq i,j,k\leq d} \partial^2_{x_i x_j}\Phi(\xi',\mathbb{P}_\xi)(\sigma_{ik}\sigma_{jk})(\xi',\mathbb{P}_\xi) \\
&+\tilde{\mathbb{E}}[\sum_{1\leq i\leq d} (\partial_\mu\Phi)_i(\xi',\mathbb{P}_\xi,\tilde{\xi}) b_i(\tilde{\xi},\mathbb{P}_\xi)] \\
&+\tilde{\mathbb{E}}[\frac{1}{2}\sum_{1\leq i,j,k\leq d} \partial_{y_i}(\partial_\mu\Phi)_j(\xi',\mathbb{P}_\xi,\tilde{\xi})\sigma_{ik}(\tilde{\xi},\mathbb{P}_\xi)\sigma_{jk}(\tilde{\xi},\mathbb{P}_\xi)] \leq 0,
\end{aligned} \tag{4.1}$$

and, for all $1 \leq j \leq d$,

$$\sum_{1\leq i\leq d} \partial_{x_i}\Phi(\xi',\mathbb{P}_\xi)\sigma_{ij}(\xi',\mathbb{P}_\xi) = 0. \tag{4.2}$$

Here $\tilde{\xi}$ denotes an independent copy of ξ, defined on a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, and $\tilde{\mathbb{E}}[\cdot]$ is the expectation w.r.t. $\tilde{\mathbb{P}}$ only concerning random variables endowed with "$\sim$".

The proof for this assertion relies on Itô's formula for mean-field flows (e.g., Buckdahn, Li, Peng, Rainer (2017)) applied to Euler-approximating flows appearing in Corollary 4.1-iv) and v).

4.1. Convex-order Comparison of Solutions

We recall that two probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ are convex-ordered $\mu \preceq_{cx} \nu$ if

$$\int_{\mathbb{R}} \phi d\mu \leq \int_{\mathbb{R}} \phi d\nu, \text{ for all } \phi : \mathbb{R} \to \mathbb{R} \text{ convex, integrable.}$$

For two real-valued random variables ξ and η defined on $(\Omega, \mathcal{F}, \mathbb{P})$, $\xi \preceq_{cx} \eta$ if $\mathbb{P}_\xi \preceq_{cx} \mathbb{P}_\eta$.

We have the following properties:

4.1. Convex-order Comparison of Solutions (continued)

Proposition 4.3

1) Let X and Y be two square-integrable random variables sharing a common expectation $\mathbb{E}[X] = \mathbb{E}[Y]$. Then the following assertions are equivalent.

   i) $X \preceq_{cx} Y$;

   ii) For every $a \in \mathbb{R}$, $\mathbb{E}[|X - a|] \leq \mathbb{E}[|Y - a|]$;

   iii) For every $a \in \mathbb{R}$, and every $\varepsilon > 0$, $\mathbb{E}[\phi_{a,\varepsilon}(X)] \leq \mathbb{E}[\phi_{a,\varepsilon}(Y)]$, where

$$\phi_{a,\varepsilon}(x) = \begin{cases} \dfrac{(x-a)^2}{2\varepsilon} + \dfrac{\varepsilon}{2}, & \text{if } x \in (a-\varepsilon, a+\varepsilon), \\ |x - a|, & \text{otherwise;} \end{cases}$$

   iv) Let $(\rho_\delta)_{\delta>0}$ be a family of standard mollifiers. For every $a \in \mathbb{R}$, $\varepsilon, \delta > 0$,

$$\mathbb{E}[\phi^\delta_{a,\varepsilon}(X)] \leq \mathbb{E}[\phi^\delta_{a,\varepsilon}(Y)], \text{ where } \phi^\delta_{a,\varepsilon} := \phi_{a,\varepsilon} * \rho_\delta.$$

4.1. Convex-order Comparison of Solutions (continued)

Proposition 4.3 (continued)

2) If X and Y are Gaussian variables, then $X \preceq_{cx} Y$ if and only if $\mathbb{E}[X] = \mathbb{E}[Y]$ and $\mathbb{E}[X^2] \leq \mathbb{E}[Y^2]$.

3) The convex order is preserved by
   i) multiplication with real constants (i.e., if $X \preceq_{cx} Y$ and $\lambda \in \mathbb{R}$, then $\lambda X \preceq_{cx} \lambda Y$);
   ii) sum of independent variables (i.e., if $(X_i)_{1 \leq i \leq m}$ are independent random variables and $(Y_i)_{1 \leq i \leq m}$ is another set of independent random variables such that $X_i \preceq_{cx} Y_i$ for all $1 \leq i \leq m$, then $\sum_{i=1}^m X_i \preceq_{cx} \sum_{i=1}^m Y_i$);
   iii) $\mathbb{L}^1$-limits (i.e., if $\{X, Y, X_i, Y_i : \ i \geq 1\} \subset \mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$ such that $X_i \to X$ in $\mathbb{L}^1$, $Y_i \to Y$ in $\mathbb{L}^1$ and $X_i \preceq_{cx} Y_i$ for all $i \geq 1$, then $X \preceq_{cx} Y$).

4) If $\phi : \mathbb{R} \to \mathbb{R}$ such that $\int_{\mathbb{R}} \phi d\mu \leq \int_{\mathbb{R}} \phi d\nu$ for every $\mu, \nu \in \mathcal{P}_2(\mathbb{R})$ with $\mu \preceq_{cx} \nu$, then ϕ is convex.

4.1. Convex-order Comparison of Solutions (continued)

We consider the set

$$\mathcal{K} := \{\mu = (\mu^1, \mu^2) \in \mathcal{P}_2(\mathbb{R}) \times \mathcal{P}_2(\mathbb{R}) : \mu^1 \preceq_{cx} \mu^2\}. \tag{4.3}$$

Proposition 4.4

The set $\mathcal{K}$ is a non-empty, closed, convex subset of $(\mathcal{P}_2(\mathbb{R}))^2$. Moreover, it has void interior (in the topology induced by W_2).

4.1. Convex-order Comparison of Solutions (continued)

We consider now the uncontrolled, one-dimensional, coefficients $b,\ \sigma_0 : \mathbb{R} \to \mathbb{R}$, and let $\sigma : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ be given by

$$\sigma(\mu) := \Big(\int_{\mathbb{R}} |\sigma_0(x)|^2 \mu(dx)\Big)^{\frac{1}{2}},\ \mu \in \mathcal{P}_2(\mathbb{R}).$$

We are interested in the near-viability of the set $\mathcal{K}$ with respect to the system driven by (b, σ). In other words, given $\xi \preceq_{cx} \eta$, what are the necessary (resp., sufficient) conditions to have $X_s^{t,\xi} \preceq_{cx} X_s^{t,\eta}$ for all $s \in [t, T]$, where

$$\begin{cases} dX_s^{t,\theta} = b(X_s^{t,\theta})ds + \sigma(\mathbb{P}_{X_s^{t,\theta}})dB_s,\ t \le s \le T, \\ X_t^{t,\theta} = \theta \in \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}). \end{cases} \tag{4.4}$$

4.1. Convex-order Comparison of Solutions (continued)

Proposition 4.5

1) If the set $\mathcal{K}$ is near-viable w.r.t (4.4), then the following two conditions hold true simultaneously:

   i) The function b is affine, i.e., $\exists\ b_0, b_1 \in \mathbb{R}$ s.t. $b(x) = b_0 + b_1 x,\ x \in \mathbb{R}$;

   ii) For every $\mu^1 \preceq_{cx} \mu^2$ with the same second order moment, σ_0 satisfies

$$\int_{\mathbb{R}} |\sigma_0(x)|^2 \mu^1(dx) \leq \int_{\mathbb{R}} |\sigma_0(x)|^2 \mu^2(dx).$$

2) Conversely, if b is affine and $|\sigma_0|^2$ is convex, then $\mathcal{K}$ is near-viable.

Thank you very much
for your attention!