Mean-Field Doubly Reflected Backward Stochastic Differential equations

S.Hamadene, LMM, Le Mans University (jww Yinggu Chen, Shandong Univ. and Tingshu Mu, LMU.)

The 9th International Colloquium on BSDEs and Mean Field Systems, Annecy, June 27-July 1, 2022.

A) Mean-field doubly reflected BSDE

We consider the following Mean-Field Doubly Reflected Backward Stochastic Differential Equation: $\forall t \le T$

$$\begin{cases} Y_t = \xi + \int_t^T f(s, Y_s, \mathbb{E}[Y_s], Z_s)\, ds + K_T^+ - K_t^+ - (K_T^- - K_t^-) \\ \qquad\qquad - \int_t^T Z_s\, dB_s \ ; \\ h(Y_t, \mathbb{E}[Y_t]) \le Y_t \le g(Y_t, \mathbb{E}[Y_t]); \\ \int_0^T (Y_t - h(Y_t, \mathbb{E}[Y_t]))dK_t^+ = \int_0^T (Y_t - g(Y_t, \mathbb{E}[Y_t]))dK_t^- = 0. \end{cases} \tag{1}$$

$B = (B_t)_{t \le T}$ is a BM and ξ, f, g and h are given data.

A solution is a quadruple of adapted processes $(Y, Z, K^{\pm}) = (Y_t, Z_t, K_t^{\pm})_{t \le T}$, $(F_t)_{t \le T}$-adapted, Y, K continuous and $K^{\pm}$ non decreasing.

a) ξ is the terminal wealth or the target wealth;

b) h is a solvency treshold;

c) g is a bonus;

d) f is the infinitesimal utility.

e) Y_t is the current value of ξ;

f) Z is the control process;

g) $K^{\pm}$ are consumption processes.

Motivation: Guaranteed life endowment with a withdrawal-bonus options.

i) A portfolio of a large number N of homogeneous life insurance policies $\ell \in \{1, \dots, N\}$.
ii) $(Y^{\ell,N}, Z^{\ell,N})$ are the characteristics of the prospective reserve of each policy .
iii) Nonlinear reserving: the driver f depends on the reserve for the particular contract ℓ and on the average reserve characteristics over the N contracts: For each $\ell = 1, \dots, N$,

a) $f(t, Y_t^{\ell,N}, (Y_t^{m,N})_{m\neq\ell}) := \alpha_t - \delta_t Y_t^{\ell,N} +$

$$\beta_t \max(\theta_t, Y_t^{\ell,N} - \tfrac{1}{N}\textstyle\sum_{k=1}^N Y_t^{k,N});$$

b) $h(Y_t^{\ell,N}, (Y_t^{m,N})_{m\neq\ell}) = \{u - c(Y_t^{\ell,N}) + \mu(\frac{1}{N}\sum_{k=1}^N Y_t^{k,N} - u)^+\} \wedge S_t;$

c) $g(Y_t^{\ell,N}, (Y_t^{m,N})_{m\neq\ell}) = \{c_2(Y_t^{\ell,N}) + c_3(\frac{1}{N}\sum_{k=1}^N Y_t^{k,N} - u))\} \vee S'_t,$

$(0 < \mu < 1)$.

iv) Sending N to infinity, yields the following forms of the driver and the obstacles:

$$f(t, Y_t, \mathbb{E}[Y_t]) := \alpha_t - \delta_t Y_t + \beta_t \max(\theta_t, Y_t - \mathbb{E}[Y_t]);$$

$$h(Y_t, \mathbb{E}[Y_t]) = \{u - c(Y_t) + \mu(\mathbb{E}[Y_t] - u)^+\} \wedge S_t;$$

$$g(Y_t, \mathbb{E}[Y_t]) = \{c_2(Y_t) + c_3(\mathbb{E}[Y_t])\} \vee S'_t$$

of the prospective reserve of a representative life insurance contract, a.k.a. the model-point among actuaries.

Pricing this type of contracts is one of the main motivations of studying the class (1) of MF-reflected BSDEs.

B) Standard Reflected BSDEs

(i) A solution for the reflected BSDE associated with $\{f(t,\omega,y,z),\xi,(h_t)_{t\le T},(g_t)_{t\le T}\}$ is a triple of $(\mathbf{F}_t)_{t\le T}$-adapted stochastic processes $(Y_t, Z_t, K_t)_{t\le T}$ such that: $\forall t \le T$,

$$\begin{cases} Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds + K_T^+ - K_t^+ - (K_T^- - K_t^-) + \int_t^T Z_s dB_s. \\ h_t \le Y_t \le g_t \\ \int_0^T (Y_s - h_s)dK_s^+ = \int_0^T (Y_s - g_s)dK_s^- = 0. \end{cases} \tag{2}$$

They are related to the valuation of recallable options, Dynkin games, min-max parabolic PDEs, etc.

(ii) Connection with the value of a Dynkin game

The process Y of (2) has the following representation (when f is Lipschitz, $h < g$ and $h_T \leq \xi \leq g_T$): $\forall t \leq T$,

$$Y_t = \operatorname*{esssup}_{\tau \geq t} \operatorname*{essinf}_{\sigma \geq t} \mathbb{E}\{\int_t^{\sigma\wedge\tau} f(s, Y_s, Z_s)ds + g(\sigma)1_{\{\sigma<\tau\}} + h(\tau)1_{\{\tau\leq\sigma,\tau<T\}} + \xi 1_{\{\tau=\sigma=T\}}|\mathcal{F}_t\}$$

$$= \operatorname*{essinf}_{\sigma \geq t} \operatorname*{esssup}_{\tau \geq t} \{\text{the same quantity}\}$$

C) Mean-Field doubly reflected BSDEs

C-i) First existence result: Fixed point argument

If (1) has a solution then Y is a fixed point of the following mapping:

$$\Phi(Y)_t = \operatorname*{esssup}_{\tau\geq t} \operatorname*{essinf}_{\sigma\geq t} \mathbb{E}\{\int_t^{\sigma\wedge\tau} f(s, Y_s, \mathbb{E}[Y_s], Z_s)ds$$

$$+ g(\sigma, Y_\sigma, \mathbb{E}[Y_t]_{t=\sigma})1_{\{\sigma<\tau\}} + h(\tau, Y_\tau, \mathbb{E}[Y_t]_{t=\tau})1_{\{\tau\leq\sigma,\tau<T\}}$$

$$+ \xi 1_{\{\tau=\sigma=T\}}|\mathcal{F}_t\}$$

$$= \operatorname*{essinf}_{\sigma\geq t} \operatorname*{esssup}_{\tau\geq t} \{\text{the same quantity}\}.$$

Theorem (CHM, '20)

Assume that:
i) f does not depend on z and Lipschitz;
ii) h, g Lipschitz, $h < g$ and $h(\xi, \mathbb{E}[\xi]) \leq \xi \leq g(\xi, \mathbb{E}[\xi])$;
iii) For some $p > 1$, $\gamma_1^g, \gamma_2^g, \beta_1^h$ and β_2^h satify

$$\Sigma := (\gamma_1^g + \gamma_2^g + \beta_1^h + \beta_2^h)^{\frac{p-1}{p}} \left[\left(\frac{p}{p-1} \right)^p (\gamma_1^g + \beta_1^h) + (\gamma_2^g + \beta_2^h) \right]^{\frac{1}{p}} < 1. \tag{3}$$

Then there exists $\delta > 0$ depending only on $p, C_f, \gamma_1^g, \gamma_2^g, \beta_1^h$ and β_2^h such that Φ is a contraction on the time interval $[T - \delta, T]$. Therefore the MFRBSDE (1) has a unique solution on $[T - \delta, T]$.

Main steps of the proof:

$\mathcal{S}^p :=$the set of continuous adapted processes ζ such that

$$\mathbb{E}[\sup_{t\leq T}|\zeta_t|^p] < \infty.$$

i) For any $Y \in \mathcal{S}^p$, $\Phi(Y) \in \mathcal{S}^p$.

ii) For $\delta > 0$, $Y, Y' \in \mathcal{S}^p$,

$$\mathbb{E}[\sup_{S\in[T-\delta,T]}|\Phi(Y)_t - \Phi(Y')_t|^p] \leq \Sigma(\delta)^p \times \mathbb{E}[\sup_{S\in[T-\delta,T]}|Y_t - Y'_t|^p]$$

where

$$\lim_{\delta\to 0}\Sigma(\delta) = \Sigma.$$

Thus for δ small enough Φ is a contraction on $\mathcal{S}^p_{[T-\delta,T]}$ and then has a fixed point Y^1.

The link between ZS Dynkin games and 2-barrier RBSDEs implies that equation (1) has a solution on $[T-\delta, T]$. As a by-product we obtain by concatenation:

Theorem (CHM, '20)

Under the same assumptions as in Theorem 1, the MFRBSDE (1) *has a unique solution on* $[0, T]$*.*

Idea of the proof: Solve the equation on $[T-2\delta, T-\delta]$ with terminal condition Y^1 and concatenate to obtain a solution on $[T-2\delta, T]$, and so on on $[T-3\delta, T-2\delta]$ up to reaching a solution on $[0, T]$.

The solution is unique since the fixed point is unique on $[T-\delta, T]$, $[T-2\delta, T-\delta]$, etc.

Remark:

i) We only have $\int_0^T |Z_s|^2 ds + K_T^{\pm} < \infty$, $\mathbb{P} - a.s.$.

ii) $\mathbb{E}[Y_s]$ can be replaced with $\mathbb{P}_{Y_s}$ where the distance between two probabilities ν_1 and ν_2 of $\mathcal{P}_p$ is given by the p-Wasserstein one.

iii) We cannot remove the condition of (3). Exemple with $\beta_1^h + \beta_2^h = 1$. Take $h(y, y') = \frac{y+y'}{2}$. If a solution exists then

$$Y_t \geq E[Y_t]$$

which is not possible in the general framework.

Remark: [continued]

iii) This result is recently generalized by Djehiche-Dumitrescu '22 to f depending also on z in considering Φ defined as follows:

$$\Phi(U)_t = \operatorname*{esssup}_{\tau\geq t}\operatorname*{essinf}_{\sigma\geq t}\mathcal{E}^{foU}_{t,\tau\wedge\sigma}\{g(\sigma, U_\sigma, \mathbb{E}[U_t]_{t=\sigma})1_{\{\sigma<\tau\}}$$

$$+ h(\tau, U_\tau, \mathbb{E}[U_t]_{t=\tau})1_{\{\tau\leq\sigma,\tau<T\}} + \xi 1_{\{\tau=\sigma=T\}}\}$$

where

$$\mathcal{E}^{foU}_{t,\tau\wedge\sigma}(\dots) = Y^U_t$$

and

$$Y^U_t = g(\sigma, U_\sigma, \mathbb{E}[U_t]_{t=\sigma})1_{\{\sigma<\tau\}} + h(\tau, U_\tau, \mathbb{E}[U_t]_{t=\tau})1_{\{\tau\leq\sigma,\tau<T\}} +$$

$$\xi 1_{\{\tau=\sigma=T\}} + \int_t^{\sigma\wedge\tau} f(s, Y^U_s, Z^U_s, \mathbb{P}_{U_s})ds - \int_t^{\sigma\wedge\tau} Z^U_s dB_s.$$

C-ii) Second existence result: Penalization

For $n, m \geq 0$, let:

$$\begin{cases} Y^{n,m} \in \mathcal{S}_c^2, \quad Z^{n,m} \in \mathcal{H}^{2,d}; \\ Y_t^{n,m} = \xi + \displaystyle\int_t^T f(s, Y_s^{n,m}, \mathbb{E}[Y_s^{n-1,m-1}], Z_s^{n,m})ds + \\ \qquad\qquad K_T^{n,m,+} - K_t^{n,m,+} - (K_T^{n,m,-} - K_t^{n,m,-}) - \displaystyle\int_t^T Z_s^{n,m} dB_s, \end{cases} \tag{4}$$

where,

$$\begin{aligned} K_t^{n,m,+} &:= m \int_0^t \left(Y_s^{n,m} - h(s, Y_s^{n-1,m-1}, \mathbb{E}[Y_s^{n-1,m-1}]) \right)^- ds; \\ K_t^{n,m,-} &:= n \int_0^t \left(Y_s^{n,m} - g(s, Y_s^{n-1,m-1}, \mathbb{E}[Y_s^{n-1,m-1}]) \right)^+ ds. \end{aligned} \tag{5}$$

$Y^{0,0}$ is the solution of

$$Y_t^{0,0} = \xi + \int_t^T f(s, Y_s^{0,0}, \mathbb{E}[Y_s^{0,0}], Z_s^{0,0})ds - \int_t^T Z_s^{0,0} dB_s, \ t \leq T, \tag{6}$$

whose existence and uniqueness is already stated (BLP, '09) and by assuming that

$$Y^{n,-1} = Y^{n,0}, \quad Y^{-1,m} = Y^{0,m}, \quad \text{and} \quad Y^{-1,-1} = Y^{0,0}. \tag{7}$$

Main assumptions

(i)

(a) $f(t, \omega, y, y', z)$ is Lipschitz ;

(b) $y' \mapsto f(t, y, y', z)$ is non-decreasing for fixed t, y, z.

(c) $y \mapsto f(t, y, y', z)$ is non-decreasing for fixed t, y, z.

(ii) (a) $\mathbb{P}-a.s.$ for any $t \leq T$, $h(\omega, t, y, y')$ and $g(\omega, t, y, y')$ are non-decreasing w.r.t y and y' and continuous w.r.t t;

(b) Lipschitz condition on g and h, i.e.,
b-i)

$$|g(t,\omega,y,y') - g(t,\omega,y_1,y_1')| \leq \gamma_1^g|y-y_1| + \gamma_2^g|y'-y_1'|$$

with $\gamma_1^g + \gamma_2^g < 1$.
b-ii)

$$|h(t,\omega,y,y') - h(t,\omega,y_1,y_1')| \leq \beta_1^h|y-y_1| + \beta_2^h|y'-y_1'|$$

with $\beta_1^h + \beta_2^h < 1$.

(c) Adapted Mokobodski's condition: There exists two process $(X_t)_{t\leq T}$ and $(\zeta_t)_{t\leq T}$ such that:

c-i) $\zeta_t > 0$, $\forall t \leq T$.

c-ii)

$$\forall t \leq T,\ X_t = X_0 + \int_0^t J_s dB_s + V_t^+ - V_t^-$$

with $J \in \mathcal{H}^{2,d}$ and $V^+, V^- \in \mathcal{S}^2_{ci}$.

c-iii)

$$h(\omega, t, y, y') \leq X_t \leq X_t + \zeta_t \leq g(\omega, t, y, y').$$

(iii) ξ is an $\mathcal{F}_T$- measurable, $\mathbb{R}$-valued r.v., $\mathbb{E}[\xi^2] < \infty$ and satisfies $\mathbb{P}$-a.s.,

$$h(T, \xi, \mathbb{E}[\xi]) \leq \xi \leq g(T, \xi, \mathbb{E}[\xi]).$$

Proposition

For any $n, m \geq 0$, $t \leq T$,

$$Y_t^{n+1,m} \leq Y_t^{n,m} \leq Y_t^{n,m+1}.$$

Proposition

There exists a constant $C \geq 0$ such that for any $n, m \geq 0$,

$$\sup_{0\leq t\leq T} \mathbb{E}[(Y_t^{n,m})^2]+\mathbb{E}\Big[\int_0^T |Z_t^{n,m}|^2 dt\Big] \leq C. \tag{8}$$

Idea of the proof:

Let $U_t^{n-1,m-1} = g(t, Y_t^{n-1,m-1}, \mathbb{E}[Y_t^{n-1,m-1}])$.

$$\begin{aligned}&\mathbb{E}[\{n \textstyle\int_0^T \left(Y_s^{n,m} - U_s^{n-1,m-1}\right)^+ ds\}^2] \\ &\leq C(1 + \mathbb{E}[\textstyle\int_0^T (|Y_s^{n,m}|^2 + |\mathbb{E}[Y_s^{n-1,m-1}]|^2 + |Z_s^{n,m}|^2) ds])\end{aligned} \tag{9}$$

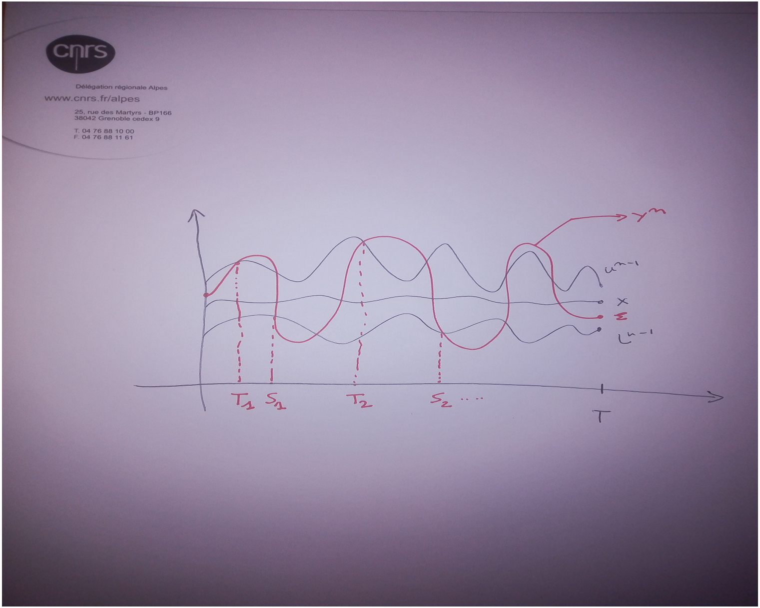
Actually let $\{T_k\}_{k\geq 1}$ and $\{S_k\}_{k\geq 0}$ defined by:

$$S_0 = 0, \quad T_k = \inf\{S_{k-1} \leq r \leq T : Y_r^{n,m} \geq U_s^{n-1,m-1}\} \wedge T,$$

$$S_k = \inf\{T_k \leq r \leq T : Y_r^{n,m} \leq L_s^{n-1,m-1}\} \wedge T, \quad k \geq 1.$$

where

$$L_t^{n-1,m-1} = h(t, Y_t^{n-1,m-1}, \mathbb{E}[Y_t^{n-1,m-1}]).$$


cnrs
Délégation régionale Alpes
www.cnrs.fr/alpes
25, rue des Martyrs - BP166
38042 Grenoble cedex 9
T. 04 76 88 10 00
F. 04 76 88 11 61
y^n
u^{n-1}
x
l^{n-1}
T_1 S_1 T_2 S_2 . . .
T

As $h < g$ then:

i) $T_k \nearrow T$ and $S_k \nearrow T$ (those sequences are even of stationary type).

ii) $Y^{n,m} \geq L^{n-1,m-1}$ on $[T_k, S_k] \cap \{T_k < S_k\}$.

Then we have:

$$Y_{T_k}^{n,m} = Y_{S_k}^{n,m} + \int_{T_k}^{S_k} f(s, Y_s^{n,m}, \mathbb{E}[Y_s^{n-1,m-1}], Z_s^{n,m}) ds \\ - \int_{T_k}^{S_k} Z_s^{n,m} dB_s - n \int_{T_k}^{S_k} \left(Y_s^{n,m} - U_s^{n-1,m-1}\right)^+ ds,$$

But

$$\sum_{k\geq 1}(Y^{n,m}_{S_k} - Y^{n,m}_{T_k}) \leq \sum_{k\geq 1}(X_{S_k} - X_{T_k}) + (\xi - X_T)^+.$$

Then summing over k, we get:

$$n\int_0^T \left(Y^{n,m}_s - U^{n-1,m-1}_s\right)^+ ds$$

$$\leq (\xi - X_T)^+ - \int_0^T \left(J_s + Z^{n,m}_s\right) \sum_{k\geq 1}\{1_{[T_k,S_k)}(s)\}dB_s$$

$$+(V^+_T - V^+_0) + \int_0^T |f(s, Y^{n,m}_s, \mathbb{E}[Y^{n-1,m-1}_s], Z^{n,m}_s)|ds.$$

which implies (9) after squarring and taking expectation.

In the same way we have:

$$\mathbb{E}[\{m \int_0^T \left(Y_s^{n,m} - L_s^{n-1,m-1}\right)^- ds\}^2]$$
$$\leq C(1 + \mathbb{E}[\int_0^T (|Y_s^{n,m}|^2 + |\mathbb{E}[Y_s^{n-1,m-1}]|^2 + |Z_s^{n,m}|^2) ds]). \tag{10}$$

Then from (4), we obtain

$$\mathbb{E}[(Y_t^{n,m})^2] \leq \widetilde{C}\mathbb{E}\left(1 + \int_t^T (Y_s^{n,m})^2 ds + \int_t^T (Y_s^{n-1,m-1})^2 ds\right),$$

where $\widetilde{C}$ is a constant independent of n and m. Finally by induction:

$$\mathbb{E}[(Y_t^{n,m})^2] \leq \gamma(t),$$

where $\gamma(t) := \widetilde{C}e^{2\widetilde{C}(T-t)}$ which is solution of

$$\gamma(t) = \widetilde{C}(1 + 2\int_t^T \gamma(s)ds).$$

Therefore estimate (8) holds true.

As a consequence

Corollary

There exists a constant C such that for any n, m,

$$\mathbb{E}[\sup_{t\leq T}(Y_t^{n,m})^2] \leq C. \tag{11}$$

Limit w.r.t. m

Proposition

For any $n \geq 0$, there exist processes (Y^n, Z^n, K^n) that satisfy the following one barrier reflected BSDE:

$$\begin{cases} Y^n \in \mathcal{S}_c^2,\ K^{n,+} \in \mathcal{S}_c^2 \text{ non-decreasing } (K_0^{n,+} = 0) \\ \qquad\qquad \text{and } Z^n \text{ belongs to } \mathcal{H}^{2,d}\,; \\ Y_t^n = \xi + \int_t^T f(s, Y_s^n, \mathbb{E}[Y_s^{n-1}], Z_s^n)ds + K_T^{n,+} - K_t^{n,+} \\ \qquad\qquad - n\int_t^T (Y_s^n - U_s^{n-1})^+ ds - \int_t^T Z_s^n dB_s, t \leq T; \\ Y_t^n \geq L_t^{n-1},\ t \leq T,\ \text{and} \quad \int_0^T (Y_t^n - L_t^{n-1})dK_t^{n,+} = 0, \end{cases} \tag{12}$$

Proposition (continued)

where:

a) $L_t^{n-1} = h(t, Y_t^n, \mathbb{E}[Y_t^{n-1}])$ *;*

b) $U_t^{n-1} = g(t, Y_t^n, \mathbb{E}[Y_t^{n-1}])$*.*

Moreover the following estimates hold true:

$$\begin{aligned} &\mathbb{E}[\sup_{0\leq t\leq T}(Y_t^n)^2] + \mathbb{E}[\int_0^T |Z_t^n|^2 dt] + \mathbb{E}[(K_T^{n,+})^2] + \\ &\mathbb{E}[n^2\left(\int_0^T \left(Y_t^n - U_t^{n-1}\right)^+ dt\right)^2] \leq C, \end{aligned} \tag{13}$$

where C is a constant which does not depend on n.

Let

$$Y^n = \lim_m \nearrow Y^{n,m}.$$

Step 1: Y^n is continuous (this is the main point).

i) By Peng's monotonic limit Y^n is rcll and $(Z^{n,m})_m$ converges to Z^n in $L^p(dt \otimes \mathbb{P})$ for any $p \in [1, 2)$.

ii) The fact that $\mathbb{E}[(K_T^{n,m,+})^2] \leq C$ and Y^n rcll imply that: For any $t \leq T$,

$$Y_t^n \geq h(t, Y_t^{n-1}, \mathbb{E}[Y_t^{n-1}]).$$

iii) Through standard reflected BSDEs, $Y^{n,m}$ has the following representation:

$$
\begin{aligned}
Y_t^{n,m} &= \operatorname{esssup}_{\tau \geq t} \mathbb{E}\{\textstyle\int_t^\tau f(s, Y_s^{n,m}, \mathbb{E}[Y_s^{n-1,m-1}], Z_s^{n,m}) ds \\
&\quad - n \int_t^\tau \left(Y_s^{n,m} - g(s, Y_s^{n-1,m-1}, \mathbb{E}[Y_s^{n-1,m-1}]) \right)^+ ds \\
&\quad + Y_\tau^{n,m} \wedge h(\tau, Y_\tau^{n,m}, \mathbb{E}[Y_t^{n,m}]_{t=\tau}) 1_{\{\tau \leq \sigma, \tau < T\}} \\
&\quad + \xi 1_{\{\tau = \sigma = T\}} | \mathcal{F}_t\}.
\end{aligned}
$$

But the Snell envelope is continuous through increasing rcll processes and then

$$
\begin{aligned}
Y_t^n \quad &= \operatorname{esssup}_{\tau \geq t} \mathbb{E}\{\textstyle\int_t^\tau f(s, Y_s^n, \mathbb{E}[Y_s^{n-1}], Z_s^n) ds \\
&-n \textstyle\int_t^\tau \left(Y_s^n - g(s, Y_s^{n-1}, \mathbb{E}[Y_s^{n-1}])\right)^+ ds \\
&+h(\tau, Y_\tau^n, \mathbb{E}[Y_t^n]_{t=\tau}) 1_{\{\tau \leq \sigma, \tau < T\}} \\
&+\xi 1_{\{\tau = \sigma = T\}} | \mathcal{F}_t\}.
\end{aligned}
$$

Next for any predictable stopping time η

$$
\{\Delta_\eta Y^n < 0\} \subset \{Y_{\eta-}^n = h(\tau, Y_{\eta-}^n, \mathbb{E}[Y_t^n]_{t=\eta-})\} \cap \{\Delta_\eta h(\dots) < 0\}.
$$

As $\beta_1^h + \beta_2^h < 1$, then Y^n does not have jumps and then Y^n is continuous. Thus the convergence of $(Y^{n,m})_m$ to Y^n is uniform and we recover classically equation (12).

Limit w.r.t. n

For any $n \geq 0$, $Y^n \leq Y^{n+1}$, then let

$$Y = \lim_n \searrow Y^n.$$

Step 1: Y is continuous (this is also the main point).

Let τ be a stopping time. For $n \geq 0$,

$$\rho_\tau^n := \inf\{s \geq \tau, Y_s^n \leq X_s\} \wedge T.$$

Then $\rho_\tau^{n+1} \leq \rho_\tau^n$. Let $\rho_\tau = \lim_n \rho_\tau^n$. We then have:

i) $Y_{\rho_\tau} = X_{\rho_\tau}$ on $\{\rho_\tau < T\}$.

ii) For any $t \in [\tau, \rho_\tau]$,

$$\begin{cases} Y_t = Y_{\rho_\tau} + \int_t^{\rho_\tau} f(s, Y_s, \mathbb{E}[Y_s], Z_s)ds - (K_{\rho_\tau}^{\tau,-} - K_t^{\tau,-}) - \int_t^{\rho_\tau} Z_s^\tau dB_s; \\ Y_t \leq g(s, Y_s, \mathbb{E}[Y_s]) \text{ and } \int_\tau^{\rho_\tau} (Y_t - g(s, Y_s, \mathbb{E}[Y_s]))dK_t^{\tau,-} = 0. \end{cases}$$

Next let

$$\theta_\tau^n := \inf\{s \geq \rho_\tau, Y_s^n \geq X_s + \zeta_s\} \wedge T.$$

Then $\theta_\tau^{n+1} \geq \theta_\tau^n$. Let $\theta_\tau = \lim_n \theta_\tau^n$. We then have:

iii) $Y_{\theta_\tau} = (X_{\theta_\tau} + \zeta_{\theta_\tau})$ on $\{\theta_\tau < T\}$.

iv) For any $t \in [\rho_\tau, \theta_\tau]$,

$$\begin{cases} Y_t = Y_{\theta_\tau} + \int_t^{\theta_\tau} f(s, Y_s, \mathbb{E}[Y_s], Z_s)ds + K_{\theta_\tau}^{\tau,+} - K_t^{\tau,+} - \int_t^{\theta_\tau} Z_s^\tau dB_s; \\ \\ Y_t \geq h(s, Y_s, \mathbb{E}[Y_s]) \text{ and } \int_{\rho_\tau}^{\theta_\tau} (Y_t - h(s, Y_s, \mathbb{E}[Y_s]))dK_t^{\tau,+} = 0. \end{cases}$$

Take now

$$\tau = 0, \gamma_1 = \rho_0, \gamma_2 = \theta_0, \gamma_3 = \rho_{\gamma_2}, \gamma_4 = \theta_{\gamma_2}, \text{etc..}$$

Then:

(i) $(\gamma_n)_n$ is of stationnay type.

(ii) $\gamma_n \leq \gamma_{n+1}$.

(iii) Y is continuous on $[0, \gamma_n]$ and then continuous on $[0, T]$.

Therefore $(Y^n)_n$ converges to Y uniformly (in $\mathcal{S}^2$). Next define

i) $Z = \lim_n Z^n$

ii) $K^+ = \lim_n K^{+,n}$

iii) $K^- = \lim_n n \int_0^t (Y_s^n - U_s^{n-1})^+ ds$.

Go back now to (12), take the limit w.r.t n to obtain:

Theorem

The quadruple $(Y, Z, K^{\pm})$ is a solution of the MFRBSDE (1).

Remark: In some cases we have also uniqueness

Thanks for your attention.