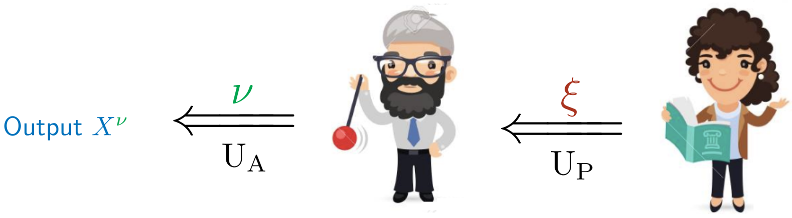
Moral hazard for time-inconsistent agents

Dylan Possamaï **Camilo Hernández**

9th International Colloquium on BSDEs and Mean Field Systems.
June 27 - July 01, 2022.

Contract theory: the big picture



Agent

Principal

Given information about a system X, $\mathbb{F}^X$.

Principal: offers contract $\xi \in \mathcal{F}^X \xrightarrow{\max} \mathrm{U_P}(X_T, \xi)$.

Agent: – accepts/rejects contract $\xi \in \Xi$,

– chooses an effort $\nu \xrightarrow{\text{control}} X^\nu \xrightarrow{\max} \mathrm{U_A}(\xi, c(X^\nu, \nu))$.

Contract theory: the big picture

Output X^{ν} $\xLeftarrow[\mathrm{U}_{\mathrm{A}}]{\nu}$

Agent

$\xLeftarrow[\mathrm{U}_{\mathrm{P}}]{\xi}$

Principal

$\mathbb{P}^{\nu}$ weak solution to: $t \in [0, T]$

$$X_t = x_0 + \int_0^t \lambda_r(X_{r\wedge\cdot}, \nu_r)\mathrm{d}r + \sigma_r(X_{r\wedge\cdot})\mathrm{d}W_r^{\nu}$$

– Continuous time.

– Controls drift.

$$\mathrm{V}_0^{\mathrm{A}}(\xi) := \sup_{\nu\in\mathcal{A}} \mathbb{E}^{\mathbb{P}^{\nu}}\Big[e^{-\alpha T}\mathrm{U}_{\mathrm{A}}(\xi) - \int_0^T e^{-\alpha r}c_r(X_{r\wedge\cdot}, \nu_r)\mathrm{d}r\Big]$$

Contract theory: the big picture

Output X^{ν} $\xLeftarrow[\mathrm{U_A}]{\nu}$  $\xLeftarrow[\mathrm{U_P}]{\xi}$ 

Agent

Principal

The contract $\xi \in \mathcal{F}_T^X$, i.e.

- asymmetry of information: access X, not Agent's effort.
- time horizon enforces a non-Markovian structure, $\xi(X_{\cdot\wedge T})$.

$$\mathrm{V}_0^{\mathrm{P}} := \sup_{\xi \in \Xi} \mathbb{E}^{\mathbb{P}^{\nu^\star}}\Big[X_T - \xi(X_{\cdot\wedge T})\Big]$$

Holmström and Milgrom '87. Sannikov '08. Cvitanić, Possamaï, and Touzi '18.

Contract theory: time-consistent preferences

$c_t(x,a)=a^2,\ \lambda_t(x,a)=a.$

$\mathrm{U_A}(x)=x.$

$$\mathrm{V_t^A}(\xi)=\sup_{\nu\in\mathcal{A}}\mathbb{E}^{\mathbb{P}^\nu}\Big[e^{-\alpha(T-t)}\xi-\int_0^T e^{-\alpha(r-t)}\nu_r^2\mathrm{d}r\Big]$$

– $\mathrm{V^A}(\xi)$ satisfies dynamic prog. principle: A's problem is time-consistent.

$$Y_t=\xi+\int_t^T\Big(\frac{Z_r^2}{2}-\alpha Y_r\Big)\mathrm{d}r-\int_t^T Z_r\mathrm{d}X_r,$$

$$\widetilde{\Xi}:=\Big\{\xi=Y_T^{y_0,Z},\ Y_t^{y_0,Z}:=y_0-\int_0^t\Big(\frac{Z_r^2}{2}-\alpha Y_r\Big)\mathrm{d}r+\int_0^t Z_r\mathrm{d}X_r\Big\}.$$

Contract theory: time-consistent preferences

$c_t(x,a)=a^2,\ \lambda_t(x,a)=a.$

$\mathrm{U_A}(x)=x.$

$$\mathrm{V^A_t}(\xi) = \sup_{\nu\in\mathcal{A}} \mathbb{E}^{\mathbb{P}^\nu}\bigg[e^{-\alpha(T-t)}\xi - \int_0^T e^{-\alpha(r-t)}\nu_r^2 \mathrm{d}r\bigg]$$

– $\mathrm{V^A}(\xi)$ satisfies dynamic prog. principle: A's problem is time-consistent.

$$Y_t = \xi + \int_t^T \left(\frac{Z_r^2}{2} - \alpha Y_r\right)\mathrm{d}r - \int_t^T Z_r \mathrm{d}X_r,$$

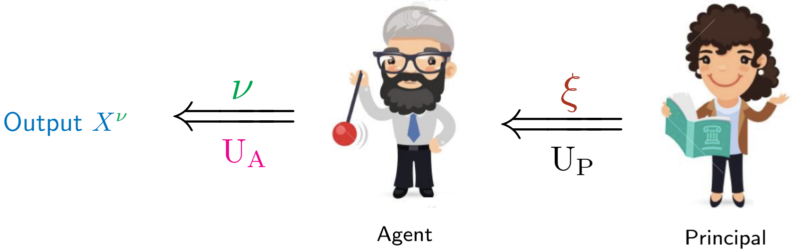
$$\widetilde{\Xi} := \left\{\xi = Y_T^{y_0,Z},\ Y_t^{y_0,Z} := y_0 - \int_0^t \left(\frac{Z_r^2}{2} - \alpha Y_r\right)\mathrm{d}r + \int_0^t Z_r \mathrm{d}X_r\right\}.$$

– P identifies all of A's optimal actions: maximisers of Hamiltonian, $a^\star(r,x,z)$.

$$\Xi = \widetilde{\Xi} \implies \mathrm{V^P_0} = C + \sup_Z \mathbb{E}^{\mathbb{P}^{\nu^\star}}\left[\int_0^T \left(\frac{Z_r^2}{2} - \alpha Y_r^{y_0,Z}\right)\mathrm{d}r\right]$$

– Standard stochastic control problem: control Z, state variables $(X, Y^{y_0,Z})$.

Time-inconsistent contract theory



Agent sees contract ξ and rejects/accepts contract

Chooses effort ν according to **time-inconsistent preferences** U_{A}.

Contract theory: a review

Inconsistent Control

Consistent planning

Time-inconsistent contract theory

Three approaches: Strotz '95

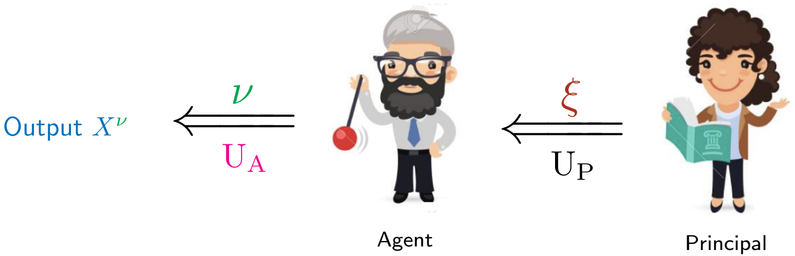
Pre-committed, Naive agent.

Consistent Planning: Game theoretic approach. Considers a non-cooperative game, where the agent plays against future versions of himself, and look for sub-game perfect Nash equilibria.

Ekeland and Lazrak; Ekeland and Pirvu; Hu, Jin, and Zhou; Björk, Khapko, and Murgoci; Czichowsky; Wei, Yong, and Zu.

- Björk, Khapko, and Murgoci '17: extended HJB (PDE). Limited to a verification argument.
- He and Jiang '19: characterisation of Markovian equilibria.

Time–inconsistent contract theory



Agent sees contract ξ and chooses equilibrium effort, $\nu^\star \in \mathcal{E}(\xi)$, according to time-inconsistent preferences, i.e.[1]

$$\mathrm{V}_t^{\mathrm{A}}(\nu^\star;\xi) := \mathrm{J}^{\mathrm{A}}(t,t,\nu^\star;\xi) := \mathbb{E}^{\mathbb{P}^\nu}\bigg[f(T-t)\mathrm{U_A}(\xi) - \int_t^T f(r-t)c_r(X_{r\wedge\cdot},\nu_r^\star)\mathrm{d}r\bigg|\mathcal{F}_t^X\bigg].$$

Principal solves

$$\mathrm{V}^{\mathrm{P}} := \sup_{\xi\in\Xi}\ \sup_{\nu\in\mathcal{E}(\xi)} \mathbb{E}^{\mathbb{P}^\nu}\Big[\mathrm{U_P}\big(X_T-\xi\big)\Big].$$

[1] Non–exponential discounting f, $f(0)=1$

What is for us an equilibrium? Continuous time

"[*The*] *problem* [*of a sophisticated agent*] *is then to find the best plan among those that* [*he*] *will actually follow.*" Strotz '95.

– Roughly speaking the same of Ekeland, Lazrak and Pirvu '06, '08, '10:

$$\varepsilon\text{-equilibrium} \;+\; \text{local property}.$$

What is for us an equilibrium? Continuous time

"[*The*] *problem* [*of a sophisticated agent*] *is then to find the best plan among those that* [*he*] *will actually follow.*" Strotz '95.

– Roughly speaking the same of Ekeland, Lazrak and Pirvu '06, '08, '10:

ε-equilibrium + local property.

Definition (H, Possamaï.'20)

Let $\nu^\star \in \mathcal{A}$, candidate. $\nu \otimes_{t+\ell} \nu^\star := \nu 1_{[t,t+\ell)} + \nu^\star 1_{[t+\ell,T]}$.

$\forall \varepsilon > 0$, $\exists \ell_\varepsilon$: $\forall (\ell, t, \nu) \in (0, \ell_\varepsilon) \times [0,T] \times \mathcal{A}$

$$\mathrm{J}^{\mathrm{A}}(t,t,\nu^\star) - \mathrm{J}^{\mathrm{A}}(t,t,\nu \otimes_{t+\ell} \nu^\star) \geq -\varepsilon \ell$$

then $\nu^\star$ is an equilibrium model.

$$\mathrm{V}^{\mathrm{A}}_t(\nu^\star;\xi) = \mathrm{J}^{\mathrm{A}}(t,t,\nu^\star;\xi),\ \nu^\star \in \mathcal{E}(\xi).$$

Consistent planning: H. and Possamaï '20

- V^{A} satisfies an extended dynamic programming principle: Agent's value alongside equilibrium is time-consistent.

  Iterating the definition for arbitrary partitions of $[0,T]$ with mesh smaller than ℓ_ε, and passing to the limit.

$$\mathrm{V}^{\mathrm{A}}_\sigma = \sup_{\nu\in\mathcal{A}} \mathbb{E}^{\mathbb{P}^\nu}\bigg[\mathrm{V}^{\mathrm{A}}_\tau - \int_\sigma^\tau \bigg(c_r(X,\nu_r) + \mathbb{E}^{\mathbb{P}^{\nu^\star}}\bigg[f'(T-r)\mathrm{U}_{\mathrm{A}}(\xi) - \int_r^T f'(u-r)c_u(X,\nu^\star_u)\mathrm{d}u\bigg|\mathcal{F}_r\bigg]\bigg)\mathrm{d}r\bigg|\mathcal{F}_\sigma\bigg]$$

Consistent planning: H. and Possamaï '20

- V^{A} satisfies an extended dynamic programming principle: Agent's value alongside equilibrium is time-consistent.

Consistent planning: H. and Possamaï '20

- V^{A} satisfies an extended dynamic programming principle: Agent's value alongside equilibrium is time-consistent.

- $\nu^\star$ is an equilibrium iff there is (Y,Z) solution to the type-I BSVIE

$$Y_t^s = \mathrm{U_A}(s,\xi) + \int_t^T h_r^\star\big(s, X_{\cdot\wedge r}, Z_r^s, Z_r^r\big)\mathrm{d}r - \int_t^T Z_r^s \mathrm{d}X_r,\ (s,t) \in [0,T]^2,$$

for which $\nu^\star$ maximises the Hamiltonian.

Let $H_t(x,z) := \sup_{a\in A}\big\{\lambda_t(x,a)\cdot z - c_t(x,a)\big\}$, $a^\star(t,x,z) \in \mathcal{M}$ denotes a maximiser in H, and $h_t^\star(s,x,z,\mathrm{z}) := \lambda_t(x,a^\star(t,x,\mathrm{z}))\cdot z - f(t-s)c_t(x,a^\star(t,x,\mathrm{z}))$

Consistent planning: H. and Possamaï '20

- V^{A} satisfies an extended dynamic programming principle: Agent's value alongside equilibrium is time-consistent.

- $\nu^\star$ is an equilibrium iff there is (Y,Z) solution to the type-I BSVIE

$$Y_t^s = \mathrm{U}_{\mathrm{A}}(s,\xi) + \int_t^T h_r^\star\big(s, X_{\cdot\wedge r}, Z_r^s, Z_r^r\big)\mathrm{d}r - \int_t^T Z_r^s \mathrm{d}X_r,\ (s,t)\in[0,T]^2,$$

for which $\nu^\star$ maximises the Hamiltonian.

Consistent planning: H. and Possamaï '20

- V^{A} satisfies an extended dynamic programming principle: Agent's value alongside equilibrium is time-consistent.

- $\nu^\star$ is an equilibrium iff there is (Y,Z) solution to the type-I BSVIE

$$Y_t^s = \mathrm{U_A}(s,\xi) + \int_t^T h_r^\star\big(s, X_{\cdot\wedge r}, Z_r^s, Z_r^r\big)\mathrm{d}r - \int_t^T Z_r^s \mathrm{d}X_r,\ (s,t)\in[0,T]^2,$$

for which $\nu^\star$ maximises the Hamiltonian.

- All in all,

$$Y_t^t = \mathrm{V}_t^{\mathrm{A}},\ \mathcal{E} = \big\{(a^\star(t, X_{\cdot\wedge t}, Z_t^t)_{t\in[0,T]}, a^\star \in \mathcal{M}\big\}.$$

Existence and uniqueness of equilibria.

Contract theory: a review

Inconsistent Control

Consistent planning

Time-inconsistent contract theory

Recap

- By refining the definition of consistent plans, i.e. $\nu^\star \in \mathcal{E}(\xi)$, we obtained an extended dynamic programming principle for the problem of the Agent.
- Infinite system (type-I extended BSVIE). It is sufficient for $\nu^\star \in \mathcal{E}(\xi)$.
- Interestingly, it is also necessary, i.e. any equilibria arises from type-I BSVIE and $\nu^\star$ maximises the Hamiltonian.
- The well-posedness of type-I BSVIEs yields the uniqueness of equilibria (up to max. of Hamiltonian).

Recap

- By refining the definition of consistent plans, i.e. $\nu^\star \in \mathcal{E}(\xi)$, we obtained an extended dynamic programming principle for the problem of the Agent.
- Infinite system (type-I extended BSVIE). It is sufficient for $\nu^\star \in \mathcal{E}(\xi)$.
- Interestingly, it is also necessary, i.e. any equilibria arises from type-I BSVIE and $\nu^\star$ maximises the Hamiltonian.
- The well-posedness of type-I BSVIEs yields the uniqueness of equilibria (up to max. of Hamiltonian).

What does this imply about the problem faced by the Principal?

- Infinitely many representations for ξ.
- ... cannot use only one and optimise over Z as before, need to understand relationships between Z and Z^s .

A solvable LQ example: Principal's Problem:

$c_t(x,a)=a^2,\ \lambda_t(x,a)=a.$
$\mathrm{U_A}(x)=x.$

$$\mathrm{J^A}(t,t,\nu;\xi)=\mathbb{E}^{\mathbb{P}^\nu}\left[f(T-t)\xi-\int_t^T f(r-t)\nu_r^2\mathrm{d}r\middle|\mathcal{F}_t^X\right]$$

A solvable LQ example: Principal's Problem:

$$Y_t^s = f(T-s)\xi + \int_t^T Z_r^r Z_r^s - f(r-s)\frac{{Z_r^r}^2}{2}\mathrm{d}r - \int_t^T Z_r^s \mathrm{d}X_r$$

$$\overline{\Xi} := \left\{ Y_T^{y_0, Z} = \frac{y_0}{f(T)} + \int_0^T \frac{f(r)}{f(T)}\frac{{Z_r^r}^2}{2}\mathrm{d}r + \int_0^T \frac{\sigma Z_r^0}{f(T)}\mathrm{d}W_r^{\nu^\star},\ (\mathfrak{Z}') \right\}$$

A solvable LQ example: Principal's Problem:

$$Y_t^s = f(T-s)\xi + \int_t^T Z_r^r Z_r^s - f(r-s)\frac{Z_r^{r\,2}}{2}\mathrm{d}r - \int_t^T Z_r^s \mathrm{d}X_r$$

$$\overline{\Xi} := \left\{ Y_T^{y_0,Z} = \frac{y_0}{f(T)} + \int_0^T \frac{f(r)}{f(T)}\frac{Z_r^{r\,2}}{2}\mathrm{d}r + \int_0^T \frac{\sigma Z_r^0}{f(T)}\mathrm{d}W_r^{\nu^\star},\ (\mathfrak{Z}') \right\}$$

We check that for $s \in [0,T]$,

$$Z_t^s = \frac{f(T-s)}{f(T)} Z_t^0 - \widetilde{Z}_t^s, \qquad (\mathfrak{Z}')$$

where $\widetilde{Z}^s$ comes from the martingale representation of

$$M_t^s := \mathbb{E}^{\mathbb{P}^\star}\left[\int_0^T \left(f(r-s) - \frac{f(T-s)f(r)}{f(T)} \right) \frac{Z_r^{r\,2}}{2}\mathrm{d}r \middle| \mathcal{F}_t \right] = M_0 + \int_0^t \sigma \widetilde{Z}_r^s \mathrm{d}W_r^\star$$

- When $f(t) := \mathrm{e}^{-\alpha t}$, $\widetilde{Z}$ vanishes... effect due to time-inconsistency.
- $\widetilde{Z}$ also vanishes whenever Z is deterministic!

$$
\begin{aligned}
\mathrm{V}^{\mathrm{P}} = \sup_{\xi \in \Xi} \mathbb{E}^{\mathbb{P}^\star}\Big[X_T - \xi\Big] &= x - \frac{y_0}{f(T)} + \sup_{\xi \in \overline{\Xi}} \mathbb{E}^{\mathbb{P}^\star}\left[\int_0^T \left(Z_r^r - \frac{f(r)}{f(T)}\frac{(Z_r^r)^2}{2}\right)\mathrm{d}r\right] \\
&\leq x - \frac{y_0}{f(T)} + \frac{1}{2}\int_0^T \frac{f(T)}{f(r)}\mathrm{d}r
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{V}^{\mathrm{P}} = \sup_{\xi \in \Xi} \mathbb{E}^{\mathbb{P}^\star}\Big[X_T - \xi\Big] &= x - \frac{y_0}{f(T)} + \sup_{\xi \in \overline{\Xi}} \mathbb{E}^{\mathbb{P}^\star}\left[\int_0^T \left(Z_r^r - \frac{f(r)}{f(T)} \frac{(Z_r^r)^2}{2}\right) \mathrm{d}r\right] \\
&\leq x - \frac{y_0}{f(T)} + \frac{1}{2} \int_0^T \frac{f(T)}{f(r)} \mathrm{d}r
\end{aligned}
$$

We can identify $f(T)/f(r)$ deterministic s.t.

$$
Z_t^t := \frac{f(T)}{f(t)},\ Z_t^s := \frac{f(T-s)}{f(T-t)} Z_t^t,\ \widetilde{Z}_t^s = 0 \implies (3') \text{ holds.}
$$

Consequently,

$$
\xi^\star = C + \int_0^T \frac{f(T)}{f(t) f(T-t)} \cdot \mathrm{d}X_t.
$$

- Agent discounts exponential utility: P has to solve a standard optimal control problem. Explicit solution when Principal also has exponential utility: linear contract.

- Agent takes utility of discounted wealth: we find optimal contract in slightly restricted class (separability in t and s for the controls). Though optimal control stays deterministic, contract is non-Markovian and non-linear, and still recovers the standard result for time-consistent agent.

- Agent discounts exponential utility: P has to solve a standard optimal control problem. Explicit solution when Principal also has exponential utility: linear contract.

- Agent takes utility of discounted wealth: we find optimal contract in slightly restricted class (separability in t and s for the controls). Though optimal control stays deterministic, contract is non-Markovian and non-linear, and still recovers the standard result for time-consistent agent.

- What about the general case?

Type–I extended BSVIE

In great generality, the problem of Agent is given by, $(s,t) \in [0,T]^2$

$$Y_t^s = \mathrm{U_A}(s,\xi) + \int_t^T h_r^\star\big(s, X_{r\wedge\cdot}, Y_r^s, Z_r^s, Y_r^r, Z_r^r\big)\mathrm{d}r - \int_t^T Z_r^s \mathrm{d}X_r.$$

Type–I extended BSVIE

In great generality, the problem of Agent is given by, $(s,t)\in[0,T]^2$

$$Y_t^s = \mathrm{U_A}(s,\xi) + \int_t^T h_r^\star\big(s, X_{r\wedge\cdot}, Y_r^s, Z_r^s, Y_r^r, Z_r^r\big)\mathrm{d}r - \int_t^T Z_r^s \mathrm{d}X_r.$$

$\mathcal{H}^{2,2}$: $Z\in\overline{\mathbb{H}}^{2,2}$ satisfying

$$Y_t^{s,y_0,Z} = y_0^s - \int_0^t h_r^\star\big(s, X_{\cdot\wedge r}, Y_r^{s,y_0,Z}, Z_r^s, Y_r^{r,y_0,Z}, Z_r^r\big)\mathrm{d}r + \int_0^t Z_r^s \mathrm{d}X_r,\ (s,t)\in[0,T]^2$$

$$\mathrm{U_A^{(-1)}}\big(0, Y_T^{0,y_0,Z}\big) = \mathrm{U_A^{(-1)}}\big(s, Y_T^{s,y_0,Z}\big),\ s\in[0,T]. \tag{3}$$

Type–I extended BSVIE

Theorem (H, Possamaï, '21)
$\overline{\Xi} := \big\{\xi = \mathrm{U}_{\mathrm{A}}^{(-1)}(0, Y_T^{0,Z}),\ Z \in \mathcal{H}^{2,2}\big\}$. *Then* $\overline{\Xi} = \Xi$. *For* $\xi \in \overline{\Xi}$

$$\mathcal{E}(\xi) = \{a^\star(t, X_{\cdot\wedge t}, Y_t^{t,Z}, Z_t^t)_{t\in[0,T]}\},\ \mathrm{V}_0^{\mathrm{A}}(\xi) = Y_0^0.$$

$\mathcal{H}^{2,2}$: $Z \in \overline{\mathbb{H}}^{2,2}$ satisfying

$$Y_t^{s,y_0,Z} = y_0^s - \int_0^t h_r^\star\big(s, X_{\cdot\wedge r}, Y_r^{s,y_0,Z}, Z_r^s, Y_r^{r,y_0,Z}, Z_r^r\big)\mathrm{d}r + \int_0^t Z_r^s \mathrm{d}X_r,\ (s,t) \in [0,T]^2$$

$$\mathrm{U}_{\mathrm{A}}^{(-1)}\big(0, Y_T^{0,y_0,Z}\big) = \mathrm{U}_{\mathrm{A}}^{(-1)}\big(s, Y_T^{s,y_0,Z}\big),\ s \in [0,T]. \qquad (\mathfrak{Z})$$

Type–I extended BSVIE

Theorem (H, Possamaï, '21)

$$\mathrm{V}^{\mathrm{P}} = \sup_{Y_0^0 \geq R} \sup_{Z \in \mathcal{H}^{2,2}} \mathbb{E}^{\mathbb{P}^\star(Z)}\Big[\mathrm{U}_{\mathrm{P}}\Big(X_T - \mathrm{U}_{\mathrm{A}}^{(-1)}\Big(T, Y_T^{T,Z}\Big)\Big)\Big],$$

$\mathcal{H}^{2,2}$: $Z \in \overline{\mathbb{H}}^{2,2}$ satisfying

$$Y_t^{s,y_0,Z} = y_0^s - \int_0^t h_r^\star\big(s, X_{\cdot\wedge r}, Y_r^{s,y_0,Z}, Z_r^s, Y_r^{r,y_0,Z}, Z_r^r\big)\mathrm{d}r + \int_0^t Z_r^s \mathrm{d}X_r,\ (s,t) \in [0,T]^2$$

$$\mathrm{U}_{\mathrm{A}}^{(-1)}\big(0, Y_T^{0,y_0,Z}\big) = \mathrm{U}_{\mathrm{A}}^{(-1)}\big(s, Y_T^{s,y_0,Z}\big),\ s \in [0,T]. \tag{3}$$

– Volterra constrained controls... Different from Viens and Zhang '17.

To sum up

- Insight: V^{A} solves a backward stochastic Volterra integral equation.
- characterisation of equilibria holds in greater generality.
- Principal's problem boils to optimal control of a Volterra forward equation with constrained Volterra controls, i.e. a family $(Z_t^s)_{(s,t)\in[0,T]^2}$, $(\mathfrak{Z})$ holds

$$Y_t^s = \mathrm{U}_{\mathrm{A}}(s,\xi) + \int_t^T h_r^\star(s, X_{r\wedge\cdot}, Y_r^s, Z_r^s, Y_r^r, Z_r^r)\mathrm{d}r - \int_t^T Z_r^s \mathrm{d}X_r.$$

- unclear how to generalise the previous examples approach.
- Idea: forget about the constraint... and reincorporate it using stochastic target control ideas.

References I

T. Björk, M. Khapko, and A. Murgoci. On time–inconsistent stochastic control in continuous time. *Finance and Stochastics*, 21(2):331–360, 2017.

J. Cvitanić, D. Possamaï, and N. Touzi. Dynamic programming approach to principal–agent problems. *Finance and Stochastics*, 22(1):1–37, 2018.

C. Czichowsky. Time–consistent mean–variance portfolio selection in discrete and continuous time. *Finance and Stochastics*, 17(2):227–271, 2013.

I. Ekeland and A. Lazrak. Being serious about non–commitment: subgame perfect equilibrium in continuous time. arXiv preprint arXiv:0604264, 2006.

I. Ekeland and T. Pirvu. Investment and consumption without commitment. *Mathematics and Financial Economics*, 2(1):57–86, 2008.

X. He and Z. Jiang. On the equilibrium strategies for time–inconsistent problems in continuous time. *SSRN preprint 3308274*, 2019.

B. Holmström and P. Milgrom. Aggregation and linearity in the provision of intertemporal incentives. *Econometrica*, 55(2):303–328, 1987.

Y. Hu, H. Jin, and X. Zhou. Time–inconsistent stochastic linear–quadratic control. *SIAM Journal on Control and Optimization*, 50(3):1548–1572, 2012.

Y. Sannikov. A continuous–time version of the principal–agent problem. *The Review of Economic Studies*, 75(3):957–984, 2008.

R. Strotz. Myopia and inconsistency in dynamic utility maximization. *The Review of Economic Studies*, 23(3):165–180, 1955.

References II

F. Viens and J. Zhang. A martingale approach for fractional Brownian motions and related path dependent PDEs. *arXiv preprint arXiv:1712.03637*, 2017.

Q. Wei, J. Yong, and Z. Zu. Time–inconsistent recursive stochastic optimal control problems. *SIAM Journal on Control and Optimization*, 55(6):4156–4201, 2017.