Moral hazard for time-inconsistent agents

Dylan Possamaï  Camilo Hernández

9th International Colloquium on BSDEs and Mean Field Systems.
June 27 - July 01, 2022.
Contract theory: a review

Inconsistent Control
Consistent planning

Time-inconsistent contract theory
Given information about a system $X$, $\mathcal{F}^X$.

Principal: offers contract $\xi \in \mathcal{F}^X \overset{\text{max}}{\rightarrow} U_P(X_T, \xi)$.

Agent: – accepts/rejects contract $\xi \in \Xi$,
– chooses an effort $\nu$ $\overset{\text{control}}{\rightarrow} X^\nu \overset{\text{max}}{\rightarrow} U_A(\xi, c(X^\nu, \nu))$. 
Contract theory: the big picture

\[ \begin{align*}
\mathbb{P}^\nu \text{ weak solution to: } t & \in [0, T] \\
X_t &= x_0 + \int_0^t \lambda_r(X_r \wedge \cdot, \nu_r)dr + \sigma_r(X_r \wedge \cdot) dW_r
\end{align*} \]

- Continuous time.
- Controls drift.

\[ V_0^A(\xi) := \sup_{\nu \in A} \mathbb{E}^\nu \left[ e^{-\alpha T} U_A(\xi) - \int_0^T e^{-\alpha r} c_r(X_r \wedge \cdot, \nu_r) dr \right] \]
Contract theory: the big picture

The contract $\xi \in \mathcal{F}_{X}^{X}$, i.e.

- asymmetry of information: access $X$, not Agent’s effort.
- time horizon enforces a non-Markovian structure, $\xi(X_{\cdot \wedge T})$.

$$V_{0}^{P} := \sup_{\xi \in \Xi} \mathbb{E}_{P^{\nu}}^{P_{\nu}^{\nu}} \left[ X_{T} - \xi(X_{\cdot \wedge T}) \right]$$

Contract theory: time-consistent preferences

\[ c_t(x, a) = a^2, \lambda_t(x, a) = a. \]
\[ U_A(x) = x. \]

\[ V^A_t(\xi) = \sup_{\nu \in \mathcal{A}} \mathbb{E}^{\nu}\left[ e^{-\alpha(T-t)}\xi - \int_0^T e^{-\alpha(r-t)}\nu_r^2 dr \right] \]

- \( V^A(\xi) \) satisfies dynamic prog.principle: A’s problem is time-consistent.

\[ Y_t = \xi + \int_t^T \left( \frac{Z_r^2}{2} - \alpha Y_r \right) dr - \int_t^T Z_r dX_r, \]

\[ \tilde{\Xi} := \left\{ \xi = Y_T^{y_0,Z}, Y_t^{y_0,Z} := y_0 - \int_0^t \left( \frac{Z_r^2}{2} - \alpha Y_r \right) dr + \int_0^t Z_r dX_r \right\}. \]
Contract theory: time-consistent preferences

\[ c_t(x, a) = a^2, \; \lambda_t(x, a) = a. \]
\[ U_A(x) = x. \]
\[ V_t^A(\xi) = \sup_{\nu \in A} \mathbb{E}^{\mathbb{P}_\nu} \left[ e^{-\alpha(T-t)} \xi - \int_0^T e^{-\alpha(r-t)} \nu_r^2 dr \right] \]

– \( V^A(\xi) \) satisfies dynamic prog. principle: \( A \)'s problem is time-consistent.

\[ Y_t = \xi + \int_t^T \left( \frac{Z_r^2}{2} - \alpha Y_r \right) dr - \int_t^T Z_r dX_r, \]

\[ \tilde{\Xi} := \left\{ \xi = Y_T^{y_0, Z}, \; Y_t^{y_0, Z} := y_0 - \int_0^t \left( \frac{Z_r^2}{2} - \alpha Y_r \right) dr + \int_0^t Z_r dX_r \right\}. \]

– \( P \) identifies all of \( A \)'s optimal actions: maximisers of Hamiltonian, \( a^*(r, x, z) \).

\[ \Xi = \tilde{\Xi} \implies V_0^P = C + \sup_{Z} \mathbb{E}^{\mathbb{P}_\nu^*} \left[ \int_0^T \left( \frac{Z_r^2}{2} - \alpha Y_r^{y_0, Z} \right) dr \right] \]

– Standard stochastic control problem: control \( Z \), state variables \( (X, Y^{y_0, Z}) \).
Agent sees contract $\xi$ and rejects/accepts contract

Chooses effort $\nu$ according to time-inconsistent preferences $U_A$. 
Contract theory: a review

Inconsistent Control
Consistent planning

Time-inconsistent contract theory
Three approaches: Strotz ’95

Pre-committed, Naive agent.

Consistent Planning: Game theoretic approach. Considers a non-cooperative game, where the agent plays against future versions of himself, and look for sub-game perfect Nash equilibria.

Ekeland and Lazrak; Ekeland and Pirvu; Hu, Jin, and Zhou; Björk, Khapko, and Murgoci; Czichowsky; Wei, Yong, and Zu.

– Björk, Khapko, and Murgoci ’17: extended HJB (PDE). Limited to a verification argument.

– He and Jiang ’19: characterisation of Markovian equilibria.
Agent sees contract $\xi$ and chooses equilibrium effort, $\nu^* \in \mathcal{E}(\xi)$, according to time-inconsistent preferences, i.e.\(^1\)

$$V_t^A(\nu^*; \xi) := J^A(t, t, \nu^*; \xi) := \mathbb{E}^{P_t} \left[ f(T - t)U_A(\xi) - \int_t^T f(r - t)c_r(X_{r\wedge \cdot}, \nu^*_r)dr \bigg| \mathcal{F}_t^X \right].$$

Principal solves

$$V^P := \sup_{\xi \in \Xi} \sup_{\nu \in \mathcal{E}(\xi)} \mathbb{E}^{P_t} \left[ U_P(X_T - \xi) \right].$$

\(^1\)Non–exponential discounting $f$, $f(0) = 1$
What is for us an equilibrium? Continuous time

“[The] problem [of a sophisticated agent] is then to find the best plan among those that [he] will actually follow.” Strotz ’95.

– Roughly speaking the same of Ekeland, Lazrak and Pirvu ’06, ’08, ’10:
  ε-equilibrium + local property.
“[The] problem [of a sophisticated agent] is then to find the best plan among those that [he] will actually follow.” Strotz ’95.

– Roughly speaking the same of Ekeland, Lazrak and Pirvu ’06, ’08, ’10:

\[ \varepsilon \text{-equilibrium} + \text{ local property.} \]

**Definition (H, Possamaï. ’20)**

Let \( \nu^* \in \mathcal{A} \), candidate. \( \nu \otimes_{t+\ell} \nu^* := \nu_{1[t,t+\ell]} + \nu^*_{1[t+\ell,T]} \).

\( \forall \varepsilon > 0, \exists \ell_\varepsilon : \forall (\ell, t, \nu) \in (0, \ell_\varepsilon) \times [0, T] \times \mathcal{A} \)

\[ J^A(t, t, \nu^*) - J^A(t, t, \nu \otimes_{t+\ell} \nu^*) \geq -\varepsilon \ell \]

then \( \nu^* \) is an equilibrium model.

\[ V^A_t(\nu^*; \xi) = J^A(t, t, \nu^*; \xi), \ \nu^* \in \mathcal{E}(\xi). \]
Consistent planning: H. and Possamaï '20

- $V^A$ satisfies an extended dynamic programming principle: Agent's value alongside equilibrium is time-consistent.

Iterating the definition for arbitrary partitions of $[0, T]$ with mesh smaller than $\ell_\varepsilon$, and passing to the limit.

$$V^A_\sigma = \sup_{\nu \in A} \mathbb{E}^{p^\nu} \left[ V^A_\tau - \int_\sigma^\tau \left( c_r(X, \nu_r) + \mathbb{E}^{p^\nu*} \left[ f'(T-r)U_A(\xi) - \int_r^T f'(u-r)c_u(X, \nu_u^*)du \bigg| \mathcal{F}_r \right] \right) dr \bigg| \mathcal{F}_\sigma \right]$$
– $V^A$ satisfies an extended dynamic programming principle: Agent’s value alongside equilibrium is time-consistent.
– $V^A$ satisfies an extended dynamic programming principle: Agent’s value alongside equilibrium is time-consistent.

– $\nu^*$ is an equilibrium iff there is $(Y, Z)$ solution to the type-I BSVIE

\[
Y_t^s = U_A(s, \xi) + \int_t^T h_r^*(s, X_r, Z_r^s, Z_r^r)dr - \int_t^T Z_r^s dX_r, \quad (s, t) \in [0, T]^2,
\]

for which $\nu^*$ maximises the Hamiltonian.

Let $H_t(x, z) := \sup_{a \in A} \{\lambda_t(x, a) \cdot z - c_t(x, a)\}$, $a^*(t, x, z) \in \mathcal{M}$ denotes a maximiser in $H$, and $h_t^*(s, x, z, z) := \lambda_t(x, a^*(t, x, z)) \cdot z - f(t - s) c_t(x, a^*(t, x, z))$
– $V_A$ satisfies an extended dynamic programming principle: Agent’s value alongside equilibrium is time-consistent.

– $\nu^*$ is an equilibrium iff there is $(Y, Z)$ solution to the type-I BSVIE

$$
Y_t^s = U_A(s, \xi) + \int_t^T h_r^*(s, X_r, Z_r^s, Z_r^r) dr - \int_t^T Z_r^s dX_r, \ (s, t) \in [0, T]^2,
$$

for which $\nu^*$ maximises the Hamiltonian.
– \( V^A \) satisfies an extended dynamic programming principle: Agent’s value alongside equilibrium is time-consistent.

– \( \nu^* \) is an equilibrium iff there is \((Y, Z)\) solution to the type-I BSVIE

\[
Y_t^s = U_A(s, \xi) + \int_t^T h^*_r(s, X_r, Z_r^s, Z_r^r)dr - \int_t^T Z_r^s dX_r, \ (s, t) \in [0, T]^2,
\]

for which \( \nu^* \) maximises the Hamiltonian.

– All in all,

\[
Y_t^t = V_t^A, \ \mathcal{E} = \{(a^*(t, X_{\land t}, Z_t^t)_{t\in[0,T]}, a^* \in \mathcal{M})\}.
\]

Existence and uniqueness of equilibria.
Contract theory: a review

Inconsistent Control
Consistent planning

Time-inconsistent contract theory
Recap

- By refining the definition of consistent plans, i.e. \( \nu^* \in \mathcal{E}(\xi) \), we obtained an extended dynamic programming principle for the problem of the Agent.

- **Infinite** system (type-I extended BSVIE). It is sufficient for \( \nu^* \in \mathcal{E}(\xi) \).

- Interestingly, it is also necessary, i.e. any equilibria arises from type-I BSVIE and \( \nu^* \) maximises the Hamiltonian.

- The well-posedness of type-I BSVIEs yields the uniqueness of equilibria (up to max. of Hamiltonian).
Recap

– By refining the definition of consistent plans, i.e. \( \nu^* \in \mathcal{E}(\xi) \), we obtained an extended dynamic programming principle for the problem of the Agent.

– Infinite system (type-I extended BSVIE). It is sufficient for \( \nu^* \in \mathcal{E}(\xi) \).

– Interestingly, it is also necessary, i.e. any equilibria arises from type-I BSVIE and \( \nu^* \) maximises the Hamiltonian.

– The well-posedness of type-I BSVIEs yields the uniqueness of equilibria (up to max. of Hamiltonian).

What does this imply about the problem faced by the Principal?

– Infinitely many representations for \( \xi \).

– ... cannot use only one and optimise over \( Z \) as before, need to understand relationships between \( Z \) and \( Z^s \).
A solvable LQ example: Principal’s Problem:

\[ c_t(x, a) = a^2, \quad \lambda_t(x, a) = a. \]
\[ U_A(x) = x. \]
\[ J^A(t, t, \nu; \xi) = \mathbb{E}^\nu \left[ f(T - t)\xi - \int_t^T f(r-t)\nu_r^2dr \right] \mathcal{F}^X_t \]
A solvable LQ example: Principal’s Problem:

\[
Y^s_t = f(T - s) \xi + \int_t^T Z^r_r Z^s_r - f(r - s) \frac{Z^r_r}{2} dr - \int_t^T Z^s_r dX_r
\]

\[\Xi := \left\{ Y^{y_0, Z}_T = \frac{y_0}{f(T)} + \int_0^T \frac{f(r) Z^r_r}{f(T)} dr + \int_0^T \frac{\sigma Z^0_r}{f(T)} dW^*_r \right\}, \quad (3') \]
A solvable LQ example: Principal’s Problem:

\[ Y_t^s = f(T - s)\xi + \int_t^T Z_r^r Z_r^s - f(r - s) \frac{Z_r^r}{2} dr - \int_t^T Z_r^s dX_r \]

\[ \Xi := \left\{ Y_T^{y_0, Z} = \frac{y_0}{f(T)} + \int_0^T \frac{f(r)}{f(T)} \frac{Z_r^r}{2} dr + \int_0^T \frac{\sigma Z_r^0}{f(T)} dW_r^\nu^*, (3') \right\} \]

We check that for \( s \in [0, T] \),

\[ Z_t^s = \frac{f(T - s)}{f(T)} Z_t^0 - \tilde{Z}_t^s \]

where \( \tilde{Z}_t^s \) comes from the martingale representation of

\[ M_t^s := \mathbb{E}^{P^\ast} \left[ \int_0^T \left( f(r - s) - \frac{f(T - s)f(r)}{f(T)} \right) \frac{Z_r^r}{2} dr \bigg| \mathcal{F}_t \right] = M_0 + \int_0^t \sigma \tilde{Z}_r^s dW_r^\nu^* \]

- When \( f(t) := e^{-\alpha t} \), \( \tilde{Z} \) vanishes… effect due to time-inconsistency.
- \( \tilde{Z} \) also vanishes whenever \( Z \) is deterministic!
\[ V^P = \sup_{\xi \in \Xi} \mathbb{E}^{P^*} \left[ X_T - \xi \right] = x - \frac{y_0}{f(T)} + \sup_{\xi \in \Xi} \mathbb{E}^{P^*} \left[ \int_0^T \left( Z_r - \frac{f(r)}{f(T)} \frac{(Z_r)^2}{2} \right) dr \right] \]

\[ \leq x - \frac{y_0}{f(T)} + \frac{1}{2} \int_0^T \frac{f(T)}{f(r)} dr \]
$$V^P = \sup_{\xi \in \Xi} \mathbb{E}^{P^*} \left[ X_T - \xi \right] = x - \frac{y_0}{f(T)} + \sup_{\xi \in \Xi} \mathbb{E}^{P^*} \left[ \int_0^T \left( Z_r^r - \frac{f(r)}{f(T)} \frac{(Z_r^r)^2}{2} \right) dr \right]$$

$$\leq x - \frac{y_0}{f(T)} + \frac{1}{2} \int_0^T \frac{f(T)}{f(r)} dr$$

We can identify $f(T)/f(r)$ deterministic s.t.

$$Z_t^T := \frac{f(T)}{f(t)}, \ Z_s^T := \frac{f(T-s)}{f(T-t)} Z_t^T, \ \tilde{Z}_t^s = 0 \implies (3') \text{ holds.}$$

Consequently,

$$\xi^* = C + \int_0^T \frac{f(T)}{f(t)f(T-t)} \cdot dX_t.$$
- **Agent discounts exponential utility**: P has to solve a standard optimal control problem. Explicit solution when Principal also has exponential utility: *linear contract*.

- **Agent takes utility of discounted wealth**: we find optimal contract in slightly restricted class (separability in $t$ and $s$ for the controls). Though optimal control stays deterministic, contract is *non-Markovian and non-linear*, and still recovers the standard result for time-consistent agent.
– **Agent discounts exponential utility**: P has to solve a standard optimal control problem. Explicit solution when Principal also has exponential utility: linear contract.

– **Agent takes utility of discounted wealth**: we find optimal contract in slightly restricted class (separability in $t$ and $s$ for the controls). Though optimal control stays deterministic, contract is non-Markovian and non-linear, and still recovers the standard result for time-consistent agent.

– What about the general case?
In great generality, the problem of Agent is given by, \((s, t) \in [0, T]^2\)

\[ Y_t^s = U_A(s, \xi) + \int_t^T h_r^*(s, X_r^\land, Y_r^s, Z_r^s, Y_r^r, Z_r^r)dr - \int_t^T Z_r^s dX_r. \]
Type–I extended BSVIE

In great generality, the problem of Agent is given by, \((s, t) \in [0, T]^2\)

\[
Y_t^s = U_A(s, \xi) + \int_t^T h_r^*(s, X_r^\wedge, Y_r^s, Z_r^s, Y_r^r, Z_r^r) \, dr - \int_t^T Z_r^s \, dX_r.
\]

\(\mathcal{H}^{2,2} : Z \in \overline{H}^{2,2}\) satisfying

\[
Y_t^{s,y_0,Z} = y_0^s - \int_0^t h_r^*(s, X_r^\wedge, Y_r^{s,y_0,Z}, Z_r^s, Y_r^r, y_0^r, Z_r^r) \, dr + \int_0^t Z_r^s \, dX_r, \ (s, t) \in [0, T]^2
\]

\[U_A^{-1}(0, Y_T^{0,y_0,Z}) = U_A^{-1}(s, Y_T^{s,y_0,Z}), \ s \in [0, T]. \quad (3)\]
Theorem (H, Possamaï, '21)

\[ \Xi := \{ \xi = U_A^{(-1)}(0, Y_T^0, Z), \ Z \in \mathcal{H}^{2,2} \} \]. Then \( \Xi = \Xi \). For \( \xi \in \Xi \)

\[ \mathcal{E}(\xi) = \{ a^*(t, X_\cdot \wedge t, Y_t^t, Z_t^t)_{t \in [0, T]} \}, \ V_0^A(\xi) = Y_0^0. \]

\( \mathcal{H}^{2,2} : \ Z \in \overline{\mathcal{H}}^{2,2} \) satisfying

\[ Y_t^{s,y_0,Z} = y_0^{s} - \int_0^t h_r^*(s, X_\cdot \wedge r, Y_r^{s,y_0,Z}, Z_r^{s,y_0,Z}) \, dr + \int_0^t Z_r^s \, dX_r, \ (s, t) \in [0, T]^2 \]

\[ U_A^{(-1)}(0, Y_T^0, y_0, Z) = U_A^{(-1)}(s, Y_T^{s,y_0,Z}), \ s \in [0, T]. \] (3)
Theorem (H, Possamaï, ’21)

\[ V^P = \sup_{Y_0^0 \geq R} \sup_{Z \in \mathcal{H}^{2,2}} \mathbb{E}^{P^*}(Z) \left[ U_P \left( X_T - U_A^{(-1)}(T, Y_T^{T,Z}) \right) \right], \]

\( \mathcal{H}^{2,2} : Z \in \overline{\mathcal{H}}^{2,2} \) satisfying

\[ Y_t^{s,y_0,Z} = y_0^s - \int_0^t h_r^*(s, X, \wedge, Y_T^s,Y_0^s,Z, Z^r_r, Y_r^{r,y_0,Z}, Z_r^{r,Z}) \, dr + \int_0^t Z_r^s \, dX_r, \ (s,t) \in [0,T]^2 \]

\[ U_A^{(-1)}(0, Y_T^{0,y_0,Z}) = U_A^{(-1)}(s, Y_T^{s,y_0,Z}), \ s \in [0,T]. \]  

– Volterra constrained controls… Different from Viens and Zhang ’17.
To sum up

– Insight: $V^A$ solves a backward stochastic Volterra integral equation.

– Characterisation of equilibria holds in greater generality.

– Principal’s problem boils to optimal control of a Volterra forward equation with constrained Volterra controls, i.e. a family $(Z^s_t)_{(s,t)\in[0,T]^2}$, (3) holds

\[ Y_t^s = U_A(s, \xi) + \int_t^T h_r^*(s, X_r^\wedge, Y_r^s, Z_r^s, Y_r^r, Z_r^r)dr - \int_t^T Z_r^s dX_r. \]

– Unclear how to generalise the previous examples approach.

– Idea: forget about the constraint... and reincorporate it using stochastic target control ideas.


