Portfolio Liquidation Games with Self-Exciting Order Flow

Ulrich Horst

– Joint work with Guanxing Fu and Xiaonyu Xia –

Humboldt-Universität zu Berlin

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Overview

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5. Conclusion

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Portfolio liquidation

- Models of optimal block trading have long been studied in the economics literature (e.g. Kyle '85, Easley and O'Hara '87...)
  - focus is on deriving endogenous impact functions from information asymmetries
- Renewed attention in the financial mathematics literature (e.g. Bertsimas & Lo '98, Almgren & Chriss '01...)
  - focus is on structural models within which to derive optimal portfolio strategies for endogenously given impact functions
  - models give rise to novel stochastic control problems:
    - ('Liquidation') constraint on the terminal state
    - singular terminal condition on the associated HJB equation
    - unknown terminal condition on the associated adjoint equation

The single player benchmark model: Graewe & H. '17

The large investor's stochastic control problem is given by

$$\operatorname*{ess\,inf}_{\xi \in L^2_{\mathcal{F}}(0,T;\mathbb{R})} \mathbb{E}\left[\int_0^T \{\eta \xi_s^2 + \xi_s Y_s + \lambda_s X_s^2\}\, ds\right]$$

subject to the state dynamics

$$\begin{cases} X_t = \mathcal{X} - \int_0^t \xi_s \, ds, \quad t \in [0,T], \\ X_T = 0, \\ Y_t = \int_0^t \{-\rho_s Y_s + \gamma \xi_s\}\, ds, \quad t \in [0,T]. \end{cases}$$

Overview

Hawkes process

- Market order flow follows Hawkes processes whose base intensities depend on the large investor's trading activities
  - Hawkes processes and stock price volatility: Bacry et al. '13, '15; El Euch et al. '18; Jaisson & Rosenbaum '15; H. & Xu '19
  - exogenous Hawkes processes and liquidation: Alfonsi & Blanc'16; Amaral & Papanicolaou '19; Cartea et al. '18
  - in our model: Hawkes processes are endogenously controlled

Hawkes market model

- market order dynamics follow Hawkes processes with rates:

$$\zeta_t^{\pm} := \mu_t + \xi_t^{\pm} + \alpha \int_0^t e^{-\beta(t-s)} dN_s^{\pm}$$

- expected number of (net) sell orders

$$\bar{Z}_t = \mathbb{E}[\bar{Z}_t^+ - \bar{Z}_t^-] = \int_0^t \mathbb{E}[\xi_s] ds + \underbrace{\alpha \int_0^t e^{-\beta(t-s)} \bar{Z}_s ds}_{=:C_t}.$$

- expected number C_t of (net) sell child orders satisfies

$$dC_t = (-(\beta - \alpha) C_t + \alpha(\mathbb{E}[\mathcal{X}] - \mathbb{E}[X_t]))\, dt, \quad C_0 = 0.$$

The mean-field type control problem

The mean-field type control problem for our large investor:

$$\operatorname*{ess\,inf}_{\xi \in L^2_{\mathcal{F}}(0,T;\mathbb{R})} \mathbb{E}\left[\int_0^T \left\{\eta_s \xi_s^2 + \xi_s Y_s + \lambda_s X_s^2\right\} ds\right]$$

subject to the following state dynamics on $[0,T]$:

$$\begin{cases} dX_t = -\xi_t\, dt, \\ dY_t = (-\rho_t Y_t + \gamma_t(\xi_t - (\beta-\alpha)C_t + \alpha(\mathbb{E}[\mathcal{X}] - \mathbb{E}[X_t]))\, dt, \\ dC_t = (-(\beta-\alpha)C_t + \alpha(\mathbb{E}[\mathcal{X}] - \mathbb{E}[X_t]))\, dt, \\ X_0 = \mathcal{X},\ X_T = 0,\ Y_0 = 0,\ C_0 = 0. \end{cases}$$

This is a non-convex optimization problem (convex for the MFG).

Overview

Many player models

The optimization problem for player i given the strategies ξ^{-i} reads:

$$\operatorname*{ess\,inf}_{\xi^i \in L^2_{\mathcal{F}}(0,T;\mathbb{R})} \mathbb{E}\left[\int_0^T \eta_s^i(\xi_s^i)^2 + \xi_s^i Y_s^i + \lambda_s^i (X_s^i)^2 \, ds\right]$$

subject to the state dynamics ($\bar{\xi}_t$ etc. denotes average quantities)

$$\begin{cases} dX_t^i = -\xi_t^i \, ds, \\ dY_t^i = \Big(-\rho_t^i Y_t^i + \gamma_t^i(\bar{\xi}_t - (\beta_t^i - \alpha_t^i)C_t^i + \alpha_t^i(\mathbb{E}[\bar{\mathcal{X}}] - \mathbb{E}[\bar{X}_t])\Big) \, dt, \\ dC_t^i = \Big(-(\beta_t^i - \alpha_t^i)C_t^i + \alpha_t^i(\mathbb{E}[\bar{\mathcal{X}}] - \mathbb{E}[\bar{X}_t])\Big) \, dt, \\ X_0^i = \mathcal{X}^i, \; X_T^i = 0, \; Y_0^i = 0, \; C_0^i = 0. \end{cases}$$

The mean field game

The corresponding MFG is then given by

$$\operatorname*{ess\,inf}_{\xi \in L^2_{\mathcal{F}}(0,T;\mathbb{R})} \mathbb{E}\left[\int_0^T \left\{\eta_t(\xi_t)^2 + \xi_t Y_t + \lambda_t (X_t)^2\right\} dt\right]$$

subject to the state dynamics

$$\begin{cases} dX_t = -\xi_t \, ds, \\ dY_t = (-\rho_t Y_t + \gamma_t(\mu_t - (\beta_t - \alpha_t)C_t + \alpha_t(\mathbb{E}[\mathcal{X}] - \nu_t)) \, dt \\ dC_t = (-(\beta_t - \alpha_t)C_t + \alpha_t(\mathbb{E}[\mathcal{X}] - \nu_t)) \, dt, \\ X_0 = \mathcal{X}, \; X_T = 0, \; Y_0 = 0, \; C_0 = 0. \end{cases}$$

and the equilibrium condition

$$\mathbb{E}[\xi^*_t(\mu,\nu)] = \mu_t, \quad \mathbb{E}[X^*_t(\mu,\nu)] = \nu_t.$$

The mean field FBSDE for open-loop equilibria

The following MF FBSDE system covers N-player and MF game.

$$
\begin{cases}
dX_t^i = -\dfrac{M_t^i - \frac{1}{N}\left\langle \widehat{B}_t^{i,(1)}, \mathcal{P}_t^i \right\rangle}{2\eta_t^i}\, dt, \\
d\mathcal{S}_t^i = \left(-A_t^i \mathcal{S}_t^i + K_t^i \chi_t + \mathcal{R}_t^i\right) dt, \\
-dM_t^i = \left(2\lambda_t^i X_t^i + \dfrac{1}{N}\mathbb{E}\left[\left\langle \widehat{B}_t^{i,(2)}, \mathcal{P}_t^i \right\rangle\right] + \left\langle \Theta, -A_t^i \mathcal{S}_t^i + K_t^i \chi_t + \mathcal{R}_t^i \right\rangle\right) dt \\
\qquad\qquad - Z_t^{M^i}\, dW_t, \\
-d\mathcal{P}_t^i = \left(-(A_t^i)^\top \mathcal{P}_t^i + \Theta \dfrac{M_t^i - \frac{1}{N}\left\langle \widehat{B}_t^{i,(1)}, \mathcal{P}_t^i \right\rangle}{2\eta_t^i}\right) dt - Z_t^{\mathcal{P}^i}\, dW_t, \\
X_0^i = \mathcal{X}^i,\ X_T^i = 0,\ \mathcal{S}_0^i = (0,0)^\top,\ \mathcal{P}_T^i = (0,0)^\top, M_T^i = ?
\end{cases}
$$

with

$$
\xi^j := \frac{M^j - \frac{1}{N}\left\langle \widehat{B}^{j,(1)}, \mathcal{P}^j \right\rangle}{2\eta^j}, \quad \chi := (\bar{\xi}, \mathbb{E}[\overline{X}], \mathbb{E}[\bar{\xi}])^\top.
$$

(No) Symmetry

Remark

- *no symmetry is assumed for the N-player game*
- *symmetry is required for the convergence to the MFG solution*

Existence

Theorem

Under a weak interaction condition, there exists a unique solution

$$(X^i, \mathcal{S}^i, M^i, \mathcal{P}^i, Z^{M^i}, Z^{\mathcal{P}^i}) \in \mathcal{H}_{a,\mathcal{F}} \times \mathbb{S}^2_{\mathcal{F}} \times L^2_{\mathcal{F}} \times \mathcal{H}_{\iota,\mathcal{F}} \times L^{2,-}_{\mathcal{F}} \times L^2_{\mathcal{F}}$$

to the above FBSDE system for positive constants $a < 1,\ \iota < 1/2$.

- $\mathcal{H}_{a,\mathscr{F}}$: the subspace of $\mathbb{S}^2_{\mathscr{F}}$ s.t. $\|y\|_a := (\mathbb{E}[\sup_{0\le t\le T}(\frac{|y_t|}{(T-t)^a})^2])^{\frac{1}{2}} < \infty$
- $L^{2,-}_{\mathscr{F}}$: the space of all progressive processes s.t. for each $\epsilon > 0$,
  $$\mathbb{E}\left[\int_0^{T-\epsilon} |y_t|^2\, dt\right] < \infty$$
- $\mathbb{S}^{2,-}_{\mathscr{F}}$: the space of all progressive continuous processes s.t.
  $$\|y\|_{\mathbb{S}^{2,-}} := (\sup_{\epsilon\ge 0}\mathbb{E}[\sup_{0\le t\le T-\epsilon} |y_t|^2])^{\frac{1}{2}} < \infty$$

The proof

The proof uses a continuation method (cf. FGHP '20). First, decouple the FBSDE system and make the ansatz

$$\widetilde{M}^i = \mathscr{A}^i \widetilde{X}^i + \mathscr{B}^i.$$

where $\mathscr{A}^i$ satisfies the singular BSDE

$$\begin{cases} d\mathscr{A}^i_t = \left(2\lambda^i_t - \dfrac{(\mathscr{A}^i_t)^2}{2\eta^i_t}\right) dt - Z^{\mathscr{A}^i}_t \, dW^i_t, \\ \lim\limits_{t \nearrow T} \mathscr{A}^i_t = +\infty \end{cases}$$

and $\mathscr{B}^i$ satisfies the linear BSDE

$$\begin{aligned} -d\mathscr{B}^i_t = \Bigg(& -\frac{\mathscr{A}^i_t \mathscr{B}^i_t}{2\eta^i_t} + \frac{\mathscr{A}^i_t}{2N\eta^i_t} \left\langle \widehat{B}^{i,(1)}_t, \widetilde{\mathcal{P}}^i_t \right\rangle + \frac{1}{N} \mathbb{E}\left[\left\langle \widehat{B}^{i,(2)}_t, \widetilde{\mathcal{P}}^i_t \right\rangle \right] \\ & + \left\langle \Theta, -A^i_t \widetilde{\mathcal{S}}^i_t + K^i_t \widetilde{\chi}_t + \mathcal{R}^i_t \right\rangle \Bigg) dt - Z^{\mathscr{B}^i}_t \, dW_t \end{aligned}$$

on $[0, T)$. One need to prove that $\mathscr{B}^i \in \mathbb{S}^{2,-}_{\mathcal{F}}$.

Continuation

The continuation step requires a mapping

$$\Phi : \left(X^i, M^i\right)_{i=1,\cdots,N} \xrightarrow[g^i(X):=\sigma X^i+g^i]{f^i(M):=\sigma\frac{M^i}{2\eta^i}+f^i} \left(\widetilde{X}^i, \widetilde{M}^i\right)_{i=1,\cdots,N}$$

to be a contraction in the right space. This holds under a weak interaction condition:

$$\underline{\lambda}^i - ... - \frac{1}{N} * - ... > 0, \quad \underline{\eta}^i - ... - \frac{1}{N} * - ... > 0.$$

Verification - without convexity

Theorem

Let

$$(X^i, \mathcal{S}^i, M^i, \mathcal{P}^i, Z^{M^i}, Z^{\mathcal{P}^i}) \in \mathcal{H}_{a,\mathcal{F}} \times S^2_{\mathcal{F}} \times L^2_{\mathcal{F}} \times \mathcal{H}_{\iota,\mathcal{F}} \times L^{2,-}_{\mathcal{F}} \times L^2_{\mathcal{F}}$$

be the unique solution to the N-player FBSDE system. Then, the processes

$$\xi^* = (\xi^{*,1}, \cdots, \xi^{*,N})$$

forms an open-loop Nash equilibrium, where

$$\xi^{*,i} = \frac{M^i - \frac{1}{N}\left\langle B^{i,(1)}, \mathcal{P}^i \right\rangle}{2\eta^i}.$$

Verification - without convexity

Theorem

Under our weak interaction condition, for any admissible strategy ξ^i the cost $J^i(\xi^i, \xi^{*,-i})$ can be decomposed into the equilibrium cost plus the cost of a round-trip strategy as

$$\begin{aligned} J^i(\xi^i, \xi^{*,-i}) =& J^i(\xi^{*,i}, \xi^{*,-i}) \\ &+ \mathbb{E}\left[\int_0^T \eta_t^i \left(\xi_t^i - \xi_t^{*,i}\right)^2 + \lambda_t^i \left(X_t^i - X_t^{*,i}\right)^2\right. \\ &\left. + \left(X_t^i - X_t^{*,i}\right) \left\langle \Theta, -A_t^i(\underline{\mathcal{S}}_t^i - \underline{\mathcal{S}}_t^{*,i}) + B_t^i(\underline{\chi}_t - \underline{\chi}_t^*)\right\rangle dt\right]. \end{aligned}$$

Moreover, the cost of the additional round-trip is non-negative.

From many player games to mean-field games

- Let the cost functions be homogeneous; for any coefficient φ^i

$$\varphi^i = \varphi\left(\mathcal{X}^i, W^i\right)$$

for independent Brownian motions $W^1, W^2, \cdots$.

- Using the Yamada-Watanabe result for mean-field FBSDE, there exists a measurable function Σ independent of i such that the solution to the mean-field FBSDE satisfies that

$$(\overline{X}^i, \overline{\mathcal{S}}^i, \overline{M}^i, \overline{\mathcal{P}}^i) = \Sigma(\mathcal{X}^i, W^i).$$

Theorem

The following convergence holds:

$$\mathbb{E}\left[\int_0^T |M_t^i - \overline{M}_t^i|^2\,dt\right] + \mathbb{E}\left[\sup_{0\le t\le T} |X_t^i - \overline{X}_t^i|^2\,dt\right] \xrightarrow{N\to\infty} 0.$$

As a result, the optimal strategy of player i in the N-player game converges to the one in MFG, i.e.,

$$\mathbb{E}\left[\int_0^T |\xi_t^{*,i,N} - \overline{\xi}_t^{*,i}|^2\,dt\right] \to 0.$$

Overview

The mean-field game

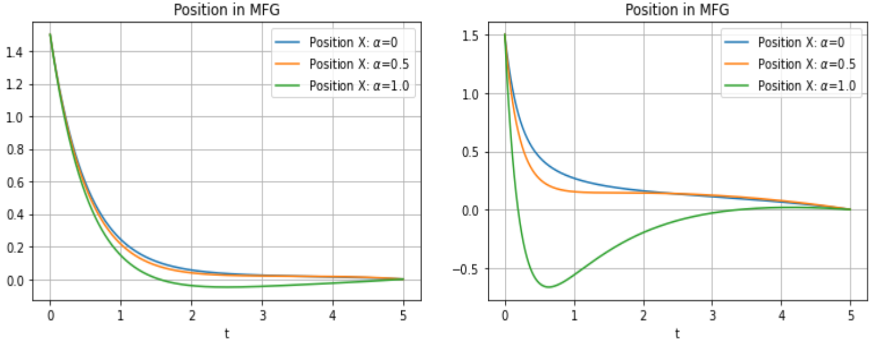

Figure: Dependence of equilibrium portfolio process on the market impact parameter α, for small (left) and larger (right) γ.

The single player mode with risk aversionl

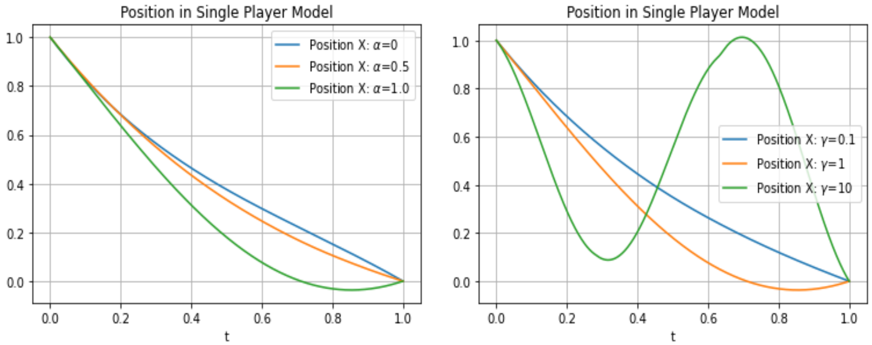

Figure: Dependence of optimal portfolio process on the market impact parameters α and γ, $\gamma = 1$(left) and $\alpha = 1$(right).

The two player model

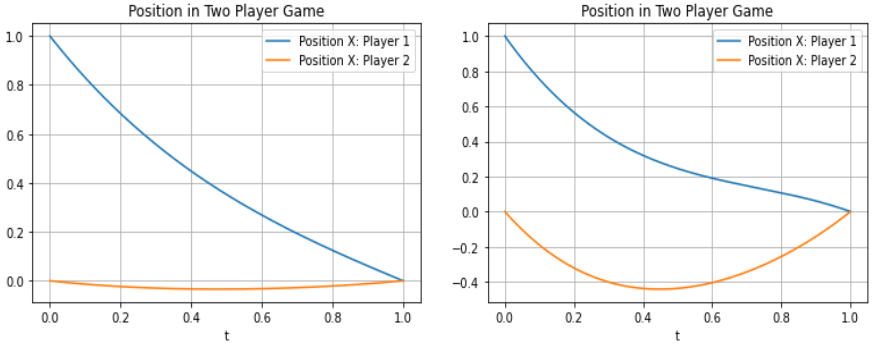

Figure: Equilibrium portfolio process in the two player game for small (left) and large (right) γ

Overview

Conclusion

- novel portfolio liquidation games with self-exciting order flow
- existence and uniqueness of solutions result for a novel mean-field FBSDE system with unknown terminal condition
- sufficient maximum principle and existence of open-loop equilibria
- quantitative analysis of equilibrium strategies

The end.

Thank you!