Well-posedness of path-dependent semilinear master equations

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Motivation for master equations

- Approximate the large system by McKean-Vlasov SDE:

$$dX(t) = b(X(t), \mathcal{L}_{(X(t),\alpha(t))}, \alpha(t))dt + \sigma(X(t), \mathcal{L}_{(X(t),\alpha(t))}, \alpha(t))dW(t)$$

- Mean field controls via dynamical programming principle $\Rightarrow$ HJB equation

$$\partial_t u(t, x, \mu) + \sup_{a \in A} H(u, a, \partial_x u, \partial_\mu u, \partial_{\tilde{x}} \partial_\mu u, \partial_x^2 u) = 0$$

- Mean field games via HJB+FP lead to

$$\partial_t u(t, x, \mu) + F(u, \partial_x u, \partial_x^2 u, \partial_\mu u, \partial_x \partial_\mu u) = 0$$

where $(x, \mu) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, and $\partial_\mu u$ is Lions' derivative.

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- Lions, Cardaliaguet '12: Lions' derivative and first-order master equation;
- Buckdahn-Li-Peng-Rainer '17, Buckdahn-Li-Peng '09, Buckdahn-Djehiche-Li-Peng '09, Carmona-Delarue '13: second order master equations;
- Bensoussan-Frehse-Yam '15, '17: measures with L^2 density functions;
- Cardadiaguet-Delarue-Lasry-Lions '19, Chassagneux-Crisan-Delarue '15: fully second order master equation;
- Mou-Zhang '20, Cosso-Gozzi-Kharroubi-Pham-Rosestolato '21, Cecchin-Delarue '22: weak solution and viscosity solution,

  ...

Path-dependent partial differential equation

- In practice, problems can be path-dependent/non-Markovian: e.g. pricing for singular options, delayed control problems, rough volatility...

- Motivated by state-dependent stochastic control, consider the following PPDE:

$$\begin{cases} \partial_t u(t,\omega) + H(u, \partial_\omega u, \partial^2_\omega u) = 0 \\ u(T,\omega) = \Phi(\omega_T), \quad \omega \in \mathcal{C}([0,T],\mathbb{R}^d), \end{cases}$$

  where $\partial_\omega u$ is vertical derivative.

- Functional Itô calculus: Dupire '09, Cont-Fournié '10, '13
- Smooth/Viscosity solutions: Wang-Peng '16, Ekren-Touzi-Zhang '14, '16, '18, Zhou '20, '21, Cosso-Gozzi-Rosestolato-Russo '21...

Mean-field PPDE

- For smooth parameters $(b_1, b_2, \sigma_1, \sigma_2)(t, \omega, \mu)$, we consider path-dependent master equation:

$$\begin{cases} \partial_t u(t,\omega,\mu) + \frac{1}{2}\mathrm{Tr}\left[\partial^2_\omega u(t,\omega,\mu)\sigma_1\sigma_1^T\right] + \frac{1}{2}\mathrm{Tr}\left[\sigma_2\sigma_2^T \int_{\mathcal{C}} \partial_{\omega'}\partial_\mu u(t,\omega,\mu,\omega')\mu(d\omega')\right] \\ \quad + b_2 \int_{\mathcal{C}} \partial_\mu u(t,\omega,\mu,\omega')\mu(d\omega') + b_1\partial_\omega u(t,\omega,\mu) \\ \quad + f(t,\omega,u(t,\omega,\mu),\partial_\omega u(t,\omega,\mu),\mu,\mathcal{L}_{u(t,W^\mu,\mu)}) = 0 \\ u(T,\omega,\mu) = \Phi(\omega_T,\mu_T). \end{cases}$$

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- Examples: say $b_1 = b_2 = 0, \sigma_1 = \sigma_2 = I$

(i) The state-dependent case: (f, Φ) has a state-dependent form:

$$f(t,\omega,y,z,\mu,\nu) = F(t,\omega(t),y,z,\mu(t),\nu),$$
$$\Phi(T,\omega,\mu) = G(\omega(T),\mu(T)).$$

(ii) PPDEs: $f(t,\omega,y,z,\mu,\nu) = H(t,\omega,y,z), \quad \Phi(T,\omega,\mu) = I(\omega_T)$.

(iii) The path-measure dependent case:

$$f(t,\omega,y,z,\mu,\nu) = J(t,y,\mu),$$
$$\Phi(T,\omega,\mu) = K(\mu_T).$$

(iv) Hybrid cases: "path-dependent" + "state-dependent"

Dupire's vertical derivative+ Lions' derivative

Definition (Dupire's horizontal derivative)

$f(t,\omega) : [0,T] \times \mathcal{D}_T^d \to \mathbb{R}$, is horizontally differentiable at (t,ω) if $\exists\, \partial_t f(t,\omega_t) \in \mathbb{R}$,

$$f(t+h,\omega_t) - f(t,\omega_t) = \partial_t f(t,\omega_t)\, h + o(h), \quad h \to 0.$$

Definition (Dupire's vertical derivative)

Suppose $f(t,\omega)$ non-anticipative. f is vertically differentiable at (t,ω), if $\exists\, \partial_\omega f(t,\omega) \in \mathbb{R}^d$,

$$f(t,\omega + x 1_{[t,T]}) - f(t,\omega) = \partial_\omega f(t,\omega) \cdot x + o(|x|), \quad x \to 0.$$

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Definition (Lions' derivative for path-measures)

$f(t,\mu) : [0,T] \times \mathcal{P}_2^D \to \mathbb{R}$ is Fréchet vertically differentiable at (t,μ) if its lift $\mathbf{f}(t,X)$ (i.e. $\mathbf{f}(t,X) := f(t,\mu)$ with $\mathcal{L}(X) = \mu$) is Fréchet differentiable at (t,X): $\exists$ measurable $\partial_\mu f(t,\mu,\tilde{\omega}) : [0,T] \times \mathcal{P}_2^D \times \mathcal{D}_T^d \to \mathbb{R}^d$

$$\mathbf{f}(t, X + \xi 1_{[t,T]}) = \mathbf{f}(t,X) + \hat{\mathbb{E}}^P[\partial_\mu f(t,\mu,X)\cdot \xi] + o(\|\xi\|_{L^2}),\ \forall \xi \in L_P^2(\mathcal{F}_t,\mathbb{R}^d).$$

Suppose

$$\begin{cases} dX(r) = a(r)dr + b(r)dB(r), \\ X_t = \gamma_t, \quad r \geq t. \end{cases} \quad \begin{cases} dX'(r) = c(r)dr + d(r)dB'(r), \\ X'_t = \eta_t, \quad r \geq t. \end{cases} \tag{1}$$

Theorem (Tang-Z. '21)

For any $(t, \gamma, \eta) \in [0, T] \times \mathcal{D}_T^d \times (\mathbb{M}_2^D)'$, and $f \in \mathscr{C}_p^{1,2,1,1}(\hat{\mathbb{D}}_{T,d})$. We have

$$\begin{aligned} & f(s, X, \mathcal{L}_{X'}) - f(t, \gamma, \mathcal{L}_\eta) \\ & = \int_t^s \partial_r f(r, X, \mathcal{L}_{X'})dr + \int_t^s \partial_\omega f(r, X, \mathcal{L}_{X'})dX(r) \\ & \quad + \frac{1}{2} \int_t^s \mathrm{Tr}\,[\partial_\omega^2 f(r, X_r, \mathcal{L}_{X'})d\langle X\rangle(r)] + \hat{\mathbb{E}}^{\tilde{P}'}[\int_t^s \partial_\mu f(r, X, \mathcal{L}_{X'}, \tilde{X}')d\tilde{X}'(r)] \\ & \quad + \frac{1}{2} \hat{\mathbb{E}}^{\tilde{P}'} \int_t^s \mathrm{Tr}\,[\partial_{\tilde{\omega}} \partial_\mu f(r, X, \mathcal{L}_{X'}, \tilde{X}')\tilde{d}(r)\tilde{d}(r)^T]dr, \quad \forall s \geq t. \end{aligned} \tag{2}$$

Solve mean-field PPDE via probability in a nutshell

- Consider the case $b_1 = b_2 = 0, \sigma_1 = \sigma_2 = 1$ firstly, i.e.

$$\begin{cases} \partial_t u(t,\omega,\mu) + \frac{1}{2}\mathrm{Tr}\left[\partial^2_\omega u(t,\omega,\mu)\right] + \frac{1}{2}\mathrm{Tr}\left[\int_{\mathcal{C}} \partial_{\omega'}\partial_\mu u(t,\omega,\mu,\omega')\mu(d\omega')\right] \\ \quad + f(t,\omega,u(t,\omega,\mu),\partial_\omega u(t,\omega,\mu),\mu,\mathcal{L}_{u(t,W^\mu,\mu)}) = 0 \\ u(T,\omega,\mu) = \Phi(\omega_T,\mu_T). \end{cases}$$

- $\omega^{\gamma_t}(\tau) = \gamma(\tau)1_{[0,t)}(\tau) + [\gamma(t) + \omega(r) - \omega(t)]1_{[t,T]}(r) \in \mathcal{D}^d_T; \quad \mathcal{L}_\eta = \mu$

- If u is the solution,

$$u(t+h,\gamma_t,\mu_t) - u(t,\gamma_t,\mu_t) = u(t+h,\gamma_t,\mu_t) \mp \hat{\mathbb{E}}[u(t+h,B^{\gamma_t},\mathcal{L}_{B^{\eta_t}})] - u(t,\gamma,\mu)$$

1. $u(t,\gamma,\mu) - u(t+h,B^{\gamma_t},\mathcal{L}_{B^{\eta_t}}) = Y^{\gamma_t,\mu_t}(t) - Y^{\gamma_t,\mu_t}(t+h) =$ " $\int_t^{t+h} f\,ds$ + martingale", need "flow property of MFBSDEs"

2. $u(t+h,\gamma_t,\mu_t) - u(t+h,B^{\gamma_t},\mathcal{L}_{B^{\eta_t}}) =$ "$\partial^2_\gamma u + \partial_{\gamma'}\partial_\mu u$ + martingale", need "partial Itô's formula"

Strong vertical derivative

Definition (strong vertical derivative)

We call $f(t,\omega)$ strongly vertically differentiable at (t,ω), if $\forall \tau \leq t$, $\exists\, \partial_{\omega_\tau} f(t,\omega)$ s.t. $\forall$ $x \in \mathbb{R}^d$,

$$f(t, \omega + x 1_{[\tau,T]}) = f(t,\omega) + \partial_{\omega_\tau} f(t,\omega) \cdot x + o(|x|), \quad \text{as } x \to 0.$$

$\partial_{\omega_\tau} f(t,\omega)$ is called strong vertical derivative of f at (τ, t, ω).

- For functionals on $\mathcal{P}_2^D$, definition of SVD is similar.

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- For functionals on $\mathcal{P}_2^D$, definition of SVD is similar.

Corollary (Partial Itô-Lions-Dupire formula, Tang-Z.'21)

Suppose $f \in \mathscr{C}^{0,2,1,1}_{s,p}(\hat{\mathbb{D}}_{T,d})$. Then we have that for any $t \le s \le v \le T$,

$$
\begin{aligned}
& f(v, X_s, \mathcal{L}_{X'_s}) - f(v, \gamma_t, \mathcal{L}_{\eta_t}) \\
&= \int_t^s \partial_{\omega_r} f(v, X_r, \mathcal{L}_{X'_r}) dX(r) + \frac{1}{2}\int_t^s \mathrm{Tr}\, [\partial^2_{\omega_r} f(v, X_r, \mathcal{L}_{X'_r}) d\langle X\rangle(r)] \\
&\quad + \hat{\mathbb{E}}^{\tilde{P}'}[\int_t^s \partial_{\mu_r} f(v, X_r, \mathcal{L}_{X'}, \tilde{X}') d\tilde{X}'(r)] + \frac{1}{2}\hat{\mathbb{E}}^{\tilde{P}'} \int_t^s \mathrm{Tr}\, [\partial_{\tilde{\omega}_r}\partial_{\mu_r} f(v, X_r, \mathcal{L}_{X'_r}, \tilde{X}'_r)\tilde{d}(r)\tilde{d}(r)^T] dr.
\end{aligned}
$$

Example

Suppose $f : [0, T] \times \mathcal{D} \to \mathbb{R}$, $F(t, x)$: $[0, T] \times \mathbb{R}^d$ smooth

(1). (State dependent case) $f(t, \omega) := F(t, \omega(t))$, $F(\cdot, \cdot) : [0, T] \times \mathbb{R}^d \to \mathbb{R}$
$\forall\, \tau_1, \tau_2, \cdots, \tau_j \in [0, t]$,

$$\partial_t f(t, \omega) = \partial_t F(t, \omega(t)), \quad \partial_{\omega_{\tau_j}} \cdots \partial_{\omega_{\tau_1}} f(t, \omega) = D_x^{\otimes j} F(t, \omega(t)).$$

(2). (Integral functionals) $f(t, \omega) := \int_0^t F(r, \omega(r))dr$. $\forall\, \tau_1, \tau_2, \cdots, \tau_j \in [0, t]$,

$$\partial_t f(t, \omega) = F(t, \omega(t)), \quad \partial_{\omega_{\tau_j}} \cdots \partial_{\omega_{\tau_1}} f(t, \omega) = \int_\tau^t D_x^{\otimes j} F(r, \omega(r))dr,$$

where $\tau = \max_{1 \le i \le j} \{\tau_i\}$.

(3). (Time delayed functionals) $\Phi(T, \omega) := F(\omega(t_0))$, $\quad t_0 \in (0, T)$,

$$\partial_{\omega_t} \Phi(T, \omega) = \partial_x F(\omega(t_0)) 1_{[0, t_0]}(t).$$

(4). (Piecewise constant functionals) Given a partition of $[0, T]$: $0 = t_0 < t_1 < \cdots t_n = T$

$$f(t, \omega) := \sum_{i=0}^{n-1} F_i(\omega(t_i)) 1_{[t_i, t_{i+1})}(t).$$

$$\partial_{\omega_\tau} f(t, \omega) = \sum_{i=0}^{n-1} DF_i(\omega(t_i)) 1_{[0, t_i]}(\tau) 1_{[t_i, t_{i+1})}(t).$$

The Feyman-Kac formula argument

- For any $(t, \gamma, \mu) \in \hat{\mathbb{D}}_{T,d}$, consider

$$Y^{\gamma_t,\eta_t}(s) = \Phi(B_T^{\gamma_t}, \mathcal{L}_{B_T^{\eta_t}}) + \int_s^T f(B_r^{\gamma_t}, Y^{\gamma_t,\eta_t}(r), Z^{\gamma_t,\eta_t}(r), \mathcal{L}_{B_r^{\eta_t}}, \mathcal{L}_{Y^{\eta_t}(r)})dr - \int_s^T Z^{\gamma_t,\eta_t}(r)dB(r), \qquad s \in [t,T], \tag{2}$$

  where $Y^{\eta_t} = Y^{\gamma_t,\eta_t}|_{\gamma=\eta}$ solves the mean-field BSDE

$$Y^{\eta_t}(s) = \Phi(B_T^{\eta_t}, \mathcal{L}_{B_T^{\eta_t}}) + \int_s^T f(B_r^{\eta_t}, Y^{\eta_t}(r), Z^{\eta_t}(r), \mathcal{L}_{B_r^{\eta_t}}, \mathcal{L}_{Y^{\eta_t}(r)})dr - \int_s^T Z^{\eta_t}(r)dB(r), \qquad s \in [t,T]. \tag{3}$$

- The decoupling field $u(t, \gamma, \mu) := Y^{\gamma_t,\mathcal{L}_{\eta_t}}(t) \in \mathbb{R}$.

Well-posedness of smooth path-dependent master equation

Proposition (Tang-Z. '21)

Suppose that $(f, \Phi) \in \mathscr{C}_{s,b}^{0,2,1,1}$. Then BSDE solution $(Y^{\gamma_t,\eta_t}, Z^{\gamma_t,\eta_t})$ is twice strongly vertically differentiable at (t, γ, μ). In particular, $u \in \mathscr{C}_{s,p}^{0,2,1,1}(\hat{\mathbb{D}}_{T,d})$.

Theorem (Tang-Z. '21)

There exists a unique $u : [0, T] \times \mathcal{C} \times \mathcal{P}_2^C \to \mathbb{R}$,

$$\begin{cases} \partial_t u(t,\omega,\mu) + \frac{1}{2}\mathrm{Tr}\,[\partial_\omega^2 u(t,\omega,\mu)\sigma_1\sigma_1^T] + \frac{1}{2}\mathrm{Tr}\,[\sigma_2\sigma_2^T \int_{\mathcal{C}} \partial_{\omega'}\partial_\mu u(t,\omega,\mu,\omega')\mu(d\omega')] \\ \quad + b_2 \int_{\mathcal{C}} \partial_\mu u(t,\omega,\mu,\omega')\mu(d\omega') + b_1\partial_\omega u(t,\omega,\mu) \\ \quad + f(t,\omega,u(t,\omega,\mu),\partial_\omega u(t,\omega,\mu),\mu,\mathcal{L}_{u(t,W^\mu,\mu)}) = 0 \\ u(T,\omega,\mu) = \Phi(\omega_T,\mu_T). \end{cases}$$

Back to PPDE

- Suppose F smooth on $\mathbb{R}^d$, consider $t_0 < T$,

$$\begin{cases} \partial_t u(t,\omega) + \frac{1}{2}\mathrm{Tr}\left[\partial_\omega^2 u(t,\omega)\right] = 0, \\ u(T,\omega) = F(\omega(t_0)), \quad (t,\omega) \in [0,T] \times \mathcal{C}. \end{cases} \tag{4}$$

  The above equation only has a viscosity solution no matter how smooth F is.

- Peng ICM '10: Which PPDEs correlate with non-Markovian BSDE ?

- Viscosity solution or more: loss of regularity? quadratic equations?

Vertical derivative for random variables

- Let $\mathbb{P}$ be a continuous martingale measure on canonical space $(\Omega_T, \mathcal{F}^B)$ with $\langle B \rangle_t \leq Kt,\ \mathbb{P} - a.s.$.
- $S_T^2 := \{f(\omega), f \in \mathscr{C}_p^2(\Omega_T), \text{i.e.}(\partial_\omega f(\omega_t), \partial_\omega f(\omega_t))$ s.t. cts.+ poly. growth$\}$.
  $\subseteq L^p(\Omega_T),\ \forall p \geq 1.$
- L_D^p the completion of S_T^2 under norm

$$\|f(\omega)\|_{L_D^2} := \sup_{[0,T]} [\hat{\mathbb{E}}_{\mathbb{P}} |f(\omega_t)|^2 dt]^{\frac{1}{2}}$$

- Consider operator

$$D := (\partial_\omega, \partial_\omega^2) : \begin{array}{rcl} S_T^2 & \longrightarrow & ([0,T] \times \Omega_T, \mathbb{R}^d \oplus \mathbb{R}^{d\times d}, [\hat{\mathbb{E}}_{\mathbb{P}} \int_0^T |\cdot|^2 dt]^{\frac{1}{2}}) \\ f(\omega) & \longmapsto & (\partial_\omega f(\omega_t), \partial_\omega^2 f(\omega_t)) \end{array}$$

Lemma 1. (Z. 22+) Differential operator D is closable in L_D^2.

General vertical derivative and its application

- Let H^1_D be the completion of S^2_T under norm
$$\|f(\omega)\|_{L^p_D} + \|(\partial_\omega f(\omega_\cdot), \partial^2_\omega f(\omega_\cdot))\|_{M^2([0,T]\times\Omega_T)}$$
- $D := (\partial_\omega, \partial^2_\omega) : H^1_D \longrightarrow ([0,T]\times\Omega_T, \mathbb{R}^d \oplus \mathbb{R}^{d\times d}, [\hat{\mathbb{E}}_{\mathbb{P}} \int_0^T |\cdot|^2 dt]^{\frac{1}{2}})$
- Itô-Dupire formula holds on H^1_D.

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- Itô-Dupire formula holds on H^1_D.

Proposition 2. (Z. '22+) For $f \in C^2_p(\mathbb{R}^{nd})$, and $0 \leq t_1 \leq \cdots \leq t_n \leq T$, then

$$\xi(\omega) := f(\omega(t_1), \cdots, \omega(t_n)) \in H^1_D.$$

In particular H^1_D is dense in $L^2(\Omega_T)$, and

$$\xi = f(0) + \int_0^T \partial_\omega f(\omega_t) dB(t) + \frac{1}{2}\int_0^T \mathrm{Tr}[\partial^2_\omega f(\omega_t)] d\langle B\rangle_t, \quad \mathbb{P}-a.s..$$

Proposition 3. (Z. '22+) For any smooth F with polynomial growth, and $t_1 \leq t_2 \leq ... \leq t_n \leq T$, there exists a unique solution $u \in C([0,T], H^1_D)$ to

$$\begin{cases} \partial_t u(t,\omega) + \frac{1}{2}\mathrm{Tr}\left[\partial^2_\omega u(t,\omega)\right] = 0, \\ u(T,\omega) = F(\omega(t_1), \omega(t_2), ..., \omega(t_n)). \end{cases} \tag{5}$$

Thanks for your attension !