A Feynman-Kac result via Markov BSDEs with generalized drivers

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9th International Colloquium on BSDEs and Mean Field Systems
Annecy, 27 June - 1 July 2022
This talk is based on a joint work with Francesco Russo:

- Issoglio E., Russo F.,
  *A Feynman-Kac result via Markov BSDEs with generalized drivers*,
  Bernoulli, Volume 26, Number 1 (2020), 728-766
Introduction and Main Result

The integral operator $A$

Solving the BSDEs & Feynman-Kac representation
The rough BSDE

We study a BSDE with generalised driver of the form

\[ Y_t = \Phi(W_T) + \int_t^T Z_r b(r, W_r) \, dr + \int_t^T f(r, W_r, Y_r, Z_r) \, dr - \int_t^T Z_r \, dW_r \]

where

- \( Y_t \in \mathbb{R}^d \)
- the coefficient \( b \) is rough, in particular \( b(t, \cdot) \) is a distribution
  - \( t \mapsto b(t) \) is a function in an \( \infty \)-dim space
  - \( b(t) \in S'(\mathbb{R}^d; \mathbb{R}^d) \)
- \( f \) is a nonlinearity which is Lipschitz continuous in \((y, z)\)
Applications of classical BSDEs

(1) hedging and pricing
(2) stochastic control problems
(3) probabilistic representation of solutions of PDEs

Rough coefficients

(1) underlying asset price has a rough dynamics (e.g. driven by SDEs with distributional coefficients) $\rightarrow$ rough BSDE
(2) Pontryagin maximum principle applied to stochastic control problems with coefficients with low regularity (continuous but not differentiable) $\rightarrow$ rough BSDE
(3) distributional coefficients are now popular in stochastic PDEs (Hairer, Gubinelli) $\rightarrow$ rough BSDE
Existing literature on BSDEs with distributional coefficients

- **Erraoui, Ouknine, Sbi (1998)** - distribution $\Phi$ as terminal condition
- **Russo, Wurzer (2017)** - fwd process $X$ is martingale solution of SDE with *distributional* generator
- **Diehl, Zhang (2017)** - BSDEs driven by Young drifts
- **Issoglio, Jing (2019)** - FBSDEs with distributional drivers
Our framework

\[
\int_t^T Z_r b(r, W_r)dr
\]

- The coefficient \( b(t) \) is a special distribution, not just any element in \( S' \).
- \( b(t) \) belongs to a fractional Sobolev space of negative order \( H_{q-\beta}(\mathbb{R}^d; \mathbb{R}^d) \)
  - where \( \beta \in (0, \frac{1}{2}) \) and \( q \in \left(\frac{d}{1-\beta}, \frac{d}{\beta}\right) \)
  - \( H_{q}^{s}(\mathbb{R}^{d}) := A^{-s/2}(L^q(\mathbb{R}^d)) \), where \( A := I - \frac{1}{2} \Delta \)
  - if \( s < 0 \) then \( H_{q}^{s}(\mathbb{R}^{d}) \subset S'(\mathbb{R}^d) \)
- We choose \( b \in C([0, T]; H_{q-\beta}^{-\beta}(\mathbb{R}^d; \mathbb{R}^d)) \)
Assumptions on other coefficients

- $\beta < \delta < 1 - \beta$; $\frac{d}{\delta} < p < q$
- $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is an element of $H^{1+\delta+2\gamma}(\mathbb{R}^d; \mathbb{R}^d)$ for some $0 < \gamma < \frac{1-\delta-\beta}{2}$
- $f : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{d \times d} \rightarrow \mathbb{R}^d$ is continuous in $(x, y, z)$ uniformly in $t$, and is Lipschitz continuous in $(y, z)$ uniformly in $t$ and $x$
- $f(t, x, 0, 0)$ is continuous in $(t, x)$;
  $\sup_{t, x} |f(t, x, 0, 0)| < \infty$ a.s.;
  $\sup_{t \in [0, T]} \|f(t, \cdot, 0, 0)\|_{L^p} < \infty$
Main Results on rough BSDE

- **Existence** \( Y_r = u(r, W_r) \) where \( u \) solves
  \[
  \begin{cases}
  \partial_t u + \frac{1}{2} \Delta u = -\nabla u^* b - f(u, \nabla u) \\
  u(T) = \Phi.
  \end{cases}
  \]

- **Uniqueness** in the class of \( Y_r = \gamma(r, W_r) \) for some \( \gamma \in C([0, T]; H^{1+\delta}_p) \).

- **Feynman-Kac (implicit) representation**
  \[
  u(s, x_0) = \mathbb{E}\left[ \Phi(x_0 + W_{T-s}) ight. \\
  + \int_s^T f(r, W_r + x_0, u(r, W_r + x_0), \nabla u(r, W_r + x_0))dr \\
  \left. + A^{W', W}_T((\nabla u^* b)(x_0 + \cdot)) - A^{W, W}_s((\nabla u^* b)(x_0 + \cdot)) \right].
  \]
Introduction and Main Result

The integral operator $A$

Solving the BSDEs & Feynman-Kac representation
The integral operator $A$ - smooth case

- How to define $\int_0^t Z_r b(r, W_r)dr$?
- $W$ is a $d$-dim Bm, $Y$ is a $d$-dim stoch process s.t. $[W, Y]$ exists
- $A^{W,Y}: C_c([0, T] \times \mathbb{R}^d; \mathbb{R}^d) \to C$ defined for $l \in C_c([0, T] \times \mathbb{R}^d; \mathbb{R}^d)$ as

$$A^{W,Y}_t(l) := \left( \int_0^t l^*(r, W_r)d[W, Y]_r \right)^*$$

Intuitive idea: $[W, Y]_t = \int_0^t Z_r^*dr$ so $A^{W,Y}_t(l) = \int_0^t Z_r l(r, W_r)dr$

Next step: extend operator to $A: E := C([0, T]; H_q^{-\beta}) \to C$
Definition of solution to rough BSDE

A continuous $\mathbb{R}^d$-valued stochastic process $Y$ is a solution of rough BSDE if

(i) $A^{W,Y}$ exists as an operator on $C([0, T]; H_q^{-\beta}) = E$

(ii) $A^{W,Y}(b)$ is a martingale-orthogonal process;

(iii) $Y_T = \Phi(W_T)$;

(iv) $M_t := Y_t - Y_0 + A_t^{W,Y}(b) + \int_0^t f(r, W_r, Y_r, \frac{d[Y,W]_r}{dr}) \, dr$ is a square-integrable $\mathcal{F}^W$-martingale

Remark: in the classical setting, this definition is equivalent to the standard definition with $(Y, Z)$ solution.
A useful representation for the integral operator $A$

Observe that

- In the special case when $Y = W$ then the operator $A$ is simply $A^W,W_t(l) = \int_0^t l(r, W_r)dr$
- In the Markovian case (i.e. if $Y_t = \gamma(t, W_t)$ for some $\gamma \in C^{0,1}$) $A^{Y,W}$ can be written in terms of $A^{W,W}$ and of the function $\gamma$. In particular for smooth $l$ we have

$$A^{Y,W}_t(l) = A^{W,W}_t(\nabla \gamma^* l)$$
Chain rule for $A$

**Proposition**

Let $l \in C_c([0, T] \times \mathbb{R}^d; \mathbb{R}^d)$. The chain rule for $A^{W,W}$ holds, namely

$$A_t^{W,W}(l) = \phi(t, W_t) - \phi(0, W_0) - \int_0^t \nabla \phi^*(r, W_r) dW_r$$

where $\phi$ is the solution of the heat equation

$$\left\{ \begin{array}{l}
\partial_t \phi + \frac{1}{2} \Delta \phi = l \\
\phi(T) = \Psi.
\end{array} \right.$$ 

**Corollary**

The chain rule holds also for $l \in C([0, T]; H_q^{-\beta})$. Indeed we can show that if $l_n \to l$ in $E$ then $A^{W,W}(l_n) \to A^{W,W}(l)$.
Extension of $A$ to rough $b$

**Proposition**

In the Markovian case when $Y_t = \gamma(t, W_t)$ we have that the operator $A^{W,Y}$ is well defined also in $E$ and for $b \in E$

$$A^{W,Y}(b) = A^{W,W}(\nabla \gamma^* b).$$

**Tools**

- **Pointwise product** $\nabla \gamma^* b$ of $\nabla \gamma^* \in C([0, T]; H^\delta_p)$ and $b \in C([0, T]; H^{-\beta}_q)$ is well defined and continuous.

- **Density** of $C_c([0, T] \times \mathbb{R}^d; \mathbb{R}^d)$ in $E = C([0, T]; H^{-\beta}_q)$
Introduction and Main Result

The integral operator $A$

Solving the BSDEs & Feynman-Kac representation
Existence and uniqueness of a solution

**Theorem**
Assume $b$ and $f$ as above.
There exists a solution to the BSDE given by $Y_r = u(r, W_r)$ where $u$ is the unique mild solution to PDE

$$
\begin{aligned}
\partial_t u + \frac{1}{2} \Delta u &= -\nabla u^* b - f(u, \nabla u) \\
u(T) &= \Phi.
\end{aligned}
$$

The solution $Y$ is unique in the class of $Y_r = \gamma(r, W_r)$ for some $\gamma \in C([0, T]; \mathcal{H}_p^{1+\delta})$. 

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Ideas for proof

**Existence**

Definition of solution $Y$ to rough BSDE:

(i) $A^{W,Y}$ exists as an operator on $C([0, T]; H^{-\beta}_q)$

(ii) $A^{W,Y}(b)$ is a martingale-orthogonal process;

(iii) $Y_T = \Phi(W_T)$;

(iv) $M_t := Y_t - Y_0 + A^{W,Y}_t(b) + \int_0^t f(r, W_r, Y_r, \frac{d[Y,W]_r}{dr}) \, dr$ is a square-integrable $\mathcal{F}^W$-martingale

**Tool**

$\int_0^t f(r, W_r, u(r, W_r), \nabla u(r, W_r)) \, dr = \int_0^t \tilde{f}(r, W_r) \, dr = A^{W,W}(\tilde{f})$

with $\tilde{f} \in C([0, T]; H^{-\beta}_p) \cap L^\infty_{loc}([0, T] \times \mathbb{R}^d; \mathbb{R}^d)$. 

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Use linearity of $A^W_W$ and chain rule

$$M_t = Y_t - Y_0 + A_t^{W,Y}(b) + A_t^{W,W}(\tilde{f})$$

$$= Y_t - Y_0 + A_t^{W,W} (\nabla^* b + \tilde{f})$$

$$= u(t, W_t) - u(0, W_0)$$

$$- u(t, W_t) + u(0, W_0) + \int_0^t \nabla^* (r, W_r) dW_r$$
Uniqueness

- Take two solutions \( Y_t^i = \gamma^i(t, W_t) \) for \( i = 1, 2 \)
- use chain rule on \( A_t^{W, Y}(b) \)
- use properties of PDE with \( \gamma^i \) on the RHS
- show that \( ||\gamma^1 - \gamma^2|| \leq 0 \)
Feynman-Kac implicit representation of $u$

**Theorem**
Assume $b$ and $f$ as above.
We have the Feynman-Kac (implicit) representation for the solution $u$ of PDE given by

$$
u(s, x_0) = \mathbb{E} \left[ \Phi(x_0 + W_{T-s}) \right. $$
$$+ \left. \int_s^T f(r, W_r + x_0, u(r, W_r + x_0), \nabla u(r, W_r + x_0))dr \right.$$
$$+ A^{W, W}_T ((\nabla u^* b)(x_0 + \cdot)) - A^{W, W}_s ((\nabla u^* b)(x_0 + \cdot)) \right]$$

for all $s \in [0, T]$ and $x_0 \in \mathbb{R}^d$. 

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References

Thank You for Your Attention.