A Feynman-Kac result via Markov BSDEs with generalized drivers

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This talk is based on a joint work with Francesco Russo:

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Introduction and Main Result

The integral operator A

Solving the BSDEs & Feynman-Kac representation

The rough BSDE

We study a BSDE with generalised driver of the form

$$Y_t = \Phi(W_T) + \int_t^T Z_r b(r, W_r) \mathrm{d}r + \int_t^T f(r, W_r, Y_r, Z_r) \mathrm{d}r - \int_t^T Z_r \mathrm{d}W_r$$

where

- $Y_t \in \mathbb{R}^d$
- the coefficient b is rough, in particular $b(t, \cdot)$ is a distribution
  - $t \mapsto b(t)$ is a function in an ∞-dim space
  - $b(t) \in \mathcal{S}'(\mathbb{R}^d; \mathbb{R}^d)$
- f is a nonlinearity which is Lipschitz continuous in (y, z)

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Applications of classical BSDEs

(1) hedging and pricing
(2) stochastic control problems
(3) probabilistic representation of solutions of PDEs

Rough coefficients

(1) underlying asset price has a rough dynamics (e.g. driven by SDEs with distributional coefficients) –> rough BSDE
(2) Pontryagin maximum principle applied to stochastic control problems with coefficients with low regularity (continuous but not differentiable) –> rough BSDE
(3) distributional coefficients are now popular in stochastic PDEs (Hairer, Gubinelli) –> rough BSDE

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Existing literature on BSDEs with distributional coefficients

- Erraoui, Ouknine, Sbi (1998) - distribution Φ as terminal condition
- Russo, Wurzer (2017) - fwd process X is martingale solution of SDE with *distributional* generator
- Diehl, Zhang (2017) - BSDEs driven by Young drifts
- Issoglio, Jing (2019) - FBSDEs with distributional drivers

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Our framework

$$\int_t^T Z_r b(r, W_r) \mathrm{d}r$$

- The coefficient $b(t)$ is a special distribution, not just any element in $\mathcal{S}'$.
- $b(t)$ belongs to a fractional Sobolev space of negative order $H_q^{-\beta}(\mathbb{R}^d; \mathbb{R}^d)$
  - where $\beta \in (0, \frac{1}{2})$ and $q \in (\frac{d}{1-\beta}, \frac{d}{\beta})$
  - $H_q^s(\mathbb{R}^d) := A^{-s/2}(L^q(\mathbb{R}^d))$, where $A := I - \frac{1}{2}\Delta$
  - if $s < 0$ then $H_q^s(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d)$
- We choose

$$b \in C([0, T]; H_q^{-\beta}(\mathbb{R}^d; \mathbb{R}^d))$$

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Assumptions on other coefficients

- $\beta < \delta < 1 - \beta$; $\quad \frac{d}{\delta} < p < q$
- $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ is an element of $H^{1+\delta+2\gamma}(\mathbb{R}^d; \mathbb{R}^d)$ for some $0 < \gamma < \frac{1-\delta-\beta}{2}$
- $f : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \to \mathbb{R}^d$ is continuous in (x, y, z) uniformly in t, and is Lipschitz continuous in (y, z) uniformly in t and x
- $f(t, x, 0, 0)$ is continuous in (t, x);
  $\sup_{t,x} |f(t, x, 0, 0)| < \infty$ a.s. ;
  $\sup_{t\in[0,T]} \|f(t, \cdot, 0, 0)\|_{L^p} < \infty$

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Main Results on rough BSDE

- Existence $Y_r = u(r, W_r)$ where u solves

$$\begin{cases} \partial_t u + \frac{1}{2}\Delta u = -\nabla u^* \, b - f(u, \nabla u) \\ u(T) = \Phi. \end{cases}$$

- Uniqueness in the class of $Y_r = \gamma(r, W_r)$ for some $\gamma \in C([0,T]; H_p^{1+\delta})$.
- Feynman-Kac (implicit) representation

$$\begin{aligned} u(s, x_0) = \mathbb{E}\Big[& \Phi(x_0 + W_{T-s}) \\ & + \int_s^T f(r, W_r + x_0, u(r, W_r + x_0), \nabla u(r, W_r + x_0)) \mathrm{d}r \\ & + A_T^{W,W}((\nabla u^* b)(x_0 + \cdot)) - A_s^{W,W}((\nabla u^* b)(x_0 + \cdot)) \Big] \end{aligned}$$

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Introduction and Main Result

The integral operator A

Solving the BSDEs & Feynman-Kac representation

The integral operator A - smooth case

- How to define $\int_0^t Z_r b(r, W_r)\mathrm{d}r$?
- W is a d-dim Bm, Y is a d-dim stoch process s.t. $[W, Y]$ exists
- $A^{W,Y} : C_c([0, T] \times \mathbb{R}^d; \mathbb{R}^d) \to \mathcal{C}$ defined for $l \in C_c([0, T] \times \mathbb{R}^d; \mathbb{R}^d)$ as

$$A_t^{W,Y}(l) := \left(\int_0^t l^*(r, W_r)\mathrm{d}[W, Y]_r\right)^*$$

Intuitive idea: $[W, Y]_t = \int_0^t Z_r^* \mathrm{d}r$ so $A_t^{W,Y}(l) = \int_0^t Z_r l(r, W_r)\mathrm{d}r$

Next step: extend operator to $A : E := C([0, T]; H_q^{-\beta}) \to \mathcal{C}$

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Definition of solution to rough BSDE

A continuous $\mathbb{R}^d$-valued stochastic process Y is a solution of rough BSDE if

(i) $A^{W,Y}$ exists as an operator on $C([0,T]; H_q^{-\beta}) = E$

(ii) $A_\cdot^{W,Y}(b)$ is a martingale-orthogonal process;

(iii) $Y_T = \Phi(W_T)$;

(iv) $M_t := Y_t - Y_0 + A_t^{W,Y}(b) + \int_0^t f\left(r, W_r, Y_r, \frac{\mathrm{d}[Y,W]_r}{\mathrm{d}r}\right)\mathrm{d}r$ is a square-integrable $\mathcal{F}^W$-martingale

Remark: in the classical setting, this definition is equivalent to the standard definition with (Y, Z) solution.

A useful representation for the integral operator A

Observe that

- In the special case when $Y = W$ then the operator A is simply $A_t^{W,W}(l) = \int_0^t l(r, W_r)\mathrm{d}r$
- In the Markovian case (i.e. if $Y_t = \gamma(t, W_t)$ for some $\gamma \in C^{0,1}$) $A^{Y,W}$ can be written in terms of $A^{W,W}$ and of the function γ.
  In particular for smooth l we have

$$A_t^{Y,W}(l) = A_t^{W,W}(\nabla\gamma^* l)$$

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Chain rule for A

Proposition

Let $l \in C_c([0,T] \times \mathbb{R}^d; \mathbb{R}^d)$. The chain rule for $A^{W,W}$ holds, namely

$$A_t^{W,W}(l) = \phi(t, W_t) - \phi(0, W_0) - \int_0^t \nabla\phi^*(r, W_r)\mathrm{d}W_r$$

where ϕ is the solution of the heat equation

$$\begin{cases} \partial_t \phi + \frac{1}{2}\Delta\phi = l \\ \phi(T) = \Psi. \end{cases}$$

Corollary

The chain rule holds also for $l \in C([0,T]; H_q^{-\beta})$.
Indeed we can show that if $l_n \to l$ in E then $A^{W,W}(l_n) \to A^{W,W}(l)$.

Extension of A to rough b

Proposition

In the Markovian case when $Y_t = \gamma(t, W_t)$ we have that the operator $A^{W,Y}$ is well defined also in E and for $b \in E$

$$A^{W,Y}(b) = A^{W,W}(\nabla \gamma^* b).$$

Tools

- Pointwise product $\nabla \gamma^* b$ of $\nabla \gamma^* \in C([0,T]; H_p^\delta)$ and $b \in C([0,T]; H_q^{-\beta})$ is well defined and continuous.
- Density of $C_c([0,T] \times \mathbb{R}^d; \mathbb{R}^d)$ in $E = C([0,T]; H_q^{-\beta})$

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Introduction and Main Result

The integral operator A

Solving the BSDEs & Feynman-Kac representation

Existence and uniqueness of a solution

Theorem

Assume b and f as above.
There exists a solution to the BSDE given by $Y_r = u(r, W_r)$ where u is the unique mild solution to PDE

$$\begin{cases} \partial_t u + \frac{1}{2}\Delta u = -\nabla u^* \, b - f(u, \nabla u) \\ u(T) = \Phi. \end{cases}$$

The solution Y is unique in the class of $Y_r = \gamma(r, W_r)$ for some $\gamma \in C([0, T]; H_p^{1+\delta})$.

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Ideas for proof

Existence

Definition of solution Y to rough BSDE:

(i) $A^{W,Y}$ exists as an operator on $C([0,T];H_q^{-\beta})$

(ii) $A_\cdot^{W,Y}(b)$ is a martingale-orthogonal process;

(iii) $Y_T = \Phi(W_T)$;

(iv) $M_t := Y_t - Y_0 + A_t^{W,Y}(b) + \int_0^t f\left(r, W_r, Y_r, \frac{\mathrm{d}[Y,W]_r}{\mathrm{d}r}\right)\mathrm{d}r$ is a square-integrable $\mathcal{F}^W$-martingale

Tool

$\int_0^t f(r, W_r, u(r,W_r), \nabla u(r,W_r))\mathrm{d}r = \int_0^t \tilde{f}(r, W_r)\mathrm{d}r = A^{W,W}(\tilde{f})$
with $\tilde{f} \in C([0,T];H_p^{-\beta}) \cap L_{loc}^\infty([0,T]\times\mathbb{R}^d;\mathbb{R}^d)$.

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Use linearity of $A^{W,W}$ and chain rule

$$\begin{aligned}
M_t =& Y_t - Y_0 + A_t^{W,Y}(b) + A^{W,W}(\tilde{f}) \\
=& Y_t - Y_0 + A_t^{W,W}(\nabla u^* b + \tilde{f}) \\
=& u(t, W_t) - u(0, W_0) \\
& - u(t, W_t) + u(0, W_0) + \int_0^t \nabla u^*(r, W_r) \mathrm{d} W_r
\end{aligned}$$

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Uniqueness

- Take two solutions $Y_t^i = \gamma^i(t, W_t)$ for $i = 1, 2$
- use chain rule on $A_t^{W,Y}(b)$
- use properties of PDE with γ^i on the RHS
- show that $\|\gamma^1 - \gamma^2\| \le 0$

Feynman-Kac implicit representation of u

Theorem

Assume b and f as above.
We have the Feynman-Kac (implicit) representation for the solution u of PDE given by

$$
\begin{aligned}
u(s, x_0) = \mathbb{E}\Big[& \Phi(x_0 + W_{T-s}) \\
& + \int_s^T f(r, W_r + x_0, u(r, W_r + x_0), \nabla u(r, W_r + x_0)) \mathrm{d}r \\
& + A_T^{W,W}((\nabla u^* b)(x_0 + \cdot)) - A_s^{W,W}((\nabla u^* b)(x_0 + \cdot)) \Big]
\end{aligned}
$$

for all $s \in [0, T]$ and $x_0 \in \mathbb{R}^d$.

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Thank You for Your Attention.

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