Quasilinear parabolic systems and FBSDEs with quadratic growth

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Thanks to Gordan Žitković for many helpful discussions!

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The FBSDE

We consider the FBSDE

$$\begin{cases} dX_t = b(t, X_t, Y_t, Z_t)dt + \sigma(t, X_t, Y_t)dB_t, \\ dY_t = -f(t, X_t, Y_t, Z_t)dt + Z_t dB_t, \\ Y_T = g(X_T), \quad X_0 = x_0. \end{cases}$$

Probabilistic setup: $(\Omega, \mathcal{F}, \mathbb{P})$, d-dimensional Brownian motion with filtration $\mathbb{F}$

Data: 4 deterministic functions

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- volatility $\sigma = \sigma(t, x, y) : [0, T] \times \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^{d \times d}$
- driver $f = f(t, x, y, z) : [0, T] \times \mathbb{R}^d \times \mathbb{R}^n \times (\mathbb{R}^d)^n \to \mathbb{R}^n$
- terminal condition $g = g(x) : \mathbb{R}^d \to \mathbb{R}^n$

Unknowns: adapted processes X, Y, Z taking values in $\mathbb{R}^d$, $\mathbb{R}^n$, and $(\mathbb{R}^d)^n$.

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Features of interest

We are interested in the situation where

1. $n > 1$, i.e. Y is multi-dimensional
2. f has quadratic growth: $|f(t, x, y, z)| \leq C(1 + |z|^2)$
3. the equation are *strongly coupled*, in the sense that $\sigma = \sigma(t, x, y)$ depends on y (but σ non-degenerate)

These three issues have received considerable attention... we highlight

- Kobylanski (2000) handles feature 2, but not 1 or 3
- Delarue (2002) handles features 1 and 3 simultaneously, but not 2
- Xing and Žitković (2018) handles 1 and 2 simultaneously, but not 3

Goal: Generalize the works above by establishing well-posedness when 1,2,3 are all present.

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The PDE

By the "4-step scheme" (Ma, Protter, and Yong 1994), solving the FBSDE boils down to solving

$$\begin{cases} \partial_t u^i + \mathrm{tr}(a(t,x,u)D^2 u^i) + f^i(t,x,u,Du) = 0, \quad (t,x) \in [0,T] \times \mathbb{R}^d, \\ u^i(T,x) = g^i(x) \quad x \in \mathbb{R}^d. \end{cases}$$

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What can go wrong?

Assume σ and g are nice. What conditions on the quadratic driver f guarantee a smooth solution?

Good news: can expect a smooth solution on $[T - \epsilon, T]$ (small-time well-posedness)

Bad news: global existence/uniqueness may fail, because

- u may blow up in finite time (blow-up)
- even if u stays bounded, Du may blow up (gradient blow-up)

More good news: if we manage to prove a gradient estimate (an a-priori estimate on $\|Du\|_{L^\infty}$), then

- can be bootstrapped to higher regularity, in particular estimates on $\|u\|_{C^{2,\alpha}}$
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The key question

Based on the previous slide, we know that understanding the PDE system boils down to understanding the following.

The key question: what conditions on f will guarantee an a-priori estimate of $\|Du\|_{L^\infty}$?

Break this up into three questions:

1. when can we get an estimate on $\|u\|_{L^\infty}$?
2. when does bound on $\|u\|_{L^\infty}$ imply bound on $\|u\|_{C^\alpha}$?
3. when does bound on $\|u\|_{C^\alpha}$ imply bound on $\|Du\|_{L^\infty}$?

We focus in this talk on the answers to 2 and 3.

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Some answers

Here are the two main structural conditions.

$$|f^i(t,x,u,p)| \leq C\big(1 + |p^i||p| + \sum_{j<i} |p^j|^2 + |p|^{2-\epsilon}\big) \qquad (H_{\mathrm{BF}})$$

$$\begin{cases} |f(t,x,u,p) - f(t,x',u',p)| \leq C(1+|p|^2)\big(|x-x'| + |u-u'|\big), \text{ and} \\ |f(t,x,u,p) - f(t,x,u,p')| \leq C(1+|p|+|p'|)|p-p'| \end{cases} \qquad (H_{\mathrm{Reg}})$$

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Under H_{BF}, an estimate on $\|u\|_{L^\infty}$ implies an estimate on $\|u\|_{C^\alpha}$.

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Back to the FBSDE

Theorem (J. 2022)

Assume that that all data is jointly continuous, and

1. $\sigma = \sigma(t, x, y)$ *is non-degnerate and Lipschitz in* (x, y)
2. $g = g(x)$ *bounded and Lipschitz*
3. $b = b(t, x, y, z)$ *Lipschitz in* (x, y, z), $|b(t, x, y, z)| \leq C(1 + |y| + |z|)$
4. f *satisfies* H_{BF} *and* H_{Reg}, *and* H_{AB} *(a technical condition to get* $\|u\|_{L^\infty} < \infty$*)*

Then there is a solution to the FBSDE

$$\begin{cases} dX_t = b(t, X_t, Y_t, Z_t)dt + \sigma(t, X_t, Y_t)dB_t, \\ dY_t = -f(t, X_t, Y_t, Z_t)dt + Z_t dB_t, \\ Y_T = g(X_T), \quad X_0 = x_0. \end{cases}$$

Comments on related literature

Our results can be viewed as...

- a generalization of the results of Delarue (2002 and 2003) to the quadratic case
  - Delarue (2002) gives existence for FBSDEs with Lipschitz data
  - Delarue (2003) gives probabilistic approach to Hölder and gradient estimates in the Lipschitz case
- a generalization of the results of Bensoussan and Frehse (2002), Xing and Žitković (2018), and Harter and Richou (2019) to the case $\sigma = \sigma(t, x, y)$ (versus $\sigma = \sigma(t, x)$)
  - Bensoussan and Frehse obtain Hölder and Sobolev estimates in a bounded domain via PDE arguments
  - Xing and Žitković (2018) obtain a Hölder estimate in the whole space via mix of PDE and probabilistic arguments
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The big picture

The Hölder estimate

- The basic idea is the same as the one used by Delarue in Lipschitz setting - combine Krylov Safonov estimates with BMO-martingale theory
- Execution is different - concept of *sliceability* is used to deal with quadratic growth

The gradient estimate

- Again, sliceability is key
- Hölder estimate implies a-priori sliceability of Z
- Probabilistic representation of Du via linear BSDE with sliceable coefficients (thanks to sliceability of Z!)
- Conclude using results from J. and Žitković 2021 (see also Delbaen and Tang 2008 and Harter and Richou 2019)

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Preliminaries

From here on, we suppose we have a nice solution u to the PDE system, and we define for $t_0, x_0 \in [0, T] \times \mathbb{R}^d$ the triple $(X^{t_0,x_0}, Y^{t_0,x_0}, Z^{t_0,x_0})$ by

$$X_t^{t_0,x_0} = x_0 + \int_{t_0}^{t} \sigma(s, X_s^{t_0,x_0}, u(s, X_s^{t_0,x_0}))dB_s, \quad t_0 \leq t \leq T$$
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If (t_0, x_0) is not important, we just write (X, Y, Z). Note that

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Notation:

$$\|Z\|_{\mathrm{bmo}}^2 = \sup_{\tau} \|\mathbb{E}[\int_{\tau}^{T} |Z|^2 dt | \mathcal{F}_{\tau}]\|_{L^\infty}, \quad \|\alpha\|_{\mathrm{bmo}^{1/2}} = \|\sqrt{|\alpha|}\|_{\mathrm{bmo}}^2$$

Z is **sliceable** if $\|Z1_{[t-\delta,t]}\|_{\mathrm{bmo}}$ is small for δ small.

Definition

A **c-Lyapunov pair** (h,k) is a smooth function $h = h(y) : \mathbb{R}^n \to \mathbb{R}$ and a constant k such that $h(0) = 0$, $Dh(0) = 0$, and for $|y| \leq c$,

$$\frac{1}{2} \sum_{i,j=1}^{n} (D^2 h(y))_{ij} z^i \cdot z^j - Dh(y) \cdot f(t,x,u,\sigma^{-1}(t,x,u)z) \geq |z|^2 - k.$$

The point is that (h,k) is a c-Lyapunov function, then

$$h(Y) + kt - \int |Z|^2 dt \text{ is a submartingale if } |Y| \leq c.$$

Preliminaries

Notation:

$$\|Z\|_{\text{bmo}}^2 = \sup_{\tau} \|\mathbb{E}[\int_{\tau}^{T} |Z|^2 dt | \mathcal{F}_{\tau}]\|_{L^\infty}, \quad \|\alpha\|_{\text{bmo}^{1/2}} = \|\sqrt{|\alpha|}\|_{\text{bmo}}^2$$

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The connection between Hölder regularity and sliceability

Lemma (Xing and Žitković)

Under H_{BF}, for any $c > 0$ there exists a c-Lyapunov pair (h, k).

Now suppose we have a c-Lyapunov pair (h, k) and $|Y| \leq c$. Then for $t - \delta \leq \tau \leq t \ldots$

$$\mathbb{E}_\tau[\int_\tau^t |Z_s|^2 ds] \leq kh + \mathbb{E}_\tau[h(u(t, X_t)) - h(u(\tau, X_\tau)]$$

$$\leq kh + C\|u\|_{C^\alpha}\mathbb{E}_\tau[\delta^{\alpha/2} + |X_t - X_\tau|^\alpha] \leq k\delta + C\delta^{\alpha/2} \leq C$$

$$\implies \|1_{[t-\delta,t]}Z\|_{\text{bmo}} \leq C\delta^{\alpha/4}$$

Thus under H_{BF},

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Sliceability implies Hölder: Suppose we have a bounded solution v to

$$\partial_t v + \mathrm{tr}(aD^2 u) + b \cdot Du + k = 0, \quad (t,x) \in (0,T) \times \mathbb{R}^d$$

Krylov-Safonov estimates show that

$$a = \frac{1}{2}\sigma\sigma^T \text{ bounded and elliptic } \& \; b, k \text{ bounded } \implies v \text{ is Hölder,}$$

at least away from $t = T$.

Ideas from Delarue (2003) show that the same argument works when

$$\sup_{(t,x)} \|b(\cdot, X_\cdot^{t,x})\|_{\mathrm{bmo}} < \infty$$

instead of b bounded.

We take this one step further by showing that if

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Because the arguments from previous slide don't use regularity of σ, they apply to quasi-linear setting. Under H_{BF}, we can write

$$\partial_t u^1 + \text{tr}(aD^2 u^i) + \tilde{b} \cdot Du^1 + \tilde{f} = 0,$$

where

$$|\tilde{b}| \leq C(1 + |Du|), \quad |\tilde{f}| \leq C(1 + |Du|^{2-\epsilon}).$$

Then since $Z^{(t,x)} = \sigma Du(\cdot, X^{(t,x)}_\cdot)$, we find

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Previous slide showed

$$Z \in \text{bmo} \implies u^1 \text{ is Hölder },$$

which in turn implies Z^1 is sliceable. It turns out similar reasoning lets us show

$$Z \in \text{bmo} \ \& \ Z^1, ..., Z^{i-1} \text{ sliceable} \implies u^i \text{ is Hölder} \ \& \ Z^i \text{ is sliceable.}$$

which lets us prove by induction that u is Hölder. The full chain of reasoning is

$$u \in L^\infty \implies Z \in \text{bmo} \implies u^1 \in C^\alpha \implies Z^1 \text{ sliceable} \implies u^2 \in C^\alpha$$
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Main estimates:

1. under H_{BF}, bound on $\|u\|_{L^\infty} \implies$ bound on $\|u\|_{C^\alpha}$
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Takeaways:

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