Viscosity Solutions to Path-Dependent Hamilton-Jacobi-Bellman Equations and Applications

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9th Colloquium on BSDEs and Mean field Systems

Annecy, June 27th, 2022

- Introduction of control problem
- Some relevant works
- Main results
- Sketch of the proof
- Applications to BSHJBEs

Introduction of control problem

We consider the following PHJB equation:

$$\begin{cases} \partial_t V(\gamma_t) + \mathbf{H}(\gamma_t, V(\gamma_t), \partial_x V(\gamma_t), \partial_{xx} V(\gamma_t)) = 0, (t, \gamma_t) \in [0, T) \times \Lambda, \\ V(\gamma_T) = \phi(\gamma_T), \quad \gamma_T \in \Lambda_T, \end{cases} \tag{1}$$

where, for every $(t, \gamma_t, r, p, l) \in [0, T] \times \Lambda \times \mathbb{R} \times \mathbb{R}^d \times \Gamma(\mathbb{R}^d)$,

$$\begin{aligned} \mathbf{H}(\gamma_t, r, p, l) &= \sup_{u \in U} [(p, b(\gamma_t, u))_{\mathbb{R}^d} + \frac{1}{2} \mathrm{tr}[l\sigma(\gamma_t, u)\sigma^\top(\gamma_t, u)] \\ &\quad + q(\gamma_t, r, \sigma^\top(\gamma_t, u)p, u)]. \end{aligned}$$

Here $\Lambda_t := C([0, t]; \mathbb{R}^d)$; $\Lambda := \bigcup_{t \in [0,T]} \Lambda_t$; $\sigma^\top$ the transpose of the matrix σ, $\Gamma(\mathbb{R}^d)$ the set of all $(d \times d)$ symmetric matrices and $(\cdot, \cdot)_{\mathbb{R}^d}$ the scalar product of $\mathbb{R}^d$. γ_t is an element of Λ_t; ∂_t is horizontal derivative, while $\partial_x, \partial_{xx}$ are first- and second-order vertical derivatives, respectively (see Dupire (2009)).

Introduction of control problem

The associated controlled equation:

$$\begin{cases} dX^{\gamma_t,u}(s) = b(X_s^{\gamma_t,u}, u(s))ds + \sigma(X_s^{\gamma_t,u}, u(s))dW(s), \\ \\ X_t^{\gamma_t,u} = \gamma_t \in \Lambda_t. \end{cases} \tag{2}$$

$\{W(t), t \geq 0\}$ is an n-dimensional standard Wiener process; the control process u takes values in some metric space (U, d). We wish to maximize a cost functional of the form:

$$J(\gamma_t, u) := Y^{\gamma_t,u}(t), \quad (t, \gamma_t) \in [0, T] \times \Lambda, \tag{3}$$

over $\mathcal{U}[t, T]$, where the process $Y^{\gamma_t,u}$ is defined by BSDE:

$$\begin{aligned} Y^{\gamma_t,u}(s) &= \phi(X_T^{\gamma_t,u}) + \int_s^T q(X_l^{\gamma_t,u}, Y^{\gamma_t,u}(l), Z^{\gamma_t,u}(l), u(l))dl \\ &\quad - \int_s^T Z^{\gamma_t,u}(l)dW(l), \quad a.s., \quad \text{all} \quad s \in [t, T]. \end{aligned} \tag{4}$$

Introduction of control problem

We define the value functional of the optimal control problem:

$$V(\gamma_t) := \text{esssup}_{u(\cdot)\in\mathcal{U}[t,T]} Y^{\gamma_t,u}(t), \quad (t,\gamma_t) \in [0,T] \times \Lambda. \tag{5}$$

We will develop a concept of viscosity solutions to PHJB equations on the space of continuous paths and show that the value functional V defined in (5) is unique viscosity solution to the PHJB equation given in (1) when the coefficients b, σ, q and ϕ only satisfy Lipschitz conditions under maximal norm $||\cdot||_0$ with respect to the path function.

The main difficulties

- the path space Λ_T is an infinite dimensional Banach space
- the maximal norm $||\cdot||_0$ is not Gâteaux differentiable

Noticing that the value functional is only Lipschitz continuous under $||\cdot||_0$ with respect to the path function, the auxiliary functional in the proof of uniqueness should includes the term $||\cdot||_0^{2m}$ or a functional which is equivalent to $||\cdot||_0^{2m}$. The lack of smoothness of $||\cdot||_0^{2m}$ makes it more difficult to define the viscosity solutions and to prove its uniqueness.

HJB equations

HJB equations in finite dimension

Crandall and Lions [TAMS,1983] Viscosity solutions for first order HJB equations.

Lions [AAM,1983a; CPDE,1983b; CPDE,1983c] Viscosity solutions for second order HJB equations.

Crandall, Ishii and Lions (1992), Fleming and Soner (2006), Yong and Zhou (1999) A detailed account for the theory of viscosity solutions.

HJB equations in infinite dimension

Fabbri, Gozzi and Święch (2017) Stochastic Optimal Control in Infinite Dimension. Dynamic Programming and HJB Equations.

Lions [AM,1988; SPDEA,1989a;JFA,1989b] Viscosity solutions of fully nonlinear second-order equations and optimal stochastic control.

Święch [CPDE,1994] "Unbounded" second order PDEs.

Święch [SICON,2020] boundary-noise and boundary-control problems.

One of the structural assumption is that the state space has to be a Hilbert space or certain Banach space with smooth norm, not including the continuous function space.

First order PHJB equations

Lukoyanov [PSIM,2007] First order cases, $\mathbf{H}$ is d_p-locally Lipschitz continuous.

Bayraktar, Keller [JFA,2018] minimax solutions for Path-dependent Hamilton-Jacobi equations in infinite dimensions.

Zhou [Automatica,2022] Viscosity solutions for fully nonlinear PHJB equations in Hilbert spaces.

Seconder order PHJB equations

Dupire (2009) Functional Itô Calculus. https://ssrn.com/abstract=1435551.

Crandall-Lions viscosity solutions
Peng (2012) The first attempt to extend Crandall-Lions framework to path-dependent case. arXiv:1106.1144v2
Tang, Zhang [DCDS,2015] Existence of the PHJB equations.

A new concept of viscosity solutions
Ekren, Keller, Touzi and Zhang [AP, 2014], Cosso, Federico, Gozzi, Rosestolato, Touzi [AP, 2018] Semi-linear PHJB equations.
Ekren, Touzi and Zhang [AP, 2016a, 2016b], Ekren [SPA,2017], Ren [AAP, 2016] Fully nonlinear case, **H** is uniformly nondegenerate.
Ren, Touzi and J. Zhang [SIAM JMA, 2017], Ren, Rosestolato [SIAM JMA, 2020] Degenerate case, **H** is d_p-uniformly continuous.

This new notion is successfully applied to the semilinear case, as this does not require anymore the passage through the maximum principle of Lions (1989). However, in the nonlinear case, this new notion faces some obstacles, which limit its applications.

Second order PHJB equations

Zhou (2021) Viscosity solutions to second order path-dependent Hamilton-Jacobi-Bellman equations and applications, arXiv:2005.05309v2.

We study viscosity solution of PHJB equation in Crandall-Lions framework by defining a smooth functional and applying the Borwein-Preiss variational principle.

Smooth functionals

Definition 1

We say $f \in C_p^{1,2}(\Lambda^t;\mathbb{R})$ for some fixed $t \in [0,T)$ if $f \in C_p^0(\Lambda^t;\mathbb{R})$ and for any $X \in \mathbf{S}_q(\mathbb{F})$ for all $q \geq 1$ satisfying X is a continuous semi-martingale on $[s,T] \subset [t,T]$, there exist $\partial_t f \in C_p^0(\Lambda^t;\mathbb{R})$, $\partial_x f \in C_p^0(\Lambda^t;\mathbb{R}^d)$ and $\partial_{xx} f \in C_p^0(\Lambda^t;\mathcal{S}(\mathbb{R}^d))$ such that, for any $\hat{s} \in [s,T]$,

$$\begin{aligned} f(X_{\hat{s}}) &= f(X_s) + \int_s^{\hat{s}} \partial_t f(X_l)dl + \frac{1}{2}\int_s^{\hat{s}} \partial_{xx} f(X_l)d\langle X\rangle(l) \\ &\quad + \int_s^{\hat{s}} \partial_x f(X_l)dX(l), \ P\text{-a.s.} \end{aligned} \tag{6}$$

We say $f \in C_p^0(\Lambda^t;K)$ if f is continuous in γ_s on Λ^t under d_∞ and grows in a polynomial way.

$$d_\infty(\gamma_s,\eta_l) := |s-l| + \sup_{0\leq\sigma\leq(s\vee l)} |\gamma_s(s\wedge\sigma) - \eta_l(l\wedge\sigma)|.$$

Smooth functionals S_m and $\Upsilon^{m,M}$

For every $m \in \mathbf{N}^+$, define $S_m : \Lambda \times \Lambda \to \mathbb{R}$ by, for every $(t, \gamma_t), (s, \eta_s) \in [0, T] \times \Lambda$,

$$S_m(\gamma_t, \eta_s) = \begin{cases} \frac{(||\gamma_t - \eta_s||_0^{2m} - |\gamma_t(t) - \eta_s(s)|^{2m})^3}{||\gamma_t - \eta_s||_0^{4m}}, & ||\gamma_t - \eta_s||_0 \neq 0; \\ 0, & ||\gamma_t - \eta_s||_0 = 0. \end{cases}$$

For every $M > 0$, define $\Upsilon^{m,M}$ and $\overline{\Upsilon}^{m,M}$ by

$$\begin{aligned} \Upsilon^{m,M}(\gamma_t, \eta_s) &= S_m(\gamma_t, \eta_s) + M|\gamma_t(t) - \eta_s(s)|^{2m}, \\ \overline{\Upsilon}^{m,M}(\gamma_t, \eta_s) &= \Upsilon^{m,M}(\gamma_t, \eta_s) + |s - t|^2. \end{aligned}$$

Smooth functionals S_m and $\Upsilon^{m,M}$

Lemma 2

For every fixed $(\hat{t}, a_{\hat{t}}) \in [0, T) \times \Lambda_{\hat{t}}$, define $S_m^{a_{\hat{t}}} : \Lambda^{\hat{t}} \to R$ by

$$S_m^{a_{\hat{t}}}(\gamma_t) := S_m(\gamma_t, a_{\hat{t}}), \quad (t, \gamma_t) \in [\hat{t}, T] \times \Lambda^{\hat{t}}.$$

Then $S_m^{a_{\hat{t}}}(\cdot) \in C_p^{1,2}(\Lambda^{\hat{t}})$. Moreover, for every $M \geq 3$,

$$||\gamma_t||_0^{2m} \leq \Upsilon^{m,M}(\gamma_t) \leq M||\gamma_t||_0^{2m}, \quad (t, \gamma_t) \in [0, T] \times \Lambda. \tag{7}$$

Since $||\gamma_t - a_{\hat{t}}||_0^6$ does not belong to $C_p^{1,2}(\Lambda^{\hat{t}})$, it cannot appear as an auxiliary functional in the proof of the uniqueness and stability of viscosity solutions. However, by the this Lemma, we can replace $||\gamma_t - a_{\hat{t}}||_0^6$ with its equivalent functional $\Upsilon^{3,3}(\gamma_t, a_{\hat{t}}) \in C_p^{1,2}(\Lambda^{\hat{t}})$.

Let $\Upsilon^{m,M}(\gamma_t)$ denote $\Upsilon^{m,M}(\gamma_t, \mathbf{0})$. Let S, Υ and $\overline{\Upsilon}$ denote S_3, $\Upsilon^{3,3}$ and $\overline{\Upsilon}^{3,3}$, respectively.

Smooth functionals S_m

We can give the concrete expression of the pathwise derivatives.

$$\partial_t S_m^{a_{\hat{t}}}(\gamma_t) = 0.$$

$$\partial_{x_i} S_m^{a_{\hat{t}}}(\gamma_t) = \begin{cases} -\frac{6m(||\gamma_t - a_{\hat{t}}||_0^{2m} - |\gamma_t(t) - a_{\hat{t}}(\hat{t})|^{2m})^2 |\gamma_t(t) - a_{\hat{t}}(\hat{t})|^{2m-2}((\gamma_t)_i(t) - (a_{\hat{t}})_i(\hat{t}))}{||\gamma_t - a_{\hat{t}}||_0^{4m}}, & ||\gamma_t - a_{\hat{t}}||_0 \neq 0, \\ 0, & ||\gamma_t - a_{\hat{t}}||_0 = 0. \end{cases}$$

Smooth functionals S_m

$$
\partial_{x_j x_i} S_m^{a_{\hat{t}}}(\gamma_t)
= \begin{cases}
\dfrac{24m^2(||\gamma_t - a_{\hat{t}}||_0^{2m} - |\gamma_t(t) - a_{\hat{t}}(\hat{t})|^{2m})|\gamma_t(t) - a_{\hat{t}}(\hat{t})|^{4m-4}((\gamma_t)_i(t) - (a_{\hat{t}})_i(\hat{t}))}{||\gamma_t - a_{\hat{t}}||_0^{4m}} \\
\times((\gamma_t)_j(t) - (a_{\hat{t}})_j(\hat{t})) - \dfrac{12m(m-1)(||\gamma_t - a_{\hat{t}}||_0^{2m} - |\gamma_t(t) - a_{\hat{t}}(\hat{t})|^{2m})^2}{||\gamma_t - a_{\hat{t}}||_0^{4m}} \\
\times|\gamma_t(t) - a_{\hat{t}}(\hat{t})|^{2m-4}((\gamma_t)_i(t) - (a_{\hat{t}})_i(\hat{t}))((\gamma_t)_j(t) - (a_{\hat{t}})_j(\hat{t})) \\
-\dfrac{6m(||\gamma_t - a_{\hat{t}}||_0^{2m} - |\gamma_t(t) - a_{\hat{t}}(\hat{t})|^{2m})^2|\gamma_t(t) - a_{\hat{t}}(\hat{t})|^{2m-2}\mathbf{1}_{\{i=j\}}}{||\gamma_t - a_{\hat{t}}||_0^{4m}}, & ||\gamma_t - a_{\hat{t}}||_0 \neq 0, \\
0, & ||\gamma_t - a_{\hat{t}}||_0 = 0.
\end{cases}
$$

Smooth norms $\Upsilon^{m,M}$

In the proof of uniqueness of viscosity solutions, we also need the following lemma.

Lemma 3

For $m \in \mathbf{N}^+$ and $M \geq 3$, we have that, for every $(t, \gamma_t), (t, \gamma'_t) \in [0, T] \times \hat{\Lambda}$,

$$(\Upsilon^{m,M}(\gamma_t + \gamma'_t))^{\frac{1}{2m}} \leq (\Upsilon^{m,M}(\gamma_t))^{\frac{1}{2m}} + (\Upsilon^{m,M}(\gamma'_t))^{\frac{1}{2m}}. \tag{8}$$

In particular, $(\Lambda_T, (\Upsilon^{m,M}(\cdot))^{\frac{1}{2m}})$ is a Banach space.

Borwein-Preiss variational principle

Let's solve the first main difficulty. We want to find the maximum point of functional f defined on (Λ^t, d_∞).

Definition 4

Let $t \in [0, T]$ be fixed. We say that a continuous functional $\rho : \Lambda^t \times \Lambda^t \to [0, +\infty)$ is a gauge-type function provided that:
(i) $\rho(\gamma_s, \gamma_s) = 0$ for all $(s, \gamma_s) \in [t, T] \times \Lambda^t$,
(ii) for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $\gamma_s, \eta_l \in \Lambda^t$, we have $\rho(\gamma_s, \eta_l) \leq \delta$ implies that $d_\infty(\gamma_s, \eta_l) < \varepsilon$.

It follows from (7) that $\overline{\Upsilon}^{m,M}$ is a gauge-type function.

$$||\gamma_t||_0^{2m} \leq \Upsilon^{m,M}(\gamma_t) \leq M||\gamma_t||_0^{2m}, \quad (t, \gamma_t) \in [0, T] \times \Lambda. \quad (7)$$

Borwein-Preiss variational principle

Lemma 5 [Borwein, Zhu (2005) Techniques of variational analysis]

Let $t \in [0, T]$ be fixed and let $f : \Lambda^t \to \mathbb{R}$ be an upper semicontinuous functional bounded from above. Suppose that ρ is a gauge-type function and $\{\delta_i\}_{i\geq 0}$ is a sequence of positive number, and suppose that $\varepsilon > 0$ and $(t_0, \gamma^0_{t_0}) \in [t, T] \times \Lambda^t$ satisfy

$$f(\gamma^0_{t_0}) \geq \sup_{(s,\gamma_s)\in[t,T]\times\Lambda^t} f(\gamma_s) - \varepsilon.$$

Then there exist $(\hat{t}, \hat{\gamma}_{\hat{t}}) \in [t, T] \times \Lambda^t$ and a sequence $\{(t_i, \gamma^i_{t_i})\}_{i\geq 1} \subset [t, T] \times \Lambda^t$ such that
(i) $\rho(\gamma^0_{t_0}, \hat{\gamma}_{\hat{t}}) \leq \frac{\varepsilon}{\delta_0}$, $\rho(\gamma^i_{t_i}, \hat{\gamma}_{\hat{t}}) \leq \frac{\varepsilon}{2^i\delta_0}$ and $t_i \uparrow \hat{t}$ as $i \to \infty$,
(ii) $f(\hat{\gamma}_{\hat{t}}) - \sum_{i=0}^{\infty} \delta_i \rho(\gamma^i_{t_i}, \hat{\gamma}_{\hat{t}}) \geq f(\gamma^0_{t_0})$, and
(iii) $f(\gamma_s) - \sum_{i=0}^{\infty} \delta_i \rho(\gamma^i_{t_i}, \gamma_s) < f(\hat{\gamma}_{\hat{t}}) - \sum_{i=0}^{\infty} \delta_i \rho(\gamma^i_{t_i}, \hat{\gamma}_{\hat{t}})$ for all $(s, \gamma_s) \in [\hat{t}, T] \times \Lambda^{\hat{t}} \setminus \{(\hat{t}, \hat{\gamma}_{\hat{t}})\}$.

We can apply $\overline{\Upsilon}^{m,M}$ to this Lemma to get a maximum of a perturbation of the auxiliary functional in the proof of uniqueness.

Definition of viscosity solutions

For every $(t, \gamma_t) \in [0, T] \times \Lambda$ and $w \in C^0(\Lambda)$, define

$$\mathcal{A}^+(\gamma_t, w) = \left\{ \varphi \in C_p^{1,2}(\Lambda^t) : 0 = (w - \varphi)(\gamma_t) = \sup_{(s,\eta_s) \in [t,T] \times \Lambda} (w - \varphi)(\eta_s) \right\},$$

$$\mathcal{A}^-(\gamma_t, w) = \left\{ \varphi \in C_p^{1,2}(\Lambda^t) : 0 = (w + \varphi)(\gamma_t) = \inf_{(s,\eta_s) \in [t,T] \times \Lambda} (w + \varphi)(\eta_s) \right\}.$$

Definition of viscosity solutions

Definition 6

$w \in C^0(\Lambda)$ is called a viscosity subsolution (resp., supersolution) to (1) if the terminal condition, $w(\gamma_T) \leq \phi(\gamma_T)$(resp., $w(\gamma_T) \geq \phi(\gamma_T)$) for all $\gamma_T \in \Lambda_T$ is satisfied, and whenever $\varphi \in \mathcal{A}^+(\gamma_s, w)$ (resp., $\varphi \in \mathcal{A}^-(\gamma_s, w)$) with $(s, \gamma_s) \in [0, T) \times \Lambda$, we have

$$\partial_t \varphi(\gamma_s) + \mathbf{H}(\gamma_s, \varphi(\gamma_s), \partial_x \varphi(\gamma_s), \partial_{xx} \varphi(\gamma_s)) \geq 0,$$

(resp., $-\partial_t \varphi(\gamma_s) + \mathbf{H}(\gamma_s, -\varphi(\gamma_s), -\partial_x \varphi(\gamma_s), -\partial_{xx} \varphi(\gamma_s)) \leq 0$).

$w \in C^0(\Lambda)$ is said to be a viscosity solution to equation (1) if it is both a viscosity subsolution and a viscosity supersolution.

Hypothesis

Hypothesis 7

$b : \Lambda \times U \to R^d$, $\sigma : \Lambda \times U \to R^{d\times n}$ $q : \Lambda \times R \times R^d \times U \to R$ and $\phi : \Lambda_T \to R$ are continuous, and there exists a constant $L > 0$ such that, for all $(t, \gamma_t, \eta_T, y, z, u)$, $(t, \gamma'_t, \eta'_T, y', z', u) \in [0, T] \times \Lambda \times \Lambda_T \times R \times \Xi \times U$,

$$|b(\gamma_t, u)|^2 \vee |\sigma(\gamma_t, u)|^2 \leq L^2(1 + ||\gamma_t||_0^2);$$
$$|b(\gamma_t, u) - b(\gamma'_t, u)| \vee |\sigma(\gamma_t, u) - \sigma(\gamma'_t, u)| \leq L||\gamma_t - \gamma'_t||_0;$$

$$|q(\gamma_t, y, z, u)| \leq L(1 + ||\gamma_t||_0 + |y| + |z|);$$
$$|q(\gamma_t, y, z, u) - q(\gamma'_t, y', z', u)|$$
$$\leq L(||\gamma_t - \gamma'_t||_0 + |y - y'| + |z - z'|);$$
$$|\phi(\eta_T) - \phi(\eta'_T)| \leq L||\eta_T - \eta'_T||_0.$$

Existence theorem

By functional Itô formula and dynamic programming principle (DPP), we get that

Theorem 8

Suppose that Hypothesis 7 holds. Then the value functional V defined by (5) is a viscosity solution to equation (1).

We also have the result of classical solutions, which show the consistency of viscosity solutions.

Theorem 9

Let V denote the value functional defined by (5). If $V \in C_p^{1,2}(\Lambda)$, then V is a classical solution of (1).

Crandall-Ishii maximum principle

Theorem 10

Let $\kappa > 0$. Let $w_1, w_2 : \Lambda \to \mathbb{R}$ be upper semicontinuous functions bounded from above and such that

$$\limsup_{||\gamma_t||_0 \to \infty} \frac{w_1(\gamma_t)}{||\gamma_t||_0} < 0; \qquad \limsup_{||\gamma_t||_0 \to \infty} \frac{w_2(\gamma_t)}{||\gamma_t||_0} < 0.$$

Let $\varphi \in C^2(\mathbb{R}^d \times \mathbb{R}^d)$ be such that

$$w_1(\gamma_t) + w_2(\eta_t) - \varphi(\gamma_t(t), \eta_t(t))$$

has a maximum over $\Lambda^{\hat{t}} \otimes \Lambda^{\hat{t}}$ at a point $(\hat{\gamma}_{\hat{t}}, \hat{\eta}_{\hat{t}})$ with $\hat{t} \in (0, T)$, where $\Lambda^t \otimes \Lambda^t := \{(\gamma_s, \eta_s) | \gamma_s, \eta_s \in \Lambda^t\}$ for all $t \in [0, T]$. Assume, moreover, $\tilde{w}_1^{\hat{t},*} \in \Phi(\hat{t}, \hat{\gamma}_{\hat{t}}(\hat{t}))$ and $\tilde{w}_2^{\hat{t},*} \in \Phi(\hat{t}, \hat{\eta}_{\hat{t}}(\hat{t}))$, and there exists a local modulus of continuity ρ_1 such that, for all $\hat{t} \leq t \leq s \leq T$, $\gamma_t \in \Lambda$, $i = 1, 2$,

$$w_i(\gamma_t) - w_i(\gamma_{t,s}) \leq \rho_1(|s - t|, ||\gamma_t||_0).$$

Crandall-Ishii maximum principle

Theorem 10 (Continued)

Then there exist the sequences $(t_k, \gamma^k_{t_k}), (s_k, \eta^k_{s_k}) \in [\hat{t}, T] \times \Lambda^{\hat{t}}$ and the sequences of functionals $\varphi_k \in C^{1,2}_p(\Lambda^{t_k})$, $\psi_k \in C^{1,2}_p(\Lambda^{s_k})$ such that φ_k, $\partial_t\varphi_k$, $\partial_x\varphi_k$, $\partial_{xx}\varphi_k$, $\psi_k, \partial_t\psi_k$, $\partial_x\psi_k, \partial_{xx}\psi_k$ are bounded and uniformly continuous, and such that

$$w_1(\gamma_t) - \varphi_k(\gamma_t)$$

has a strict global maximum 0 at $\gamma^k_{t_k}$ over Λ^{t_k},

$$w_2(\eta_t) - \psi_k(\eta_t)$$

has a strict global maximum 0 at $\eta^k_{s_k}$ over Λ^{s_k}, and

$$\left(t_k, \gamma^k_{t_k}(t_k), w_1(\gamma^k_{t_k}), \partial_t\varphi_k(\gamma^k_{t_k}), \partial_x\varphi_k(\gamma^k_{t_k}), \partial_{xx}\varphi_k(\gamma^k_{t_k})\right)$$
$$\xrightarrow{k \to \infty} \left(\hat{t}, \hat{\gamma}_{\hat{t}}(\hat{t}), w_1(\hat{\gamma}_{\hat{t}}), b_1, \nabla_{x_1}\varphi(\hat{\gamma}_{\hat{t}}(\hat{t}), \hat{\eta}_{\hat{t}}(\hat{t})), X\right),$$

Crandall-Ishii maximum principle

Theorem 10 (Continued)

$$\left(s_k, \eta^k_{s_k}(s_k), w_2(\eta^k_{s_k}), \partial_t\psi_k(\eta^k_{s_k}), \partial_x\psi_k(\eta^k_{s_k}), \partial_{xx}\psi_k(\eta^k_{s_k})\right)$$
$$\xrightarrow{k\to\infty} \left(\hat{t}, \hat{\eta}_{\hat{t}}(\hat{t}), w_2(\hat{\eta}_{\hat{t}}(\hat{t})), b_2, \nabla_{x_2}\varphi(\hat{\gamma}_{\hat{t}}(\hat{t}), \hat{\eta}_{\hat{t}}(\hat{t})), Y\right),$$

where $b_1 + b_2 = 0$ and $X, Y \in \mathcal{S}(\mathbb{R}^d)$ satisfy the following inequality:

$$-\left(\frac{1}{\kappa} + |A|\right) I \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq A + \kappa A^2,$$

and $A = \nabla^2_x\varphi(\hat{\gamma}_{\hat{t}}(\hat{t}), \hat{\eta}_{\hat{t}}(\hat{t}))$. Here $\nabla_{x_1}\varphi$ and $\nabla_{x_2}\varphi$ denote the standard first order derivative of φ with respect to the first variable and the second variable, respectively.

Comparison theorem

By Crandall-Ishii maximum principle and Borwein-Preiss variational principle, we get that

Theorem 11

Suppose Hypothesis 7 holds. Let $W_1 \in C^0(\Lambda)$ (resp., $W_2 \in C^0(\Lambda)$) be a viscosity subsolution (resp., supersolution) to equation (1) and let there exist constant $L > 0$ such that, for any $(t, \gamma_t), (s, \eta_s) \in [0, T] \times \Lambda$,

$$|W_1(\gamma_t)| \vee |W_2(\gamma_t)| \leq L(1 + ||\gamma_t||_0); \tag{9}$$

$$\begin{aligned} & |W_1(\gamma_t) - W_1(\eta_s)| \vee |W_2(\gamma_t) - W_2(\eta_s)| \\ \leq \quad & L(1 + ||\gamma_t||_0 + ||\eta_s||_0)|s - t|^{\frac{1}{2}} + L||\gamma_t - \eta_s||_0. \end{aligned} \tag{10}$$

Then $W_1 \leq W_2$.

Uniqueness theorem

By existence theorem and comparison theorem we get that

Theorem 12

Let Hypothesis 7 hold. Then the value functional V defined by (5) is the unique viscosity solution to (1) in the class of functionals satisfying (9) and (10).

Proof of Theorem 11

Step 1. Definitions of the auxiliary functions.

We assume the converse result that $(\tilde{t}, \tilde{\gamma}_{\tilde{t}}) \in [T - \bar{a}, T) \times \Lambda$ exists such that $\tilde{m} := W_1(\tilde{\gamma}_{\tilde{t}}) - W_2(\tilde{\gamma}_{\tilde{t}}) > 0$.

We define for any $(t, \gamma_t, \eta_t) \in (T - \bar{a}, T] \times \Lambda \times \Lambda$,

$$\begin{aligned}\Psi(\gamma_t, \eta_t) &= W_1(\gamma_t) - W_2(\eta_t) - \beta \Upsilon(\gamma_t, \eta_t) - \beta^{\frac{1}{3}} |\gamma_t(t) - \eta_t(t)|^2 \\ &\quad - \varepsilon \frac{\nu T - t}{\nu T} (\Upsilon(\gamma_t) + \Upsilon(\eta_t)).\end{aligned}$$

Proof of Theorem 11

From Lemma 5 it follows that, for every $(t_0, \gamma^0_{t_0}, \eta^0_{t_0}) \in [\tilde{t}, T] \times \Lambda^{\tilde{t}} \times \Lambda^{\tilde{t}}$ satisfy

$$\Psi(\gamma^0_{t_0}, \eta^0_{t_0}) \geq \sup_{(s,\gamma_s,\eta_s)\in[\tilde{t},T]\times\Lambda^{\tilde{t}}\times\Lambda^{\tilde{t}}} \Psi(\gamma_s, \eta_s) - \frac{1}{\beta},$$

there exist $(\hat{t}, \hat{\gamma}_{\hat{t}}, \hat{\eta}_{\hat{t}}) \in [\tilde{t}, T] \times \Lambda^{\tilde{t}} \times \Lambda^{\tilde{t}}$ and a sequence $\{(t_i, \gamma^i_{t_i}, \eta^i_{t_i})\}_{i\geq 1} \subset [\tilde{t}, T] \times \Lambda^{\tilde{t}} \times \Lambda^{\tilde{t}}$ such that for all $(s, \gamma_s, \eta_s) \in [\hat{t}, T] \times \Lambda^{\hat{t}} \times \Lambda^{\hat{t}} \setminus \{(\hat{t}, \hat{\gamma}_{\hat{t}}, \hat{\eta}_{\hat{t}})\}$,

$$\Psi_1(\gamma_s, \eta_s) < \Psi_1(\hat{\gamma}_{\hat{t}}, \hat{\eta}_{\hat{t}}),$$

where

$$\Psi_1(\gamma_t, \eta_t) := \Psi(\gamma_t, \eta_t) - \sum_{i=0}^{\infty} \frac{1}{2^i}[\Upsilon(\gamma^i_{t_i}, \gamma_t) + \Upsilon(\eta^i_{t_i}, \eta_t) + |s - t_i|^2].$$

Proof of Theorem 11

Step 2. There exists $M_0 > 0$ independent of β such that

$$||\hat{\gamma}_{\hat{t}}||_0 \vee ||\hat{\eta}_{\hat{t}}||_0 < M_0, \tag{11}$$

and the following result holds true:

$$\beta||\hat{\gamma}_{\hat{t}} - \hat{\eta}_{\hat{t}}||_0^6 + \beta|\hat{\gamma}_{\hat{t}}(\hat{t}) - \hat{\eta}_{\hat{t}}(\hat{t})|^4 \to 0 \text{ as } \beta \to \infty.$$

Step 3. There exists $N > 0$ such that $\hat{t} \in (T - \bar{a}, T)$ for all $\beta \geq N$.

Proof of Theorem 11

Step 4. Maximum principle.

We put, for $(t, \gamma_t, \eta_t) \in (T - \bar{a}, T] \times \Lambda \times \Lambda$,

$$\begin{aligned} w_1(\gamma_t) &= W_1(\gamma_t) - 2^5 \beta \Upsilon(\gamma_t, \hat{\xi}_{\hat{t}}) - \varepsilon \frac{\nu T - t}{\nu T} \Upsilon(\gamma_t) - \varepsilon \overline{\Upsilon}(\gamma_t, \hat{\gamma}_{\hat{t}}) \\ &\quad - \sum_{i=0}^{\infty} \frac{1}{2^i} \overline{\Upsilon}(\gamma^i_{t_i}, \gamma_t), \end{aligned}$$

$$\begin{aligned} w_2(\eta_t) &= -W_2(\eta_t) - 2^5 \beta \Upsilon(\eta_t, \hat{\xi}_{\hat{t}}) - \varepsilon \frac{\nu T - t}{\nu T} \Upsilon(\eta_t) - \varepsilon \overline{\Upsilon}(\eta_t, \hat{\eta}_{\hat{t}}) \\ &\quad - \sum_{i=0}^{\infty} \frac{1}{2^i} \Upsilon(\eta^i_{t_i}, \eta_t), \end{aligned}$$

where $\hat{\xi}_{\hat{t}} = \frac{\hat{\gamma}_{\hat{t}} + \hat{\eta}_{\hat{t}}}{2}$.

Proof of Theorem 11

Step 5. Calculation and completion of the proof.

By a simple calculation, we can let $\beta > 0$ be large enough such that

$$\begin{aligned} c \quad \leq \quad & -\frac{\varepsilon}{\nu T}(\Upsilon(\hat{\gamma}_{\hat{t}}) + \Upsilon(\hat{\eta}_{\hat{t}})) \\ & +\varepsilon\frac{\nu T - \hat{t}}{\nu T}(342L + 36)L(1 + ||\hat{\gamma}_{\hat{t}}||_0^6 + ||\hat{\eta}_{\hat{t}}||_0^6) + \frac{c}{4}. \end{aligned}$$

Recalling $\nu = 1 + \frac{1}{2T(342L+36)L}$ and $\bar{a} = \frac{1}{2(342L+36)L} \wedge T$, the following contradiction is induced:

$$c \leq \frac{\varepsilon}{\nu T} + \frac{c}{4} \leq \frac{c}{2}.$$

BSHJBE

$$
\begin{cases}
d\bar{V}(t,x) = -\sup_{u\in U}[(\nabla_x \bar{V}(t,x), b(W_t,x,u))_{\mathbb{R}^d} \\
+\frac{1}{2}\mathrm{tr}(\nabla_x^2 \bar{V}(t,x)\bar{\sigma}(W_t,x,u)\bar{\sigma}^\top(W_t,x,u)) \\
+\mathrm{tr}(\bar{\sigma}^\top(W_t,x,u)\nabla_x p(t,x)) + \bar{q}(W_t,x,\bar{V}(t,x),p(t,x) \\
+\bar{\sigma}^\top(W_t,x,u)\nabla_x \bar{V}(t,x),u)] + p(t,x)dW(t),\ (t,x)\in[0,T]\times\mathbb{R}^m, \\
\bar{V}(T,x) = \bar{\phi}(W_T,x), \quad x\in\mathbb{R}^m, \quad P\text{-a.s.}
\end{cases}
\tag{12}
$$

BSHJBE

Define the pair of $\mathcal{F}_t$-adapted processes

$$(\overline{V}(t,x), p(t,x)) := (V(W_t,x), \partial_\gamma V(W_t,x)), \quad (t,x) \in [0,T] \times \mathbb{R}^m, \tag{13}$$

Theorem 13

Let b, σ, q, ϕ satisfy Hypothesis 7. Then the $\mathcal{F}_t$-adapted process $\overline{V}(t,x) := V(W_t,x)$ defined by (13) is a unique viscosity solution to BSHJB equation (12).

Further work

- Infinite horizon optimal control problems and elliptic equations
- Stochastic differential game
- Stochastic evolution equations
- mean field problems
- Minimax solution

- **Thank you for your attention!**