Set-Valued Backward SDEs and Related Set-Valued Stochastic Analysis

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Based on the joint works with Çağın Ararat and Wenqian Wu

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Outline

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Examples of Set-valued Dynamics

- “Reacheable Sets"
  - Any higher dimensional controlled dynamic systems (could be both forward and backward(!))
  - The backward version could be used to characterize "DPP" for “time-inconsistent" problems (Karnam-Ma-Zhang, 2017).
  - Non-zero Sum/Mean-field Games with multiple equilibria (Feinstein-Rudloff-Zhang, '20, Iseri-Zhang, '21, ...)
- Set-valued Dynamic Risk Measures
  - Systemic Risks (Hamel-Heyde-Rudloff, '11, Feinstein-Rudloff-Weber, '17, Ararat-Hamel-Rudloff '17, Ararat-Rudloff '19, Biagini-Fouque-Fritelli-Meyer-Brandis, '19)
  - Multi-portfolio time consistency (Feinstein-Rudloff, '15)
  - Set-valued Risk measures and BSdI/E (d= difference) (Ararat-Feinstein 2019)

Set-valued SDEs vs SDIs

The Main Point :

A set of processes is NOT necessarily a set-valued process !

In other words, a set-valued process is a process taking values in a (metric) space of subsets in a vector space, rather than a collection of trajectories in this space.

Some obvious technical issues :

- The algebraic structure among sets ?
- Measurability ?
- (Stochastic) Analysis ?

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First Glance of Set-valued Analysis

- $(\mathbb{X}, \rho)$ — a metric space
- $\mathscr{P}(\mathbb{X}) := 2^{\mathbb{X}}$ — all nonempty subsets of $\mathbb{X}$.
- $\mathscr{C}(\mathbb{X}) \subset \mathscr{P}(\mathbb{X})$ — all *closed* subsets of $\mathbb{X}$,
- $\mathscr{K}(\mathbb{X})$ — all *compact, convex* subsets of $\mathbb{X}$.

Definition (Minkowski addition and scalar multiplication)

Let $A, B \in \mathscr{K}(\mathbb{X})$ and $\alpha \in \mathbb{R}$, we define

$$\begin{aligned} A + B &= \{x \in \mathbb{X} : x = a + b,\ a \in A, b \in B\}; \\ \alpha A &= \{x \in \mathbb{X} : x = \alpha a,\ a \in A\}. \end{aligned}$$

Note : $A - A := A + (-1)A \neq \{0\}$!. That is, $-A$ is NOT the "inverse" of A(!). Thus $\mathscr{K}(\mathbb{X})/\mathscr{C}(\mathbb{X})$ is NOT a vector space.

Set Differences

- *Minkovski difference/geometric difference/inf-residuation*

$$A \dot{-} B = \{x \in \mathbb{X} \mid x + B \subset A\}, \quad A, B \in \mathscr{K}(\mathbb{X}),$$

  - $A \dot{-} A = \{0\}$, but $(A \dot{-} B) + B \subset A$ (and "=" may fail !)

- *Hukuhara difference (1967)*

$$A \ominus B = C \quad \Longleftrightarrow \quad A = B + C, \;\; A, B \in \mathscr{K}(\mathbb{X}).$$

  - $A \ominus B$ exists $\Longleftrightarrow \forall a \in \mathrm{ext}(A)$, $\exists x \in \mathbb{X}$ s.t. $a \in x + B \subset A$.
    - $a \in \mathrm{ext}(A)$ (called an *extreme point* of A) if it cannot be written as a strict convex combination of two points in A.
  - If exists, $A \ominus B$ is unique, closed, convex, and $= A \dot{-} B$.
  - Some properties of "$\ominus$" requires "cancellation law". (I.e., $A + C = B + C \implies A = B$), which is true if $\mathbb{X}$ is locally compact, A, B are closed, convex, and C is compact.

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Topological Structure on $\mathscr{K}(\mathbb{X})$

- Recall the *Hausdorff distance* on $\mathscr{C}(\mathbb{X})/\mathscr{K}(\mathbb{X})$:

$$h(A, B) \quad := \quad \max\{\sup_{x\in A} d(x, B), \ \sup_{x\in B} d(x, A)\}, \qquad (1)$$

- $(\mathscr{K}(\mathbb{X}), h)$ is a Polish space (if $\mathbb{X}$ is)
- If $\mathbb{X} = \mathbb{R}^d$, denote $\mathscr{X} = \mathscr{K}(\mathbb{R}^d)$ and
  - $\mathscr{B}(\mathscr{X}) := \sigma(\mathscr{K}(\mathbb{R}^d))$ — (Borel) σ-algebra on $(\mathscr{X}, h)$;
  - define $\|A\| := h(A, \{0\}) = \sup\{|a|, a \in A\}$,
    - Then $\|\cdot\|$ is a "norm", and $h(A, B) = \|A \ominus B\|$, if $\ominus$ exists.
    - **Warning :** $(\mathscr{X}, \|\cdot\|)$ is NOT a Banach space, as it is not even a vector space.

Set-Valued Mappings and Selections

Denote $\mathscr{O}$ = all open sets in $\mathbb{R}^d$.

- For $V \in \mathscr{O}$, define $\mathscr{O}(V) := \{K \in \mathscr{K} : K \cap V \neq \emptyset\}$;
- Let $(\mathbb{T}, \mathcal{F}, \mu)$ be a measure space. A mapping $F : \mathbb{T} \to \mathscr{K}$ is called *(weakly) measurable* if $F^{-}(V) := \{t \in \mathbb{T} : F(t) \in \mathscr{O}(V)\} \in \mathcal{F}$, $\forall V \in \mathscr{O}$.
- Denote all measurable mappings $F : \mathbb{T} \to \mathscr{K}$ by $\mathscr{M}(\mathbb{T}, \mathscr{K})$.
- A "measurable selector" of F is a function $f : \mathbb{T} \mapsto \mathbb{X}$, such that $f(t) \in F(t)$, a.e. $t \in \mathbb{T}$, and $f \in \mathbb{L}^0(\mathbb{T}; \mathbb{R}^d)$. Denote the collection of selectors of F by $S(F)$.
- Clearly, $F = G$ if and only if $S(F) = S(G)$.

Measurability vs. Decomposibility

Definition

A set $M \subset \mathbb{L}^0(\mathbb{T};\mathbb{R}^d)$ is called decomposable w.r.t. $\mathcal{F}$ if for any $f_1, f_2 \in M$ and $A \in \mathcal{F}$, the function $\mathbf{1}_A f_1 + \mathbf{1}_{A^c} f_2 \in M$.

- For $C \subset \mathbb{L}^0(\mathbb{T},\mathbb{R}^d)$, we denote $dec\{C\}$ (resp. $\overline{dec}\{C\}$) to be the *decomposable hull* of C (resp. closure of $dec\{C\}$).

For $p \geq 1$, define $S_p(F) = S(F) \cap \mathbb{L}^p(\mathbb{T};\mathscr{X})$, and

$$\mathscr{A}_p(\mathbb{T},\mathscr{X}) := \{F \in \mathscr{M}(\mathbb{T};\mathscr{X}) : S_p(F) \neq \emptyset\}.$$

Theorem

Let M be a nonempty subset of $\mathbb{L}^p(\mathbb{T},\mathbb{R}^d)$ where $p \geq 1$, such that for each $t \in \mathbb{T}$, $M(t) \in \mathscr{X}$. Then there exists $F \in \mathscr{A}_p(\mathbb{T},\mathscr{X})$ such that $M = S_p(F)$ if and only if M is decomposable.

Set-Valued Random Variables and Stochastic Processes

- SV r.v. $Z : \Omega \mapsto \mathscr{X}$ and SV process $\Phi : [0, T] \times \Omega \mapsto \mathscr{X}$ are defined naturally as SV measurable functions.
  - $S_{\mathcal{G}}(Z)$ — $\mathcal{G}$-measurable selectors, $\mathcal{G} \subseteq \mathcal{F}$;
  - $S(\Phi)$ — all $\mathscr{B}([0, T]) \otimes \mathcal{F}$-measurable selectors
  - $S_{\mathbb{F}}(\Phi)$ ($S_{\mathscr{P}}(\Phi)$) — all $\mathbb{F}$-adapted (progressive) selectors.
- Z is *p-integrally bounded* if $\mathbb{E}[\|Z\|^p] = \mathbb{E}[h^p(Z, \{0\})] < \infty$ (i.e., $S_p(Z)$ is a bounded set in $\mathbb{L}^p$).
- If $Z \in \mathscr{A}_1(\Omega, \mathscr{X})$, then $\mathbb{E}[Z] := \int_\Omega Z(\omega)\mathbb{P}(d\omega)$ (Aumann).
  - If $F \in \mathscr{A}_1(\mathbb{T}, \mathbb{R}^d)$, the *Aumann Integral* of F is defined by $\int_{\mathbb{T}} F(t)\mu(dt) := cl\Big\{\int_{\mathbb{T}} f(t)\mu(dt) : f \in S_1(F)\Big\} := cl(J(S_1(F)))$.
  - It holds that $\int_{\mathbb{T}} F(t)\mu(dt) = \int_{\mathbb{T}} coF(t)dt$, hence convex.

Conditional Expectations

- For $\mathcal{G} \subset \mathcal{F}$ and $Z \in \mathscr{A}^1_{\mathcal{F}}(\Omega)$, we define $\mathbb{E}(Z|\mathcal{G}) \in \mathscr{A}^1_{\mathcal{G}}(\Omega)$ via the Aumann integral identity

$$\int_A \mathbb{E}(Z|\mathcal{G})(\omega)\mathbb{P}(d\omega) = \int_A Z(\omega)\mathbb{P}(d\omega), \qquad A \in \mathcal{G}. \tag{2}$$

- If $F \in \mathscr{A}^1(\Omega)$, then $\exists!$ $\mathbb{E}[F|\mathcal{G}] \in \mathscr{A}^1_{\mathcal{G}}(\Omega)$, s.t.

$$S_1(\mathbb{E}[F|\mathcal{G}]) = \mathrm{cl}_{\mathbb{L}}\{\mathbb{E}[f|\mathcal{G}] : f \in S_1(F)\}. \tag{3}$$

- $\mathbb{E}[\cdot|\mathcal{G}]$ satisfy all the natural properties in terms of Minkovski addition and scalar multiplication.
- If $X_1 \ominus X_2$ exists. Then, $\mathbb{E}[X_1 \ominus X_2|\mathcal{G}]$ exists and

$$\mathbb{E}[X_1 \ominus X_2|\mathcal{G}] = \mathbb{E}[X_1|\mathcal{G}] \ominus \mathbb{E}[X_2|\mathcal{G}]. \tag{4}$$

Set-Valued Martingales

- A *set-valued* $(\mathbb{P},\mathbb{F})$*-martingale* $M=\{M_t\}_{t\ge 0}$ is $M\in\mathscr{L}^1_{\mathbb{F}}$ such that $M_s=\mathbb{E}[M_t|\mathcal{F}_s]$, for $0\le s\le t$.
  - Set-Valued "sub-" (or "super-") martingales ? Order ?
  - A set-valued mg M has decomposable $S_{\mathcal{F}_t}(M_t)$ for each $t\ge 0$.
- For a set-valued mg M, denote the *martingale selectors* by

$$MS(M) := \{\text{all } \mathbb{F}\text{-mg } f=\{f_t\} \text{ s.t. } f_t\in S_{\mathcal{F}_t}(M_t),\ t\ge 0\}.$$

$$P_t[MS(M)] := \{f_t : f\in MS(M)\}\in\mathscr{P}(\mathbb{L}^1_{\mathcal{F}_t}(\Omega,\mathbb{R}^d)).$$

  - **Note :** $S_{\mathcal{F}_t}(M_t)$ and $P_t(MS(M))$ are quite different ! In particular, the former is *decomposable*, but the latter is not.
  - The following relations holds

$$S_{\mathcal{F}_t}(M_t)=\overline{dec}\{P_t[MS(M)]\},\qquad t\ge 0. \tag{5}$$

Set-Valued Stochastic Integrals (Aumann-Itô)

- Let $B = \{B_t\}_{t\in\mathbb{T}:=[0,T]}$ be a m-dim B.M. on $(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F})$.
- Define linear mappings on $\mathbb{L}^2_{\mathbb{F}}(\mathbb{T}; \mathbb{R}^d)$ and $\mathbb{L}^2_{\mathbb{F}}(\mathbb{T}; \mathbb{R}^{d\times m})$:

$$J(\phi) := \int_0^T \phi_t dt, \qquad \mathcal{J}(\psi) := \int_0^T \psi_t dB_t.$$

- For $K \in \mathscr{P}(\mathbb{L}^2_{\mathbb{F}}(\mathbb{T}; \mathbb{R}^d))$ and $K' \in \mathscr{P}(\mathbb{L}^2_{\mathbb{F}}(\mathbb{T}; \mathbb{R}^{d\times m}))$, define $J(K) := \{J(\phi) : \phi \in K\}, \mathcal{J}(K') := \{\mathcal{J}(\psi) : \psi \in K'\}$.

Theorem/Definition

For $\Phi \in \mathscr{L}^{2,d}_{\mathbb{F}}(\mathbb{T})$ and $\Psi \in \mathscr{L}^{2,d\times m}_{\mathbb{F}}(\mathbb{T})$, $\exists$ $\Gamma, Z \in \mathscr{L}^{2,d}_{\mathcal{F}_T}(\Omega)$ s.t.

$$S_{\mathcal{F}_T}[\Gamma] = \overline{dec}\{J[S_{\mathbb{F}}(\Phi)]\}, \text{ and } S_{\mathcal{F}_T}[Z] = \overline{dec}\{\mathcal{J}[S_{\mathbb{F}}(\Psi)]\}.$$

Denote $\Gamma := \int_0^T \Phi dt$ and $Z := \int_0^T \Psi dB_t$.

Remarks

- Both $S_{F_T}(\int_0^T \Phi dt)$ and $S_{F_T}(\int_0^T \Psi_t dB_t)$ are decomposable (hence $\mathcal{F}_T$-measurable), but none of $J(S_{\mathbb{F}}(\Phi))$ and $\mathcal{J}(S_{\mathbb{F}}(\Psi))$ is, unless singleton. (see Kisielewicz ('13, p105) for the ex's)
- The *Indefinite stochastic integral* $\int_0^t \Psi dB := \int_0^T \mathbf{1}_{[0,t]}\Psi_s dB_s$ is well-defined, and has the "*temporal additivity*" :

$$\int_0^T \Psi_s dB_s = \int_0^t \Psi_s dB_s + \int_t^T \Psi_s dB_s, \quad t \in [0, T].$$

Generalized Aumann-Itô integral

Let $G \in \mathscr{P}(\mathbb{L}^2_{\mathbb{F}}(\mathbb{T}; \mathbb{R}^{d\times m}))$. Denote $\mathcal{J}^t_B(G) = \{\int_0^t g_s dB_s : g \in G\}$, $t \in \mathbb{T}$. Then, $\exists!\ \Phi_t \in \mathscr{L}^2_{\mathcal{F}_t}(\Omega)$ s.t. $S_{\mathcal{F}_t}(\Phi_t) = \overline{dec}\{\mathcal{J}^t_B(G)\}$. We call Φ_t the *generalized stochastic integral* and denote it by $\int_0^t G \circ dB_s$.

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Hukuhara Difference for Stochastic Integrals

Proposition (Ararat-M.-Wu, 2020)

- If $\Phi, \Psi \in \mathscr{A}(\mathbb{T}, \mathscr{X})$. Then
  - $\Phi + \Psi, \Phi \ominus \Psi$ (if exists) $\in \mathscr{A}(\mathbb{T}, \mathscr{X})$,
  - $S(\Phi + \Psi) = S(\Phi) + S(\Psi)$, $S(\Phi \ominus \Psi) = S(\Phi) \ominus S(\Psi)$.
- If $\Phi, \Psi \in \mathscr{P}(\mathbb{L}^2_{\mathbb{F}}(\mathbb{T}, \mathscr{X}))$, s.t., $\Phi \ominus \Psi$ exists, then
  - $\mathcal{J}^t_s(\Phi) \ominus \mathcal{J}^t_s(\Psi) = \mathcal{J}^t_s(\Phi \ominus \Psi)$, $0 \leq s < t \leq T$.
    $\Longrightarrow \int_0^T \Phi \circ dB_s \ominus \int_0^T \Psi \circ dB_s := \int_0^T (\Phi \ominus \Psi) \circ dB_s$, $\mathbb{P}$-a.s.
  - If Φ, Ψ are convex and square integrably bounded, then all stochastic integrals above can be in the Aumann-Itô sense.
  - $\int_0^T \Phi \circ dB_s \ominus \int_0^T \Psi \circ dB_s = \{0\} \Longrightarrow \int_0^T \Phi \circ dB_s = \int_0^T \Psi \circ dB_s$.
  - $\int_0^t \Phi \circ dB_s \ominus \int_0^t \Psi \circ dB_s = \{0\}$, $\mathbb{P}$-a.s., $\forall t \in \mathbb{T} \Longrightarrow \Phi \equiv \Psi$.

Some Important Estimates

- Define

$$\begin{cases} H(\Phi, \Psi) := \mathbb{E}h(\Phi, \Psi), & \Phi, \Psi \in \mathscr{L}^1_{\mathcal{F}_T}(\Omega; \mathscr{X}); \\ \mathcal{H}_2(\Phi, \Psi) := [\mathbb{E}h^2(\Phi, \Psi)]^{1/2}, & \Phi, \Psi \in \mathscr{L}^2_{\mathcal{F}_T}(\Omega; \mathscr{X}) \end{cases}$$

- Then for $\mathcal{G} \subset \mathcal{F}$, it holds that
  - $H(\mathbb{E}[\Phi|\mathcal{G}], \mathbb{E}[\Psi|\mathcal{G}]) \leq H(\Phi, \Psi)$ (Kisielewicz '13)
  - $h^2(\mathbb{E}[\Phi|\mathcal{G}], \mathbb{E}[\Psi|\mathcal{G}]) \leq \mathbb{E}[h^2(\Phi, \Psi)|\mathcal{G}]$, $\mathbb{P}$-a.s. (Ararat-M.-Wu)
  - $\mathcal{H}_2(\mathbb{E}[\Phi|\mathcal{G}], \mathbb{E}[\Psi|\mathcal{G}]) \leq \mathcal{H}_2(\Phi, \Psi)$.
  - $h^2\Big(\int_t^T \Phi_s ds, \int_t^T \Psi_s ds\Big) \leq (T-t)\int_t^T h^2(\Phi_s, \Psi_s) ds$, $\mathbb{P}\text{-a.s.}$

Outline

Set-Valued Martingale Representation Theorem

- Assume $\mathbb{F} = \mathbb{F}^B$ and let M be a set-valued $(\mathbb{L}^2)$ $\mathbb{F}$-martingale. Then for each $f \in MS(M)$, $\exists!\ g^f \in \mathbb{L}^2_{\mathbb{F}}(\mathbb{T}; \mathbb{R}^{d\times m})$, such that $f_t = \int_0^t g^f_s dB_s$, $t \in \mathbb{T}$, $\mathbb{P}$-a.s.
- Denote $\mathcal{G}^M := \{g^f : f \in MS(M)\} \in \mathscr{P}(\mathbb{L}^2_{\mathbb{F}}(\mathbb{T}; \mathbb{R}^{d\times m}))$.

Theorem (Kisielevicz, 2014)

Assume $\mathbb{F} = \mathbb{F}^B$, where B is a $\mathbb{R}^m$-valued Brownian motion. Then for every set-valued martingale $M = \{M_t\}_{t\in[0,T]} \in \mathscr{L}^{2,d}_{\mathbb{F}}(\mathbb{T})$, with $M_0 = \{0\}$, there exists $\mathcal{G}^M \in \mathscr{P}(\mathbb{L}^2_{\mathbb{F}}(\mathbb{T}; \mathbb{R}^{d\times m}))$, such that $M_t = \int_0^t \mathcal{G}^M \circ dB_s$, $\mathbb{P}$-a.s. $t \in \mathbb{T}$.

- **Note** : The set $\mathcal{G}^M$ is likely **not** decomposable, thus the stochastic integral can only be in the generalized sense.

Simplest Possible Set-Valued BSDEs

$$Y_t = \mathbb{E}\Big[\xi + \int_t^T F(s, Y_s, \cdots)ds \Big| \mathcal{F}_t\Big], \qquad t \in \mathbb{T} = [0, T].$$

$$Y_t = \xi + \int_t^T F(s, Y_s, \cdots)ds - \int_t^T Z \circ dB_s, \qquad t \in \mathbb{T},$$

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where $\xi \in \mathscr{L}^2_{\mathcal{F}_T}(\Omega)$, $F : \mathbb{T} \times \Omega \times \mathscr{X} \mapsto \mathscr{X}$ is a multifunction to be specified, and the stochastic integral is in generalized sense !

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Well-postedness (First Take)

Consider the following simplest SVBSDE :

$$Y_t = \mathbb{E}\Big[\xi + \int_t^T F(s, \cdot, Y_s)ds \Big| \mathscr{F}_t\Big], \qquad t \in [0, T]. \qquad (6)$$

Main Idea :

- Consider the mapping

$$(\Phi(Y))_t = \mathbb{E}\Big[\xi + \int_t^T F(s, Y_s)ds \Big| \mathscr{F}_t\Big], \qquad t \in \mathbb{T}.$$

- Find an appropriate space $\mathcal{Y}$, so that $\Phi : \mathcal{Y} \mapsto \mathcal{Y}$, and is a contraction.

Well-postedness (First Take)

- Consider $(\mathscr{K}(\mathbb{L}^2), H_{\mathbb{L}^2})$, where $\mathbb{L}^2 = \mathbb{L}^2(\Omega; \mathbb{R}^d)$.
- Note : $H_{\mathbb{L}^2}$ is not $\mathcal{H}_2$(!), and one shows that

$$H^2_{\mathbb{L}^2}\Big\{\mathbb{E}\Big[\int_\tau^t \Psi_s ds\Big|\mathcal{F}_t\Big], \mathbb{E}\Big[\int_\tau^t \Psi'_s ds\Big|\mathcal{F}_t\Big]\Big) \le (t-\tau)\int_\tau^t \mathcal{H}^2_2(\Psi_s, \Psi'_s)ds, \qquad 0 \le \tau < t \le T. \tag{7}$$

- Now consider $\mathcal{Y} = \mathbb{C}(\mathbb{T}, \mathscr{K}(\mathbb{L}^2))$ with the metric :

$$D_\lambda(X, Y) := \sup_{t\in[0,T]} e^{-\lambda(T-t)} H_{\mathbb{L}^2}(X_t, Y_t), \qquad \lambda > 0.$$

  Then $(\mathcal{Y}, D_\lambda)$ is a complete metric space.
- Using (7) to show that $\Phi : \mathcal{Y} \mapsto \mathcal{Y}$ is a contraction for λ large.

What about Z ?

Consider again the simplest form :

$$Y_t + \int_t^T Z \circ dB_s = \xi + \int_t^T F(s, Y_s)ds, \qquad t \in \mathbb{T}, \ a.s. \qquad (8)$$

Lemma (Aarat-M.-Wu, '21)

Given $\xi \in \mathscr{L}^2_{\mathcal{F}_T}(\Omega; \mathscr{X})$ and $\Phi \in \mathscr{L}^2_{\mathbb{F}}(\mathbb{T}; \mathscr{X})$, there exists a unique $(Y, Z) \in \mathscr{L}^2_{\mathbb{F}}(\mathbb{T}; \mathscr{X}) \times \mathscr{P}(\mathbb{L}^2_{\mathbb{F}}(\mathbb{T} \times \mathbb{R}^{d\times m}))$, such that

$$Y_t = \mathbb{E}\Big[\xi + \int_t^T \Phi_s ds \Big| \mathcal{F}_t\Big] = \Big[\xi + \int_t^T \Phi_s ds\Big] \ominus \int_t^T Z \circ dB_s, \ \ t \in \mathbb{T}.$$

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Picard Iteration

- Define the iteration sequence $\{(Y^{(n)}, Z^{(n)})\}$ (Lemma)
- Denote $\Delta Y_t^{(n)} := Y_t^{(n)} \ominus Y_t^{(n-1)}$, $\Delta Z_t^{(n)} = Z_t^{(n)} \ominus Z_t^{(n-1)}$
- Argue that (with $M_{r,T}^{(n)} := \int_r^T Z^{(n)} \circ dB_s$)

$$\Delta Y_t^{(n)} + \Delta M_{t,T}^{(n)} = \int_t^T [F(s, Y_s^{(n-1)}) \ominus F(s, Y_s^{(n-2)})]ds.$$

- With some assumptions on F one shows that

$$\mathbb{E}\|\triangle Y_t^{(n)}\|^2 \leq \|\Delta Y_t^{(n)} + \triangle M_{t,T}^{(n)}\|^2 \leq TK^2 \int_t^T \mathbb{E}\|\triangle Y_s^{(n-1)}\|^2 ds.$$

$$\Longrightarrow \mathbb{E}\|\Delta Y_t^{(n)}\|^2 \leq \frac{CK^{n-1}T^{2(n-1)}}{(n-1)!} =: a_n^2, \text{ where } \sum_{n=0}^{\infty} a_n < \infty.$$

Finally ...

- Using properties of Hukuhara difference to show that

$$\|Y_t^{(n)} \ominus Y_t^{(m)}\|_H \le \sum_{k=m}^{n-1} \|\Delta Y_t^{(k)}\|_H \le \sum_{k=m}^{n-1} a_k.$$

- Since $h(A, B) \le \|A \ominus B\|$, $\{Y_t^{(n)}\}$ is Cauchy in $\mathscr{L}^2_{\mathcal{F}_T}(\Omega; \mathscr{X})$, hence $\exists Y \in \mathscr{L}^2_{\mathcal{F}_T}(\Omega; \mathscr{X})$, such that

$$\sup_{t\in[0,T]} \|Y_t^{(n-1)} \ominus Y_t\|_H^2 \le \sum_{k=n-1}^{\infty} a_k \to 0, \qquad n \to \infty.$$

- Argue that Y satisfies the SVBSDE

$$Y_t = \xi + \mathbb{E}\Big[\int_t^T f(s, Y_s)ds \Big| \mathcal{F}_t\Big], \quad t \in \mathbb{T} \implies \textit{Done!}$$

Some hidden subtleties

- Existence of $\Delta Y^{(n)}$ and $\Delta Z^{(n)}$?
  - $\Delta Y^{(n-1)}$ exists $\Longrightarrow \Delta[F(t, Y_t^{(n-1)})]$ exists.
  - $\Delta Y_t^{(n)} + \Delta M_t^{(n)} = \int_t^T [\Delta F(s, Y_s^{(n-1)})]ds$
  - (MRT) $\Delta M_t^{(n)} = \int_0^t \Delta Z^{(n)} \circ dB_s$.
- Assume $Y^{(n)} \ominus Y^{(m)}$ exists $\forall n, m \oplus \lim_{n,m} \| Y^{(n)} \ominus Y^{(m)} \| = 0$. (I.e., $\{Y^{(n)}\}$ is Cauchy in $(\mathscr{X}, h)$). Do we actually know that $\exists Y$ such that $Y^{(n)} \ominus Y$ exists for all n?

Proposition (Ararat-M.-Wu, '21)

Assume that $\{A_n\}_{n\geq 1}, A, B \in \mathscr{K}(\mathbb{R}^d)$, $h(A_n, A) \to 0$, as $n \to \infty$. and $A_n \ominus B$ exists for all n. Then, $A \ominus B$ exists.

Some Serious Issues

Two theorems from Kisielewicz's 2020 book (Springer) :

Unboundedness of Aumann-Itô SV-Integrals (Corollary 5.3.2)

For every nonempty decomposable set $K \subset \mathbb{L}^2(\mathbb{T}, \mathscr{X})$ and every $0 \leq s < t \leq T$, the Itô set-valued integral $\int_s^t K_s dB_s$ is square integrably bounded *if and only if K is a singleton*.

Decomposition of Unity (Lemma 3.3.4)

For every set $K \subset \mathbb{L}^2(\mathbb{T} \times \Omega, \mathscr{X})$, and every partition $\pi : 0 = \tau_0 < \tau_1 < \cdots \tau_n = T$ of $[0, T]$, it holds that

$$K \subseteq \mathbf{1}_{[0,\tau_1]}K + \mathbf{1}_{(\tau_1,\tau_2]}K + \cdots + \mathbf{1}_{(\tau_{n-1},T]}K.$$

If K is *decomposable*, then the equality holds.

Also ...

A result from recent paper by J.P. Zhang and Kouji Yano (2020) :

Singleton Test (Lemma 3.1)

For any set-valued random variable $F \in L^1(\Omega, \mathcal{K}(\mathscr{X}))$ and any $a \in \mathscr{X}$, the expectation $\mathbb{E}(F) = \{a\}$ if and only if F degenerates to a random singleton $\{f\}$ with $\mathbb{E}(f) = a$.

Representability of Set-Valued Martingale (Theorem 3.1)

Let M be a set-valued $\mathbb{F}$-martingale. Then

$$M_t = \mathbb{E}[M_0] + \int_0^t G_s dB_s \quad \Longleftrightarrow \quad M_t = C + \int_0^t g_s dB_s,$$

for some $g \in \mathbb{L}^2_{\mathbb{F}}(\Omega; \mathscr{X})$ and $C \in \mathcal{K}(\mathscr{X})$.

Some Bad News

- Kisielewicz' set-valued MRT essentially holds only for vector-valued martingales (singleton), since $M_0 = \{0\}$;
- If MRT holds and M is square-integrably bounded, then Z cannot be measurable/decomposible unless it is a singleton(!).
- But a generalized integral is NOT temporally additive, that is,

$$\int_0^T Z \circ dB_s \subsetneq \int_0^t Z \circ dB_s + \int_t^T Z \circ dB_s,$$

thus the whole scheme would collapse in general!

The Remedies

- Consider only BSDEs without Z
- Use other set-valued stochastic integrals (e.g., "trajectory integrals") that are easily temporally additive
  - The danger : the integral may not be a set-valued process, therefore even more hopeless for MRT.
- Find new definition of set-valued stochastic integrals that are
  - non-singleton expected value (truly set-valued) ;
  - temporally additive ; and
  - MRT compatible !

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Outline

What do we know

- Aumann-Itô : $\underbrace{\int \underbrace{Z_s}_{decomposable} dB_s}_{decomposable}$ (MRT X, Addtive √)
- Trajectory : $\underbrace{\int \underbrace{Z_s}_{non-decomposable} dB_s}_{non-decomposable}$ (MRT X, Addtive √)
- G-Aumann-Itô : $\underbrace{\int \underbrace{Z_s}_{non-decomposable} \circ dB_s}_{decomposable}$ (MRT√, Addtive X)

Main Delemmas

- If MRT holds, then the integrand Z cannot be decomposable ;
- If integrand Z is not decomposable, then the additivity fails ;
- If the initial value M_0 is a singleton (e.g., $\{0\}$), then the martingale must be degenerate (singleton) ;
- If the MRT holds with a standard Aumman-Itô integral, then the martingale must be a constant set "pushed" by a vector-valued martingale.

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First Try (Aarat-M.-Wu, '21)

- Consider $\mathbb{R}_t := \mathbb{L}^2_{\mathcal{F}_t}(\Omega, \mathbb{R}^d) \times \mathbb{L}^2_{\mathbb{F}}([t, T] \times \Omega, \mathbb{R}^{d\times m})$, $t \in \mathbb{T}$.
- Define a mapping $F^t \colon \mathbb{R}_0 \mapsto \mathbb{R}_t$ by

$$F^t(x, z) := \Big(x + \int_0^t z_s dB_s, z^t\Big), \quad (x, z) \in \mathbb{R}_0,$$

where $z^t := (z_u)_{u\in[t,T]}$, the restriction of z onto $[t, T]$.

- For $t \in [0, T]$ and $(\xi, z^t) \in \mathbb{R}_t$, define, for $u \in [0, T]$,

$$\mathcal{J}^t_u(\xi, z^t) := \mathbb{E}[\xi|\mathcal{F}_u]\mathbf{1}_{[0,t)}(u) + \Big\{\xi + \int_t^u z^t_s dB_s\Big\}\mathbf{1}_{[t,T]}(u).$$

Then, for any $t \in \mathbb{T}$, $\mathcal{J}^t(\xi, z^t)$ is an $\mathbb{F}$-martingale on $\mathbb{T}$,

- (Time Consistency) $\mathcal{J}^t \circ F^t = \mathcal{J}^s \circ F^s = \mathcal{J}^0$ on $\mathbb{R}_0$, $s, t \in \mathbb{T}$.

- For $\mathcal{R} \subset \mathbb{R}_0$, $t \geq 0$, $\exists$ SV r.v $I_0^t(\mathcal{R}) \in \mathbb{L}^2_{\mathcal{F}_t}(\Omega, \mathscr{C}(\mathbb{R}^d))$, s.t.

$$S^2_{\mathcal{F}_t}(I_0^t(\mathcal{R})) = \overline{\text{dec}}_{\mathcal{F}_t}(\mathcal{J}_t^0[\mathcal{R}]). \tag{9}$$

  - We call $I_0^t(\mathcal{R}) := \int_{0-}^t \mathcal{R} \circ dB$ the *generalized stoch. integral*.
  - This integral tracks the initial values of the mg's in $\mathcal{J}^0[\mathcal{R}]$(!).
- Let M be a $\mathbb{L}^2$-SV mg, and $MS(M)$ its ($\mathbb{L}^2$-)mg selectors. By standard MRT we can define, for each $t \in \mathbb{T}$,

$$\mathcal{R}_t^M := \{(\xi, z) \in \mathbb{R}_t : \mathcal{J}^t(\xi, z) \in MS(M)\}; \quad \mathcal{R}^M := \mathcal{R}_0^M.$$

Theorem (MRT, Ararat-M.-Wu, 2021)

$$M_t = \int_{0-}^t \mathcal{R}^M \circ dB, \quad \mathbb{P}\text{-a.s.}, t \in \mathbb{T}.$$

Moreover, $S^2_{\mathcal{F}_t}(M_t) = \overline{\text{dec}}_{\mathcal{F}_t}(P_t[MS(M)]) = \overline{\text{dec}}_{\mathcal{F}_t}(\mathcal{J}_t^0[\mathcal{R}^M])$.

Connection with SV BSDEs

- The definition of the new stochastic SV integral can lead to a representation of the SVBSDE (6) :

$$Y_t + \int_{0-}^{T} \mathcal{Z} \circ dB = \xi + \int_t^T f(s, Y_s) ds + \int_{0-}^{t} \mathcal{Z} \circ dB. \quad (10)$$

where the pair $(Y, \mathcal{Z}) \in Y \in \mathscr{L}^2_{\mathbb{F}}([0, T] \times \Omega, \mathscr{K}(\mathbb{R}^d)) \times \mathbb{R}_0$ can be defined as the solution, if $Y_0 = \pi_\xi[\mathcal{Z}]$ and one can argue that such solution $(Y, \mathcal{Z})$ exists and is unique.

The Main Issues

- The representation is only defined "a.s." for each $t \in \mathbb{T}$.
- No "path-regularity" for the (*indefinite*) integral $t \mapsto I_0^t(\mathcal{R})$
- **"Temporal additivity" ?** $(M_t \overset{?}{=} "I_{s-}^t(\mathcal{R}^M)" + M_s)$

Indefinite Integrals — A "Path" View

- Recall : for $\mathcal{R} \subset \mathbb{R}_0$, $\mathcal{J}^0[\mathcal{R}] \subset \mathbb{L}^2_{\mathcal{F}_T}(\Omega, \mathbb{C}^d_T)$.
- $I^{\mathcal{R}} = I_0^T(\mathcal{R}) \in \mathscr{M}(\Omega; \mathcal{C}(\mathbb{C}^d_T))$, s.t. $S^2_{\mathcal{F}_T}(I^{\mathcal{R}}) = \overline{\text{dec}}_{\mathcal{F}_T}(\mathcal{J}^0[\mathcal{R}])$.
- Define, for $t \in \mathbb{T}$, $I^{\mathcal{R}}_t(\omega) := \overline{P_t[I^{\mathcal{R}}(\omega)]}, \quad \omega \in \Omega$.
  - Note that $I^{\mathcal{R}}(\omega) \subset \mathbb{C}^d_T$ is closed, bdd, but not necessarily weakly compact, thus $P_t[I^{\mathcal{R}}(\omega)]$ is not necessarily closed(!).

Proposition (Ararat-M. (2022))

- For each $t \in \mathbb{T}$, $I^{\mathcal{R}}_t$ is an $\mathcal{F}_T$-measurable SV random variable.
- The set-valued mapping $(t, \omega) \mapsto I^{\mathcal{R}}_t(\omega)$ on $\mathbb{T} \times \Omega$ is $\mathscr{B}(\mathbb{T}) \otimes \mathcal{F}_T$-measurable.
- $S^2_{\mathcal{F}_T}(I^{\mathcal{R}}_t) = \text{cl}_{\mathbb{L}^2(\Omega)}\{y_t : y \in S^2_{\mathcal{F}_T}(I^{\mathcal{R}})\}$.

Moreover ...

- For $t \in \mathbb{T}$, define the $\mathbb{E}[I_t^{\mathcal{R}}|\mathcal{F}_t] : \Omega \to \mathcal{C}(\mathbb{R}^d)$ s.t.

$$S^2_{\mathcal{F}_t}(\mathbb{E}[I_t^{\mathcal{R}}|\mathcal{F}_t]) = \overline{\text{dec}}_{\mathcal{F}_t}\{\mathbb{E}[\xi|\mathcal{F}_t] : \xi \in S^2_{\mathcal{F}_T}(I_t^{\mathcal{R}})\}$$
$$= \mathrm{cl}_{\mathbb{L}^2}\{\mathbb{E}[\xi|\mathcal{F}_t] : \xi \in S^2_{\mathcal{F}_T}(I_t^{\mathcal{R}})\}.$$

- (R. Wang (2001)) $\exists!$ *optional set-valued process* $({}^{o}I_t^{\mathcal{R}})_{t\in[0,T]}$ s.t. $\mathbb{E}[I_\tau^{\mathcal{R}}|\mathcal{F}_\tau] = {}^{o}I_\tau^{\mathcal{R}}$, a.s., for every $\mathbb{F}$-stopping time τ. ,

Theorem (Ararat-M. (2022))

- For $t \in \mathbb{T}$, $S^2_{\mathcal{F}_t}({}^{0}I_t^{\mathcal{R}}) = \overline{\text{dec}}_{\mathcal{F}_t}(\mathcal{J}_t^0[\mathcal{R}])$.
- ${}^{o}I_t^{\mathcal{R}} = \mathbb{E}[I_t^{\mathcal{R}}|\mathcal{F}_t] = \int_{0-}^t \mathcal{R} \circ dB \quad \mathbb{P}\text{-a.s.}$
- Let M be a convex, $\mathbb{L}^2$-integrably bounded SV mg. Then $M_t = \mathbb{E}[I_t^{\mathcal{R}^M}|\mathcal{F}_t] = {}^{o}I_t^{\mathcal{R}^M}$, $\mathbb{P}$-a.s.

Temporal Additivity of Indefinite Integrals

- Let $\Pi_n = \{\pi = (t_i)_{i=0}^n\}$ be the set of partitions on $\mathbb{T}$.
- Let $\mathcal{Y} \subset \mathbb{L}^0(\mathbb{T}, \mathbb{R}^d)$. For $\pi \in \Pi_n$ and $\{y^i\}_{i=0}^n \subset \mathcal{Y}$, define

$$y_t^\pi = y_t^0 \mathbf{1}_{\{0\}}(t) + \sum_{i=1}^n y_t^i \mathbf{1}_{(t_{i-1}, t_i]}(t), \;\; t \in \mathbb{T}.$$

- $\mathcal{Y}$ is called *temporally decomposable* if $\forall n \in \mathbb{N}$, $\pi \in \Pi_n$, and $\{y^i\}_{i=0}^n \subset \mathcal{Y} \Longrightarrow y^\pi \in \mathcal{Y}$.
  - Denote the "temporally decomposable hull" of $\mathcal{Y}$ by $\text{temp}(\mathcal{Y})$.
  - For $A \in \mathscr{B}(\mathbb{T})$, $\mathcal{Y} \subset \mathcal{Y}\mathbf{1}_A + \mathcal{Y}\mathbf{1}_{A^c}$ ("=" if $\mathcal{Y}$ is t-decomposable)
  - $\text{temp}(\mathcal{Y}) = \text{temp}(\mathcal{Y}\mathbf{1}_{[0,t]}) + \text{temp}(\mathcal{Y}\mathbf{1}_{(t,T]})$.

Hope

- Define $I^{\mathcal{R}} \in \mathscr{M}(\Omega, \mathcal{C}(\mathbb{D}_T^-))$ s.t. $S^2_{\mathcal{F}_T}(I^{\mathcal{R}}) = \overline{\text{dec}}(\text{temp}(\mathcal{J}_0[\mathcal{R}]))$.
- $t \mapsto {}^o I_t^{\mathcal{R}} := \mathbb{E}_t[\overline{P_t(I^{\mathcal{R}})}]$ is *temp-additive*, and $M_t = {}^o I_t^{\mathcal{R}^M}$, a.s.

Main Difficulties

- **Observe** : $I^{\mathcal{R}}$ will now be $\mathcal{C}(\mathbb{D}_T^-)$ -valued, instead of $\mathcal{C}(\mathbb{C}_T^d)$-valued, to suit the temporal decomposability.
- **Uniform topology or Skorokhod topology ?**
  - $\mathbb{D}_T^-$ with Skorohod topology is *not a topological vector space*
  - $(\mathbb{D}_T^-, \|\cdot\|_\infty)$ is a Banach Space but *not separable*
- **Why separable ?**
  - needed for, e.g., measurability $\Longleftrightarrow$ decomposibility

Some Discoveries

- $(\mathbb{D}_T^-, \|\cdot\|_\infty)^*$ is weak*-separable(!)
- Scalar measurability, Aumann-Pettis integration theory , ..., etc., for multi-functions on non-separable Banach spaces

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Pettis Integration

- Let $\mathbb{X}$ be a Banach space. A function $f : (\Omega, \mathcal{F}, \mu) \mapsto \mathbb{X}$ is called *scalarly measurable/integrable* if $\omega \mapsto \langle x^*, f(\omega) \rangle$ is measurable/integrable for each $x^* \in \mathbb{X}^*$.
- f is called *Pettis integrable* if it is scalarly integrable $\oplus$ $\forall A \in \mathcal{F}$, $\exists \int_A f d\mu \in \mathbb{X}$ such that

$$\langle x^*, \int_A f d\mu \rangle = \int_A \langle x^*, f \rangle d\mu, \quad x^* \in \mathbb{X}^*.$$

- **Note :** If $\mathbb{X}$ is separable, then f is Pettis integrable $\Longleftrightarrow$ $\{\langle x^*, f \rangle : x^* \in \mathbb{B}_{\mathbb{X}^*}\}$ is uniformly integrable. But If $\mathbb{X}$ is nonseparable, then the equivalence holds only when $\mathbb{X}$ has the *μ-Pettis Integral Property (μ-PIP)* (i.e., every scalarly bounded and measurable function is Pettis integrable).

The space $\mathbb{X} = \mathbb{D}_T^-$, while non-separable, is μ-PIP(!)

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The space $\mathbb{X} = \mathbb{D}^-_T$, while non-separable, is μ-PIP(!)

The Space $Pe(\mu, \mathbb{X})$

- Let $Pe(\mu, \mathbb{X})$ be the space of all Pettis integrable $\mathbb{X}$-valued functions, with the *strong topology* defined by the norm :

$$\|f\|_{Pe} := \sup_{x^* \in \mathbb{B}_{\mathbb{X}^*}} \int_\Omega |\langle x^*, f(\omega) \rangle| \mu(d\omega).$$

- We identify $(Pe(\mu, \mathbb{X}))^* \simeq \mathbb{L}^\infty(\mu) \times \mathbb{X}^*$, in the sense that the dual product on $Pe(\mu, \mathbb{X})$ is defined by the linear mapping :

$$f \mapsto \langle h \otimes x^*, f \rangle_{Pe} := \int_\Omega h(\omega) \langle x^*, f(\omega) \rangle \mu(d\omega).$$

- For $M \subset Pe(\mu, \mathbb{X})$, we denote by $\mathrm{cl}_{Pe}(M)$ ($\mathrm{cl}^w_{Pe}(M)$) the closure of M w.r.t. the *strong* (*weak*) topology on $Pe(\mu, \mathbb{X})$.
- The space $Pe(\mu, \mathbb{X})$ with its weak topology is a Hausdorff topological vector space.

Aumann-Pettis Set-Valued Integrals

- $\mathbb{X}$ – a Banach space such that $\mathbb{X}^*$ is weak* separable.
  - $\mathscr{C}_w(\mathbb{X})$ — weakly closed subsets of $\mathbb{X}$,
  - $\mathscr{K}_w(\mathbb{X})$ — convex weakly compact subsets of $\mathbb{X}$.
- For $F\colon (\Omega, \mathcal{F}, \mu) \to \mathscr{C}_w(\mathbb{X})$, let $S_{Pe}(F) := S(F) \cap Pe(\mu, \mathbb{X})$, and $\omega \mapsto \delta^*(x^*, F(\omega)) := \sup\{\langle x^*, x \rangle : x \in F(\omega)\}$, $\omega \in \Omega$, be the *support function* of F.
- F is called *scalarly measurable/integrable* if the support function $\delta^*(x^*,F(\cdot))\in\mathbb{L}^0(\Omega)/\mathbb{L}^1(\Omega)$, $x^* \in \mathbb{X}^*$.
- F is called *Aumann-Pettis integrable* if $S_{Pe}(F) \neq \emptyset$.
- F is called *Pettis integrable* if it is scalarly integrable and $\forall A \in \mathcal{F}$, $\exists \int_A F d\mu \in \mathscr{K}_w(\mathbb{X})$ such that

$$\delta^*\Big(x^*, \int_A F d\mu\Big) = \int_A \delta^*(x^*, F) d\mu, \qquad x^* \in \mathbb{X}^*.$$

Known Results (Cascales-Ladets-Rodríguez, '10)

Assume that $\mathbb{X}$ in non-separable, but $\mathbb{X}^*$ is weak* separable. Let $F\colon \Omega \to \mathscr{K}_w(\mathbb{X})$ be a set-valued function. (Think : $\mathbb{X} = \mathbb{D}_T^-$!)

- F is Pettis integrable $\Longleftrightarrow$ every scalarly measurable selector of F is Pettis integrable.
- F is Pettis integrable $\Longrightarrow \exists \{f_n\}_{n\in\mathbb{N}} \subset S_{Pe}(F)$ such that

$$F(\omega) = \mathrm{cl}^w_{\mathbb{X}}\{f_n(\omega)\colon n \in \mathbb{N}\}, \quad \omega \in \Omega.$$

  - Such a property is known as Castaing Representation
  - As a consequence, "Pettis" $\Longrightarrow$ "Aumann-Pettis" !
- F is Pettis integrable $\Longrightarrow \{\delta^*(x^*, F)\colon x^* \in \mathbb{B}_{\mathbb{X}^*}\}$ is uniformly integrable. The converse holds if $\mathbb{X}$ has μ-PIP.

Moreover ...

Theorem (Ararat-M. (2022))

Assume that $\mathbb{X}$ is non-separable, but $\mathbb{X}^*$ is weak*-separable. Assume further that $B_{\mathbb{X}^*}$ (unit ball in $\mathbb{X}^*$) is weak*-separable. Let $M \subset Pe(\mu, \mathbb{X})$ be weakly closed decomposable set. Then

- If $M = \mathrm{cl}^w_{Pe}\,\mathrm{dec}\{f_n : n \in \mathbb{N}\}$ for some sequence $(f_n)_{n\in\mathbb{N}}$ in $Pe(\mu, \mathbb{X})$, then there exists an Aumann-Pettis integrable SV function $F : \Omega \to \mathscr{C}_w(\mathbb{X})$ such that $M = \mathrm{cl}^w_{Pe} S_{Pe}(F)$.
- If $M = \mathrm{cl}^w_{Pe} S_{Pe}(F)$ for some Pettis integrable set-valued function $F : \Omega \to \mathscr{K}_w(\mathbb{X})$, then there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $Pe(\mu, \mathbb{X})$ such that $M = \mathrm{cl}^w_{Pe}\,\mathrm{dec}\{f_n : n \in \mathbb{N}\}$ and $F(\omega) = \mathrm{cl}^w_{\mathbb{X}}\{f_n(\omega) : n \in \mathbb{N}\}$ for every $\omega \in \Omega$.

Temporally Additive SV Stochastic Integral

Given $(\Omega, \mathcal{F}, \mathbb{P})$, consider $\mathcal{J}^0\colon \mathbb{R}_0 \to \mathbb{L}^2_{\mathcal{F}_T}(\Omega, \mathbb{C}_T) \subset Pe_{\mathcal{F}_T}(\mathbb{P}, \mathbb{D}^-_T)$.

- If $\mathcal{R} \subset \mathbb{R}_0$, then $\exists\{(x^n, z^n)\}_{n\in\mathbb{N}} \subset \mathcal{R}$, such that
  - $\mathrm{cl}\mathcal{R} = \mathrm{cl}\{(x^n, z^n)\colon n \in \mathbb{N}\}$
  - $\mathrm{cl}\mathcal{J}^0[\mathcal{R}] = \mathrm{cl}\{\mathcal{J}^0(x^n, z^n)\colon n \in \mathbb{N}\}$
  - $\mathrm{cl}^w_{Pe}\mathcal{J}^0[\mathcal{R}] = \mathrm{cl}^w_{Pe}\{\mathcal{J}^0(x^n, z^n)\}_{n\in\mathbb{N}}$,
  - $\mathrm{cl}^w_{Pe}\,\mathrm{dec}\,\mathrm{temp}_{\mathbb{Q}}\,\mathcal{J}^0[\mathcal{R}] = \mathrm{cl}^w_{Pe}\,\mathrm{dec}\,\mathrm{temp}_{\mathbb{Q}}\{\mathcal{J}^0(x^n, z^n)\}_{n\in\mathbb{N}}$
- Since $\mathrm{temp}_{\mathbb{Q}}\{\mathcal{J}^0(x^n, z^n)\colon n \in \mathbb{N}\}$ is countable, there exists an Aumann-Pettis integrable SV r.v. $\Phi^{\mathcal{R}}\colon \Omega \to \mathscr{C}_w(\mathbb{D}^-_T)$ such that

$$\mathrm{cl}^w_{Pe}\,\mathrm{dec}\,\mathrm{temp}_{\mathbb{Q}}\,\mathcal{J}^0[\mathcal{R}] = \mathrm{cl}^w_{Pe} S_{Pe}(\Phi^{\mathcal{R}}).$$

- We call $\Phi^{\mathcal{R}}$ the *stochastic Aumann-Pettis integral of $\mathcal{R}$* and denote it by $\Phi^{\mathcal{R}} := \fint_{0-}^{T} \mathcal{R} \circ dB$.

Temporally Additive Indefinite Integral

- For $t \in \mathbb{T}$, let $J^t(\mathcal{R}) := \mathcal{J}^0_{t\wedge\cdot}(\mathcal{R})$; $J^t_{\mathbb{Q}}(\mathcal{R}) := \bigcup_{q\in\mathbb{Q},q<t} J^q[\mathcal{R}]$
- $\mathrm{cl}^w_{Pe} \operatorname{dec} \operatorname{temp}_{\mathbb{Q}} J^t_{\mathbb{Q}}[\mathcal{R}] = \mathrm{cl}^w_{Pe} \operatorname{dec} \operatorname{temp}_{\mathbb{Q}} \bigcup_{q\in\mathbb{Q},q<t} \{J^q(x^n, z^n)\}_{n\in\mathbb{N}}$.
- $\exists$ Aumann-Pettis integrable SV r.v. $\tilde{\Phi}^t(\mathcal{R})\colon \Omega \to \mathscr{C}_w(\mathbb{D}^-_T)$ s.t. $\mathrm{cl}^w_{Pe} S_{Pe}(\tilde{\Phi}^t(\mathcal{R})) = \mathrm{cl}^w_{Pe} \operatorname{dec} \operatorname{temp}_{\mathbb{Q}} J^t_{\mathbb{Q}}[\mathcal{R}]$.
- Similarly, for $s < t$, define $J^t_s(\mathbb{R}_s) := \mathcal{J}^0_{t\wedge\cdot} \circ F^{-s}(\mathbb{R}_s)$, where $F^{-s} : \mathbb{R}_s \ni (\xi, z) \mapsto (\mathbb{E}[\xi], z^s \oplus z) \in \mathbb{R}_0$, $\xi = \mathbb{E}[\xi] + \int_0^s z^s_u dB_u$.
- $\exists$ Aumann-Pettis integrable $\tilde{\Phi}^t_s(\mathcal{R})\colon \Omega \to \mathscr{C}_w(\mathbb{D}^-_T)$ s.t. $\mathrm{cl}^w_{Pe} S_{Pe}(\tilde{\Phi}^t_s(\mathcal{R})) = \mathrm{cl}^w_{Pe} \operatorname{dec} \operatorname{temp}_{\mathbb{Q}} J^t_s[\mathcal{R}]$.
- Denote $\tilde{\Phi}^t_s(\mathcal{R}) = \fint_{s-}^t \mathcal{R} \circ dB$, $0 \le s < t \le T$, we expect

$$\fint_{0-}^t \mathcal{R} \circ dB = \fint_{s-}^t \mathcal{R} \circ dB + \fint_{0-}^s \mathcal{R} \circ dB.$$

Important Facts

- $\mathbb{D}_T^-$ is a *Banach space* (hence a topological vector space)
- $(\mathbb{D}_T^-)^*$ is weak* separable and $\mathbb{P}$-*PIP* for any $\mathbb{P} \in \mathcal{P}(\mathbb{D}_T^-)$
- $B_{(\mathbb{D}_T^-)^*}$ is weak* separable (not trivial !)
- $\text{temp}(\mathcal{Y}) = \text{temp}(\mathcal{Y}\mathbf{1}_{[0,t)}) + \text{temp}(\mathcal{Y}\mathbf{1}_{[t,T]})$
- $dec(A+B) = dec(A) + dec(B), \qquad A, B \in \mathscr{C}_w(\mathbb{D}_T^-)$
- $\mathrm{cl}^w_{Pe}(A+B) = \mathrm{cl}^w_{Pe}(A) + \mathrm{cl}^w_{Pe}(B), \qquad A, B \in \mathscr{K}_w(\mathbb{D}_T^-)$
- $S_{Pe}(X+Y) = S_{Pe}(X) + S_{Pe}(Y), \qquad X, Y \in \mathscr{M}(\mathbb{T}, \mathbb{D}_T^-)$.
- (Time Consistency)
  - $\mathcal{J}^t \circ F^t = \mathcal{J}^s \circ F^s = \mathcal{J}^0$ on $\mathbb{R}_0$, $s, t \in \mathbb{T}$.
  - $F^t[\mathcal{R}_0] = \mathcal{R}_t$, $F^{-t}[\mathcal{R}_t] = \mathcal{R}_0$
- … … … …

A Story ...

Weak* separability of dual unit ball of D[0,1]

Asked 7 days ago Modified 4 days ago Viewed 95 times

5

Let $D[0,1]$ be the space of all right-continuous left-limited functions $f\colon [0,1] \to \mathbb{R}$ equipped with the supremum norm $f \mapsto \|f\|_\infty = \sup_{t\in[0,1]} |f(t)|$. This is a non-separable Banach space whose dual $D[0,1]^*$ is known to be separable in the weak* topology; see, e.g., Chapter 41, p. 1756 of

Johnson, W. B. (ed.); Lindenstrauss, J. (ed.), Handbook of the geometry of Banach spaces. Volume 2, Amsterdam: North-Holland. xii, 1007-1866 (2003). ZBL1013.46001.

Is the unit ball in $D[0,1]^*$ separable in the weak* topology?

functional-analysis banach-spaces weak-topology

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asked May 19 at 13:39
Çağın Ararat
53 ▲1

[The question was posted on the website : *Mathstackexchange*.]

A Story ...

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A WEAK* SEPARABLE $C(K)^*$ SPACE WHOSE UNIT BALL IS NOT WEAK* SEPARABLE

A. AVILÉS, G. PLEBANEK, AND J. RODRÍGUEZ

ABSTRACT. We provide a ZFC example of a compact space K such that $C(K)^*$ is w^*-separable but its closed unit ball $B_{C(K)^*}$ is not w^*-separable. All previous examples of such kind had been constructed under CH. We also discuss the measurability of the supremum norm on that $C(K)$ equipped with its weak Baire σ-algebra.

1. INTRODUCTION

Let K be a compact space (all our topological spaces are assumed to be Hausdorff) and let $C(K)$ be the Banach space of all continuous real-valued functions on K (equipped with the supremum norm). One can consider the following list of properties related to the separability in K and $C(K)^*$:

A Story ...

1 Answer Sorted by: Highest score (default)

1

$(D([0,1]), \|\cdot\|_\infty)$ is a commutative C^*-algebra, so it is isometrically isomorphic to $C(\Delta)$ by the Gelfand map, where Δ is the character space of $D([0,1])$.

Let $h_{1+} \in \Delta$ be defined by $h_{1+}(f) = f(1)$ for all $f \in (D([0,1])$. Every $h \in \Delta \setminus \{h_{1+}\}$ is either of the form

$$\forall f \in D([0,1]) \quad h_{c+}(f) = f(c+) = \lim_{x \to c+} f(x)$$

for some $c \in [0,1)$ or of the form

$$\forall f \in D([0,1]) \quad h_{c-}(f) = f(c-) = \lim_{x \to c-} f(x)$$

for some $c \in (0,1]$. Let $K = \{(c,1) : c \in [0,1]\} \cup \{(c,-1) : c \in (0,1]\}$ with the weak parallel line topology. It is relatively straightforward to show that Δ is homeomorphic to K. K is a separable, compact, Hausdorff space.

Since Δ and K are homeomorphic, $C(\Delta)$ and $C(K)$ are isometrically isomorphic as Banach spaces. Lastly, see the implications in the first page of the paper https://doi.org/10.48550/arXiv.1112.5710 : since K is separable, the unit ball of $(C(K))^*$ is weak* separable.

Conclusion

- We argue that the current set-valued stochastic analysis may have fundamental difficulties in studying a set-valued BSDE.
- A successful remedy might have to contain a new notion of set-valued stochastic integrals that satisfies following requirements :
  - It is a set-valued process (decomposable) allowing non-decomposable integrand ;
  - it is temporally additive ; and
  - It permits a (set-valued) martingale representation theorem.
- We proposed possible new definition of set-valued stochastic integral that meets the desired the properties.
- There is a long, but hopeful way ahead...

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THANK YOU VERY MUCH !