FBSDEs: Initiation, Development, and Beyond

Jiongmin Yong

(University of Central Florida)

June, 2022

(Celebrating Jin Ma's 65th Birthday)

Outline

0. Pre-History

1. BSDEs and the Initiation of FBSDEs

2. Stochastic Optimal Control Method

3. The Four-Step Scheme

4. Black's Conjecture

5. BSPDEs

6. Beyond

0. Pre-History

0. Pre-History

1978, Fudan University

0. Pre-History

1978, Fudan University

1982–1985, supervised by Xunjing Li

0. Pre-History

1978, Fudan University

1982–1985, supervised by Xunjing Li

1982–1983, working under Xunjing Li

0. Pre-History

1978, Fudan University

1982–1985, supervised by Xunjing Li

1982–1983, working under Xunjing Li

1983–1986, Purdue, under Leonard D. Berkovitz;

0. Pre-History

1978, Fudan University

1982–1985, supervised by Xunjing Li

1982–1983, working under Xunjing Li

1983–1986, Purdue, under Leonard D. Berkovitz;

(1988–2003, Fudan, Xunjing Li)

0. Pre-History

1978, Fudan University

1982–1985, supervised by Xunjing Li

1982–1983, working under Xunjing Li

1983–1986, Purdue, under Leonard D. Berkovitz;

(1988–2003, Fudan, Xunjing Li)

1987–1992, U. of Minnesota, under Naresh Jain

0. Pre-History

1978, Fudan University

1982–1985, supervised by Xunjing Li

1982–1983, working under Xunjing Li

1983–1986, Purdue, under Leonard D. Berkovitz;

(1988–2003, Fudan, Xunjing Li)

1987–1992, U. of Minnesota, under Naresh Jain

1992, Purdue and U. of Minnesota, IMA "Control Year"
Avner Friedman

1. BSDEs and the Initiation of FBSDEs

1. BSDEs and the Initiation of FBSDEs

- 1972, Bismut introduced linear BSDEs.

1. BSDEs and the Initiation of FBSDEs

- 1972, Bismut introduced linear BSDEs.
- 1990, Pardoux–Peng initiated nonlinear BSDEs.

$$Y(t) = \xi + \int_t^T g(s, Y(s), Z(s))ds - \int_t^T Z(s)dW(s), \qquad t \in [0, T], \tag{1}$$

1. BSDEs and the Initiation of FBSDEs

- 1972, Bismut introduced linear BSDEs.
- 1990, Pardoux–Peng initiated nonlinear BSDEs.

$$Y(t) = \xi + \int_t^T g(s, Y(s), Z(s))ds - \int_t^T Z(s)dW(s), \qquad t \in [0, T], \tag{1}$$

Theorem (Pardoux–Peng) *Under proper conditions,* $\xi \in L^p_{\mathcal{F}_T}(\Omega; \mathbb{R}^m)$, *BSDE* (1) *admits a unique adapted solution* $(Y(\cdot), Z(\cdot))$.

1. BSDEs and the Initiation of FBSDEs

- 1972, Bismut introduced linear BSDEs.
- 1990, Pardoux–Peng initiated nonlinear BSDEs.

$$Y(t) = \xi + \int_t^T g(s, Y(s), Z(s))ds - \int_t^T Z(s)dW(s), \qquad t \in [0, T], \tag{1}$$

Theorem (Pardoux–Peng) *Under proper conditions,* $\xi \in L^p_{\mathcal{F}_T}(\Omega; \mathbb{R}^m)$, *BSDE* (1) *admits a unique adapted solution* $(Y(\cdot), Z(\cdot))$.

- Many further development, ...

- Early 1990s, Fabio Antonelli started to investigate following:

$$\begin{cases} U_t = J_t + \displaystyle\int_0^t f_s(U_s, V_s)dA_s, \\ V_t = \mathbb{E}\Big[\displaystyle\int_t^T g_s(U_s, V_s)dC_s + Y \;\Big|\; \mathcal{F}_t \Big]. \end{cases} \tag{2}$$

A, C — finite variation processes
(U, V) — unknown

- Early 1990s, Fabio Antonelli started to investigate following:

$$\begin{cases} U_t = J_t + \int_0^t f_s(U_s, V_s)dA_s, \\ V_t = \mathbb{E}\Big[\int_t^T g_s(U_s, V_s)dC_s + Y \;\Big|\; \mathcal{F}_t \Big]. \end{cases} \tag{2}$$

A, C — finite variation processes
(U, V) — unknown

Theorem (Antonelli) (i) *Let* $(U, V) \mapsto (f(U, V), g(U, V))$ *be Lipschitz continuous with the Lipschitz constant or time duration* T *being small enough. Then,* (2) *admits a unique adapted solution.*

- Early 1990s, Fabio Antonelli started to investigate following:

$$\begin{cases} U_t = J_t + \int_0^t f_s(U_s, V_s) dA_s, \\ V_t = \mathbb{E}\Big[\int_t^T g_s(U_s, V_s) dC_s + Y \;\Big|\; \mathcal{F}_t \Big]. \end{cases} \tag{2}$$

A, C — finite variation processes
(U, V) — unknown

Theorem (Antonelli) (i) *Let* $(U, V) \mapsto (f(U, V), g(U, V))$ *be Lipschitz continuous with the Lipschitz constant or time duration* T *being small enough. Then,* (2) *admits a unique adapted solution.*

(ii) *If the Lipschitz constant or the time duration* T *is not small enough, system* (2) *might not have an adapted solution on* $[0, T]$.

- Early 1990s, Fabio Antonelli started to investigate following:

$$\begin{cases} U_t = J_t + \int_0^t f_s(U_s, V_s)dA_s, \\ V_t = \mathbb{E}\Big[\int_t^T g_s(U_s, V_s)dC_s + Y \;\Big|\; \mathcal{F}_t\Big]. \end{cases} \tag{2}$$

A, C — finite variation processes
(U, V) — unknown

Theorem (Antonelli) (i) *Let $(U, V) \mapsto (f(U, V), g(U, V))$ be Lipschitz continuous with the Lipschitz constant or time duration T being small enough. Then,* (2) *admits a unique adapted solution.*

(ii) *If the Lipschitz constant or the time duration T is not small enough, system* (2) *might not have an adapted solution on* $[0, T]$.

Completed in 1991, published in 1993.

2. Stochastic Optimal Control Method

2. Stochastic Optimal Control Method

- 1992, Jin invited me to join the research,

2. Stochastic Optimal Control Method

- 1992, Jin invited me to join the research,

$$\begin{cases} X(t) = x + \int_0^t b(X(s), Y(s), Z(s))ds \\ \qquad\qquad + \int_0^t \sigma(X(s), Y(s), Z(s))dW(s), \\ Y(t) = g(X(T)) + \int_t^T \widehat{b}(X(s), Y(s), Z(s))ds \\ \qquad\qquad + \int_t^T \widehat{\sigma}(X(s), Y(s), Z(s))dW(s), \end{cases} \tag{3}$$

No restriction on the size of T and the Lipschitz constant.

2. Stochastic Optimal Control Method

- 1992, Jin invited me to join the research,

$$\begin{cases} X(t) = x + \int_0^t b(X(s), Y(s), Z(s))ds \\ \qquad\qquad + \int_0^t \sigma(X(s), Y(s), Z(s))dW(s), \\ Y(t) = g(X(T)) + \int_t^T \widehat{b}(X(s), Y(s), Z(s))ds \\ \qquad\qquad + \int_t^T \widehat{\sigma}(X(s), Y(s), Z(s))dW(s), \end{cases} \tag{3}$$

No restriction on the size of T and the Lipschitz constant.

- FBSDE (3) is a two-point boundary value problem for SDEs.

For ODE case: **shooting method**. Consider:

$$\begin{cases} \dot{x}(t) = b(x(t), y(t)), \\ \dot{y}(t) = \widehat{b}(x(t), y(t)), \\ x(0) = x_0, \qquad y(T) = g(x(T)). \end{cases} \tag{4}$$

For ODE case: **shooting method**. Consider:

$$\begin{cases} \dot{x}(t) = b(x(t), y(t)), \\ \dot{y}(t) = \widehat{b}(x(t), y(t)), \\ x(0) = x_0, \qquad y(T) = g(x(T)). \end{cases} \tag{4}$$

Instead, consider:

$$\begin{cases} \dot{x}(t) = b(x(t), y(t)), \\ \dot{y}(t) = \widehat{b}(x(t), y(t)), \\ x(0) = x_0, \qquad y(0) = y_0, \end{cases} \tag{5}$$

with y_0 being a parameter to be chosen.

For ODE case: **shooting method**. Consider:

$$\begin{cases} \dot{x}(t) = b(x(t), y(t)), \\ \dot{y}(t) = \widehat{b}(x(t), y(t)), \\ x(0) = x_0, \qquad y(T) = g(x(T)). \end{cases} \tag{4}$$

Instead, consider:

$$\begin{cases} \dot{x}(t) = b(x(t), y(t)), \\ \dot{y}(t) = \widehat{b}(x(t), y(t)), \\ x(0) = x_0, \qquad y(0) = y_0, \end{cases} \tag{5}$$

with y_0 being a parameter to be chosen.

(i) For any (x_0, y_0), solve (5) to get $(x(\cdot\,; x_0, y_0), y(\cdot\,; x_0, y_0))$, indicating the dependence on (x_0, y_0).

For ODE case: **shooting method**. Consider:

$$\begin{cases} \dot{x}(t) = b(x(t), y(t)), \\ \dot{y}(t) = \widehat{b}(x(t), y(t)), \\ x(0) = x_0, \qquad y(T) = g(x(T)). \end{cases} \tag{4}$$

Instead, consider:

$$\begin{cases} \dot{x}(t) = b(x(t), y(t)), \\ \dot{y}(t) = \widehat{b}(x(t), y(t)), \\ x(0) = x_0, \qquad y(0) = y_0, \end{cases} \tag{5}$$

with y_0 being a parameter to be chosen.

(i) For any (x_0, y_0), solve (5) to get $(x(\cdot\,; x_0, y_0), y(\cdot\,; x_0, y_0))$, indicating the dependence on (x_0, y_0).

(ii) Select a "bullet" y_0 so that the "target" is hit:

$$y(T; x_0, y_0) = g(x(T; x_0, y_0)).$$

For FBSDE (3), consider

$$\begin{cases} X(r) = x + \int_t^r b(X(s), Y(s), u(s))ds \\ \qquad\qquad + \int_t^r \sigma(X(s), Y(s), u(s))dW(s), \\ Y(r) = y - \int_t^r \widehat{b}(X(s), Y(s), u(s))ds \\ \qquad\qquad - \int_t^r \widehat{\sigma}(X(s), Y(s), u(s))dW(s), \end{cases} \tag{6}$$

with the cost functional

$$J(t, x, y; u(\cdot)) = \mathbb{E}|Y(T) - g(X(T))|^2. \tag{7}$$

For FBSDE (3), consider

$$\begin{cases} X(r) = x + \int_t^r b(X(s), Y(s), u(s))ds \\ \qquad\qquad + \int_t^r \sigma(X(s), Y(s), u(s))dW(s), \\ Y(r) = y - \int_t^r \widehat{b}(X(s), Y(s), u(s))ds \\ \qquad\qquad - \int_t^r \widehat{\sigma}(X(s), Y(s), u(s))dW(s), \end{cases} \tag{6}$$

with the cost functional

$$J(t, x, y; u(\cdot)) = \mathbb{E}|Y(T) - g(X(T))|^2. \tag{7}$$

Problem (C). Find $u(\cdot) \in \mathcal{U}[t, T]$, such that

$$J(t, x, y; \bar{u}(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}[t,T]} J(t, x, y; u(\cdot)) \equiv V(t, x, y). \tag{8}$$

For FBSDE (3), consider

$$\begin{cases} X(r) = x + \int_t^r b(X(s), Y(s), u(s))ds \\ \qquad\qquad + \int_t^r \sigma(X(s), Y(s), u(s))dW(s), \\ Y(r) = y - \int_t^r \widehat{b}(X(s), Y(s), u(s))ds \\ \qquad\qquad - \int_t^r \widehat{\sigma}(X(s), Y(s), u(s))dW(s), \end{cases} \tag{6}$$

with the cost functional

$$J(t,x,y;u(\cdot)) = \mathbb{E}|Y(T) - g(X(T))|^2. \tag{7}$$

Problem (C). Find $u(\cdot) \in \mathcal{U}[t, T]$, such that

$$J(t,x,y;\bar{u}(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}[t,T]} J(t,x,y;u(\cdot)) \equiv V(t,x,y). \tag{8}$$

$V(\cdot\,,\cdot\,,\cdot)$ — **value function** of Problem (C).

• If FBSDE (3) admits an adapted solution (X, Y, Z), then by choosing $y = Y(0)$ and $\bar{u}(\cdot) = Z(\cdot)$,

$$J(0, x, Y(0); \bar{u}(\cdot)) = |Y(T) - g(X(T))|^2 = V(0, x, y) = 0.$$

• If FBSDE (3) admits an adapted solution (X, Y, Z), then by choosing $y = Y(0)$ and $\bar{u}(\cdot) = Z(\cdot)$,

$$J(0, x, Y(0); \bar{u}(\cdot)) = |Y(T) - g(X(T))|^2 = V(0, x, y) = 0.$$

• If Problem (C) admits an optimal triple $(\bar{X}(\cdot), \bar{Y}(\cdot), \bar{u}(\cdot))$ for some $(0, x, y)$ with

$$V(0, x, y) = 0, \tag{9}$$

then $(\bar{X}(\cdot), \bar{Y}(\cdot), \bar{u}(\cdot))$ is an adapted solution of FBSDE (3).

• If FBSDE (3) admits an adapted solution (X, Y, Z), then by choosing $y = Y(0)$ and $\bar{u}(\cdot) = Z(\cdot)$,

$$J(0, x, Y(0); \bar{u}(\cdot)) = |Y(T) - g(X(T))|^2 = V(0, x, y) = 0.$$

• If Problem (C) admits an optimal triple $(\bar{X}(\cdot), \bar{Y}(\cdot), \bar{u}(\cdot))$ for some $(0, x, y)$ with

$$V(0, x, y) = 0, \tag{9}$$

then $(\bar{X}(\cdot), \bar{Y}(\cdot), \bar{u}(\cdot))$ is an adapted solution of FBSDE (3).

Proposition (Ma-Y). *FBSDE* (3) *is globally solvable if and only if Problem (C) admits an optimal control at some* $(0, x, y)$ *which is a* **nodal point** *of* $V(\cdot\,, \cdot\,, \cdot)$.

For Problem (C), $V(\cdot\,,\cdot\,,\cdot)$ satisfies the HJB equation:

$$\begin{cases} V_t(t,x,y) + H(t,x,y,V_x,V_y,V_{xx},V_{xy},V_{yy}) = 0, \\ \qquad\qquad (t,x,y) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^m, \\ V(T,x,y) = |y - g(x)|^2, \qquad (x,y) \in \mathbb{R}^n \times \mathbb{R}^m, \end{cases} \tag{10}$$

where the *Hamiltonian* H is given by the following:

$$\begin{aligned} & H(t,x,y,V_x,V_y,V_{xx},V_{xy},V_{yy}) \\ & = \inf_{u\in\mathbb{R}^{m\times d}} \Big\{ \langle\, V_x, b(x,y,u)\,\rangle + \langle\, V_y, \widehat{b}(x,y,u)\,\rangle \\ & \qquad + \frac{1}{2}\mathrm{tr}\left[\sigma(x,y,u)\sigma(x,y,u)^\top V_{xx}\right] \\ & \qquad + \mathrm{tr}\left[\sigma(x,y,u)\widehat{\sigma}(x,y,u)^\top V_{xy}\right] \\ & \qquad + \frac{1}{2}\mathrm{tr}\left[\widehat{\sigma}(x,y,u)\widehat{\sigma}(x,y,u)^\top V_{yy}\right] \Big\}. \end{aligned}$$

Consider the following FBSDE:

$$\begin{cases} X(t) = x + \int_0^t b(X(s), Y(s))ds + \int_0^t \sigma(X(s), Y(s))dW(s), \\ Y(t) = g(X(T)) + \int_t^T \widehat{b}(X(s), Y(s))ds \\ \qquad + \int_t^T \widehat{\sigma}(X(s), Y(s), Z(s))dW(s). \end{cases} \tag{11}$$

This is equivalent to the following:

$$\begin{cases} X(t) = x + \int_0^t b(X(s), Y(s))ds + \int_0^t \sigma(X(s), Y(s))dW(s), \\ Y(t) = \mathbb{E}\Big[g(X(T)) + \int_t^T \widehat{b}(X(s), Y(s))ds \mid \mathcal{F}_t\Big]. \end{cases} \tag{12}$$

This is comparable with the equation studied by Antonelli.

Theorem (Ma–Y). *Under proper conditions, including*

$$\sigma(x,y)\sigma(x,y)^\top \geqslant \nu > 0, \qquad \forall (x,y) \in \mathbb{R}^{n+m},$$

and

$$\widehat{\sigma}(x,y,\mathbb{R}^{m\times d}) = \mathbb{R}^{m\times d}, \tag{13}$$

for any $T > 0$ and $x \in \mathbb{R}$, FBSDE (11) is solvable.

Theorem (Ma–Y). *Under proper conditions, including*

$$\sigma(x,y)\sigma(x,y)^\top \geqslant \nu > 0, \qquad \forall (x,y) \in \mathbb{R}^{n+m},$$

and

$$\widehat{\sigma}(x,y,\mathbb{R}^{m\times d}) = \mathbb{R}^{m\times d}, \tag{13}$$

for any $T > 0$ and $x \in \mathbb{R}$, FBSDE (11) is solvable.

Our paper was completed in the early 1993, which was listed as #1117 in the IMA preprint series.

3. The Four-Step Scheme

3. The Four-Step Scheme

Invariant Embedding **(Decoupling)**

3. The Four-Step Scheme

Invariant Embedding (Decoupling)

In solving LQ problem, one faces to the following:

$$\begin{cases} \dot{\bar{x}}(t) = A\bar{x}(t) + B\bar{u}(t), \\ \dot{\bar{y}}(t) = -A^\top \bar{y}(t) - Q\bar{x}(t), \\ \bar{x}(0) = x_0, \qquad \bar{y}(T) = G\bar{x}(T), \end{cases}$$

with

$$R\bar{u}(t) + B^\top \bar{x}(t) = 0.$$

Then, we obtain the following coupled system:

$$\begin{cases} \dot{\bar{x}}(t) = A\bar{x}(t) - BR^{-1}B^\top \bar{y}(t), \\ \dot{\bar{y}}(t) = -A^\top \bar{y}(t) - Q\bar{x}(t), \\ \bar{x}(0) = x_0, \qquad \bar{y}(T) = G\bar{x}(T), \end{cases}$$

To decouple, we set

$$\bar{y}(t) = P(t)\bar{x}(t).$$

To decouple, we set

$$\bar{y}(t) = P(t)\bar{x}(t).$$

Then by assuming everything is fine, we have

$$\begin{aligned}
&-A^\top P(t)\bar{x}(t) - Q\bar{x}(t) = \dot{\bar{y}}(t) \\
&= \dot{P}(t)\bar{x}(t) + P(t)[A\bar{x}(t) - BR^{-1}B^\top P(t)\bar{x}(t)] \\
&= \left[\dot{P}(t) + P(t)A - P(t)BR^{-1}B^\top P(t)\right]\bar{x}(t).
\end{aligned}$$

To decouple, we set

$$\bar{y}(t) = P(t)\bar{x}(t).$$

Then by assuming everything is fine, we have

$$\begin{aligned} &-A^\top P(t)\bar{x}(t) - Q\bar{x}(t) = \dot{\bar{y}}(t) \\ &= \dot{P}(t)\bar{x}(t) + P(t)[A\bar{x}(t) - BR^{-1}B^\top P(t)\bar{x}(t)] \\ &= \Big[\dot{P}(t) + P(t)A - P(t)BR^{-1}B^\top P(t)\Big]\bar{x}(t). \end{aligned}$$

Thus, $P(\cdot)$ should be the solution to the **Riccati** equation:

$$\begin{cases} \dot{P}(t) + P(t)A + A^\top P(t) - P(t)BR^{-1}B^\top P(t) + Q = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad t \in [0, T], \\ P(T) = G. \end{cases} \tag{14}$$

Consider the following general **FBSDE**:

$$
\begin{cases}
dX(t) = b(t, X(t), Y(t), Z(t))dt \\
\qquad\qquad\qquad\qquad +\sigma(t, X(t), Y(t))dW(t), \\
dY(t) = -g(t, X(t), Y(t), Z(t))dt + Z(t)dW(t), \\
X(0) = x, \qquad Y(T) = h(X(T)).
\end{cases}
\tag{15}
$$

Consider the following general **FBSDE**:

$$\begin{cases} dX(t) = b(t, X(t), Y(t), Z(t))dt \\ \qquad\qquad\qquad\qquad +\sigma(t, X(t), Y(t))dW(t), \\ dY(t) = -g(t, X(t), Y(t), Z(t))dt + Z(t)dW(t), \\ X(0) = x, \qquad Y(T) = h(X(T)). \end{cases} \tag{15}$$

Let

$$Y(t) = \theta(t, X(t)).$$

Consider the following general **FBSDE**:

$$\begin{cases} dX(t) = b(t, X(t), Y(t), Z(t))dt \\ \qquad\qquad\qquad\qquad +\sigma(t, X(t), Y(t))dW(t), \\ dY(t) = -g(t, X(t), Y(t), Z(t))dt + Z(t)dW(t), \\ X(0) = x, \qquad Y(T) = h(X(T)). \end{cases} \tag{15}$$

Let

$$Y(t) = \theta(t, X(t)).$$

Then by Itô's formula, for $i = 1, 2, \cdots, m$,

$$\begin{aligned} &-g^i(t, X(t), \theta(t, X(t)), Z(t))dt + Z^i(t)dW(t) = dY^i(t) \\ &= \Big[\theta^i_t(t, X(t)) + \theta^i_x(t, X(t))b(t, X(t), \theta(t, X(t)), Z(t)) \\ &\quad + \frac{1}{2}\mathrm{tr}\,[\theta^i_{xx}(t, X(t))(\sigma\sigma^\top)(t, X(t), \theta(t, X(t)))\Big]dt \\ &\quad + \theta^i_x(t, X(t))\sigma(t, X(t), \theta(t, X(t)))dW(t). \end{aligned}$$

Hence, if $\theta(\cdot\,,\cdot)$ is a right choice, we should have

$$Z(t) = \theta_x(t, X(t))\sigma(t, X(t), \theta(t, X(t))),$$

which suggests:

$$z = p\sigma(t, x, \theta). \tag{16}$$

Here p will be the row vector, generically representing θ_x.

Hence, if $\theta(\cdot\,,\cdot)$ is a right choice, we should have

$$Z(t) = \theta_x(t, X(t))\sigma(t, X(t), \theta(t, X(t))),$$

which suggests:

$$z = p\sigma(t, x, \theta). \tag{16}$$

Here p will be the row vector, generically representing θ_x. Consequently, we should set

$$\begin{cases} \theta^i_t(t,x) + \theta^i_x(t,x)b(t,x,\theta(t,x),\theta_x(t,x)\sigma(t,x,\theta(t,x))) \\ \qquad +\frac{1}{2}\mathrm{tr}\left[\theta^i_{xx}(t,x)\sigma(t,x,\theta(t,x))\sigma(t,x,\theta(t,x))^\top\right] \\ \qquad +g^i(t,x,\theta(t,x),\theta_x(t,x)\sigma(t,x,\theta(t,x))) = 0, \\ \qquad\qquad\qquad (t,x) \in [0,T]\times\mathbb{R}^n, \quad 1 \leqslant i \leqslant m, \\ \theta(T,x) = g(x), \qquad x \in \mathbb{R}^n. \end{cases} \tag{17}$$

Let $\theta(\cdot\,,\cdot)$ be the classical solution. Then we solve SDE:

$$\begin{cases} dX(t) = b\big(t, X(t), \theta(t, X(t)), \theta_x(t, X(t))\sigma(t, X(t), \theta(t, X(t)))\big)dt \\ \qquad\qquad +\sigma(t, X(t), \theta(t, X(t)))dW(t), \\ X(0) = x. \end{cases} \tag{18}$$

Let $\theta(\cdot\,,\cdot)$ be the classical solution. Then we solve SDE:

$$\begin{cases} dX(t) = b\big(t, X(t), \theta(t, X(t)), \theta_x(t, X(t))\sigma(t, X(t), \theta(t, X(t)))\big)dt \\ \qquad\qquad +\sigma(t, X(t), \theta(t, X(t)))dW(t), \\ X(0) = x. \end{cases} \tag{18}$$

Then by setting

$$\begin{cases} Y(t) = \theta(t, X(t)), \\ Z(t) = \theta_x(t, X(t))\sigma(t, X(t), \theta(t, X(t))), \qquad t \in [0, T], \end{cases} \tag{19}$$

Let $\theta(\cdot\,,\cdot)$ be the classical solution. Then we solve SDE:

$$\begin{cases} dX(t) = b\big(t, X(t), \theta(t, X(t)), \theta_x(t, X(t))\sigma(t, X(t), \theta(t, X(t)))\big)dt \\ \qquad\qquad +\sigma(t, X(t), \theta(t, X(t)))dW(t), \\ X(0) = x. \end{cases} \tag{18}$$

Then by setting

$$\begin{cases} Y(t) = \theta(t, X(t)), \\ Z(t) = \theta_x(t, X(t))\sigma(t, X(t), \theta(t, X(t))), \qquad t \in [0, T], \end{cases} \tag{19}$$

DONE!

Theorem (Ma–Protter–Y). *Under suitable conditions, including*

$$\sigma(t,x,y)\sigma(t,x,y)^{\top} \geqslant \delta I, \qquad \forall (t,x,y) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^m, \quad (20)$$

for some $\delta > 0$. Then FBSDE (15) *can be solved by the following* **Four-Step Scheme***:*

Theorem (Ma–Protter–Y). *Under suitable conditions, including*

$$\sigma(t,x,y)\sigma(t,x,y)^\top \geqslant \delta I, \qquad \forall (t,x,y) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^m, \quad (20)$$

for some $\delta > 0$*. Then FBSDE* (15) *can be solved by the following* **Four-Step Scheme***:*

Step 1. Set $z = p\sigma(t,x,\theta)$.

Theorem (Ma–Protter–Y). *Under suitable conditions, including*

$$\sigma(t,x,y)\sigma(t,x,y)^\top \geqslant \delta I, \qquad \forall (t,x,y) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^m, \quad (20)$$

for some $\delta > 0$. Then FBSDE (15) can be solved by the following **Four-Step Scheme***:*

Step 1. Set $z = p\sigma(t,x,\theta)$.

Step 2. Solve parabolic system (17) to get $\theta(\cdot\,,\cdot)$.

Theorem (Ma–Protter–Y). *Under suitable conditions, including*

$$\sigma(t,x,y)\sigma(t,x,y)^\top \geqslant \delta I, \qquad \forall (t,x,y) \in [0,T]\times\mathbb{R}^n\times\mathbb{R}^m, \quad (20)$$

for some $\delta > 0$. Then FBSDE (15) can be solved by the following **Four-Step Scheme***:*

Step 1. Set $z = p\sigma(t,x,\theta)$.

Step 2. Solve parabolic system (17) to get $\theta(\cdot\,,\cdot)$.

Step 3. Solve FSDE (18) to get $X(\cdot)$.

Theorem (Ma–Protter–Y). *Under suitable conditions, including*

$$\sigma(t,x,y)\sigma(t,x,y)^\top \geqslant \delta I, \qquad \forall (t,x,y) \in [0,T]\times\mathbb{R}^n\times\mathbb{R}^m, \quad (20)$$

for some $\delta > 0$. Then FBSDE (15) can be solved by the following **Four-Step Scheme***:*

Step 1. Set $z = p\sigma(t,x,\theta)$.

Step 2. Solve parabolic system (17) to get $\theta(\cdot\,,\cdot)$.

Step 3. Solve FSDE (18) to get $X(\cdot)$.

Step 4. Set

$$\begin{cases} Y(t) = \theta(t,X(t)), \\ Z(t) = \theta_x(t,X(t))\sigma(t,X(t),\theta(t,X(t)), \qquad t\in[0,T], \end{cases}$$

Theorem (Ma–Protter–Y). *Under suitable conditions, including*

$$\sigma(t,x,y)\sigma(t,x,y)^\top \geqslant \delta I, \qquad \forall (t,x,y) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^m, \quad (20)$$

for some $\delta > 0$. Then FBSDE (15) can be solved by the following **Four-Step Scheme***:*

Step 1. Set $z = p\sigma(t,x,\theta)$.

Step 2. Solve parabolic system (17) to get $\theta(\cdot\,,\cdot)$.

Step 3. Solve FSDE (18) to get $X(\cdot)$.

Step 4. Set

$$\begin{cases} Y(t) = \theta(t, X(t)), \\ Z(t) = \theta_x(t, X(t))\sigma(t, X(t), \theta(t, X(t)), \qquad t \in [0,T], \end{cases}$$

A Punch Line: Jin named the method.

Theorem (Ma–Protter–Y). *Under suitable conditions, including*

$$\sigma(t,x,y)\sigma(t,x,y)^\top \geqslant \delta I, \qquad \forall (t,x,y) \in [0,T]\times\mathbb{R}^n\times\mathbb{R}^m, \quad (20)$$

for some $\delta > 0$. Then FBSDE (15) can be solved by the following **Four-Step Scheme***:*

Step 1. Set $z = p\sigma(t,x,\theta)$.

Step 2. Solve parabolic system (17) to get $\theta(\cdot\,,\cdot)$.

Step 3. Solve FSDE (18) to get $X(\cdot)$.

Step 4. Set

$$\begin{cases} Y(t) = \theta(t,X(t)), \\ Z(t) = \theta_x(t,X(t))\sigma(t,X(t),\theta(t,X(t)), \qquad t\in[0,T], \end{cases}$$

A Punch Line: Jin named the method.

This paper was listed #1146 in the IMA preprint series, published in 1994 in PTRF, earlier than our first paper in the series.

4. Black's Conjecture.

4. Black's Conjecture.

June of 1992, IMA.

4. Black's Conjecture.

June of 1992, IMA.

Black's Conjecture: There should be an analytic relationship between the consol rate and the short rate.

4. Black's Conjecture.

June of 1992, IMA.

Black's Conjecture: There should be an analytic relationship between the consol rate and the short rate.

The mathematical formulation of *infinite-horizon consol rate* problem:

Problem (IHCR) Find an adapted, locally square-integrable process (X, Y) such that

$$\begin{cases} X(t) = x + \int_0^t b(X(s), Y(s))ds + \int_0^t \sigma(X(s), Y(s)dW(s), \\ Y(t) = \mathbb{E}\Big[\int_t^\infty \exp\Big(-\int_t^s h(X(r))dr\Big)ds \mid \mathcal{F}_t\Big], \\ \qquad t \in [0, \infty). \end{cases} \tag{21}$$

The above system is equivalent to the following FBSDE:

$$\begin{cases} dX(t) = b(X(t), Y(t))dt + \sigma(X(t), Y(t))dW(t), \quad t \in [0, \infty), \\ dY(t) = \big[h(X(t))Y(t) - 1\big]dt + \langle\, Z(t), dW(t)\,\rangle, \quad t \in [0, \infty), \\ X(0) = x, \qquad Y(t) \text{ is uniformly bounded.} \end{cases} \tag{22}$$

This is an FBSDE in an infinite horizon.

Theorem (Duffie–Ma–Y). *Under proper conditions. Problem (IHCR) admits at least one nodal solution* $(X(\cdot), Y(\cdot))$. *In other words, there is a* C^2 *bounded function* $\theta(\cdot)$ *so that* $X(\cdot), Y(\cdot))$ *solves* (21) *and*

$$Y(t) = \theta(X(t)).$$

Moreover, nodal solution is unique.

5. BSPDEs

$$
\begin{cases}
X(t) = x + \displaystyle\int_0^t b(s, X(s), Y(s), Z(s))ds \\
\qquad\qquad + \displaystyle\int_0^t \sigma(s, X(s), Y(s))dW(s), \\
Y(t) = h(X(T)) + \displaystyle\int_t^T g(s, X(s), Y(s), Z(s))ds \\
\qquad\qquad - \displaystyle\int_t^T Z(s)dW(s).
\end{cases}
\tag{23}
$$

In the above, all the involved functions are random fields.

Try to write $Y(t) = u(t, X(t))$, for some random field $u(\cdot\,, \cdot)$.

Suppose

$$du(t,x) = p(t,x)dt + \langle\, q(t,x), dW(t)\,\rangle, \quad (t,x)\in[0,T]\times\mathbb{R}^n, \quad (24)$$

with $p(\cdot\,,\cdot)$ and $q(\cdot\,,\cdot)$ undetermined. Then by a generalized Itô-Ventzell formula, we have

$$\begin{aligned}
& u(t,X(t)) = u(T,X(T)) - \int_t^T \Big\{ p(s,X(s)) \\
& + u_x(s,X(s))b(s,X(s),Y(s),Z(s)) \\
& + \frac{1}{2}\mathrm{tr}\Big[u_{xx}(s,X(s))\sigma(s,X(s),Y(s))\sigma(s,X(s),Y(s))^\top \Big] \\
& + \mathrm{tr}\Big[q_x(s,X(s))\sigma(s,X(s)) \Big] \Big\} ds \\
& - \int_t^T \Big(q(s,X(s))^\top + u_x(s,X(s)))\sigma(s,X(s),Y(s)) \Big) dW(s).
\end{aligned}$$

We choose (u, q) so that

$$Y(t) = u(t, X(t)).$$

Then

$$\begin{aligned}
&Z(t) = q(t, X(t))^\top + u_x(t, X(t))\sigma(t, X(t), u(t, X(t))),\\
&g\big(t, X(t), Y(t), Z(t))\big)\\
&= -\Big\{p(t, X(t)) + u_x(t, X(t))b(t, X(t), Y(t), Z(s))\\
&\quad + \frac{1}{2}\mathrm{tr}\left[u_{xx}(s, X(s))\sigma(s, X(s), Y(s))\sigma(s, X(s), Y(s))^\top\right]\\
&\quad + \mathrm{tr}\left[q_x(s, X(s))\sigma(s, X(s))\right]\Big\}.
\end{aligned}$$

Thus, we need to solve BSPDE:

$$\begin{cases} du(t,x) + \Big\{\frac{1}{2}\mathrm{tr}\Big[u_{xx}(t,x)\sigma(t,x,u(t,x))\sigma(t,x,u(t,x))^\top\Big] \\ +u_x(t,x)b(t,x,u(t,x),q(t,x)^\top + u_x(t,x)\sigma(t,x,u(t,x))) \\ +\mathrm{tr}\Big[q_x(t,x)\sigma(t,x,u(t,x))\Big] \\ +g(t,x,u(t,x),q(t,x)^\top + u_x(t,x)\sigma(t,x,u(t,x)))\Big\}dt \\ \quad -\langle\, q(t,x), dW(t)\,\rangle = 0, \\ u(T,x) = h(x). \end{cases} \tag{25}$$

If the above BSPDE admits a classical adapted solution (u, q), then we can try to solve the following FSDE:

$$\begin{cases} dX(t) = b\big(t, X(t), u(t, X(t)), \\ \qquad\qquad q(t, X(t)) + u_x(t, X(t))\sigma(t, X(t), u(t, X(t)))\big)dt \\ \qquad\qquad +\sigma(t, X(t), u(t, X(t)))dW(t), \\ X(0) = x. \end{cases} \tag{26}$$

If this can also go through, we could finally set

$$\begin{cases} Y(t) = u(t, X(t)), \\ Z(t) = q(t, X(t)) + u_x(t, X(t))\sigma(t, X(t), u(t, X(t)). \end{cases} \tag{27}$$

Hence, at least formally, we still have the following Four-Step Scheme:

Step 1. Define

$$z = q + p\sigma(t, x, y), \quad \forall (t, x, p, q) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^{1\times d} \times \mathbb{R}^{1\times n}.$$

Step 2. Solve BSPDE (25).

Step 3. Solve FSDE (26).

Step 4. Set $Y(\cdot)$ and $Z(\cdot)$ by (27).

This motivated us to study BSPDEs. As a first step, we consider

$$
\begin{aligned}
u(t,x) = h(x) + \int_t^T \Big\{ \nabla \cdot \Big[A(s,x) u_x(s,x) \Big] + u_x(s,x) a(s,x) \\
+ a_0(s,x) u(s,x) + \mathrm{tr} \Big[B(s,x) q_x(s,x) \Big] \\
+ \langle\, b_0(s,x), q(s,x) \,\rangle + g_0(s,x) \Big\} ds \\
- \int_t^T q(s,x) dW(s),
\end{aligned} \tag{28}
$$

Theorem (Ma–Y). *Under proper conditions, including*

$$
\begin{aligned}
\big[B(\partial_{x_i} B^\top) \big]^\top = B(\partial_{x_i} B^\top), \\
\text{a.e. } (t,x) \in [0,T] \times \mathbb{R}^n, \text{ a.s. }, \ 1 \leqslant i \leqslant n.
\end{aligned} \tag{29}
$$

BSPDE (28) *is well-posed.*

• 2000, Ying Hu (of Université de Rennes 1, Frances) joined Jin and myself, working on one-dimensional semilinear BSPDEs.

• 2012, Ma–Yin–Zhang, established the equivalence of the well-posedness of random coefficient FBSDEs and the existence of the so-called random decoupling field, via the solution of BSPDEs.

• 2013, Du–Tang–Zhang removed the technical condition (29).

• 2013, Du–Zhang studied multi-dimensional semilinear case.

• 2014, Du–Chen studied BSPDEs, allowing quadratic growth of (u_x, q) in the semilinear term.

• For the general BSPDEs, it seems to be still open for the cases that the differential operators $\mathcal{L}$ and $\mathcal{M}$ are nonlinear in u and/or in q.

In about 1997, Jin and I decided to write a book summarizing the updated theory of FBSDEs. The book was published in 1999.

6. Beyond

- FBSDEs with reflections.
- Numerical solutions of FBSDEs
- Weak solutions of FBSDEs
- Many, many more,...

Great Achievements, Jin!

Congratulations! Bro!

Numerical Algebra, Control and Optimization (NACO):

A journal of **American Institute of Mathematical Sciences (AIMS)**.

Numerical Algebra, Control and Optimization (NACO):

A journal of **American Institute of Mathematical Sciences (AIMS)**.

Will publish a special issue dedicating **Jin Ma**

"Stochastic Analysis, Mathematical Finance, and Related Topics",

edited by Jianfeng Zhang and Song Yao

Numerical Algebra, Control and Optimization (NACO):

A journal of **American Institute of Mathematical Sciences (AIMS)**.

Will publish a special issue dedicating **Jin Ma**

"Stochastic Analysis, Mathematical Finance, and Related Topics",

edited by Jianfeng Zhang and Song Yao

NACO, current EiC: **Jiongmin Yong** (UCF)

co-EiCs: **Ren-Cang Li** (UT Arlington)

Jiawang Nie (UCSD)

Everyone is welcome to submit papers!

Thank You Very Much!