FBSDEs: 
Initiation, Development, and Beyond

Jiongmin Yong
(University of Central Florida)
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(Celebrating Jin Ma’s 65th Birthday)
Outline

0. Pre-History
1. BSDEs and the Initiation of FBSDEs
2. Stochastic Optimal Control Method
3. The Four-Step Scheme
4. Black’s Conjecture
5. BSPDEs
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0. Pre-History
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1992, Purdue and U. of Minnesota, IMA “Control Year”
    Avner Friedman
1. BSDEs and the Initiation of FBSDEs

- 1972, Bismut introduced linear BSDEs.
- 1990, Pardoux–Peng initiated nonlinear BSDEs.

\[ Y(t) = \xi + \int_T^t g(s, Y(s), Z(s)) \, ds - \int_T^t Z(s) \, dW(s), \quad t \in [0, T] \] (1)

**Theorem (Pardoux–Peng)**

Under proper conditions, \( \xi \in L^p_{FT}(\Omega; \mathbb{R}^m) \), BSDE (1) admits a unique adapted solution \((Y(\cdot), Z(\cdot))\).
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• Many further development, ...
Early 1990s, Fabio Antonelli started to investigate following:

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\begin{align*}
U_t &= J_t + \int_0^t f_s(U_s, V_s) dA_s, \\
V_t &= \mathbb{E} \left[ \int_t^T g_s(U_s, V_s) dC_s + Y \right| \mathcal{F}_t].
\end{align*}
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(2)

A, C — finite variation processes 
(U, V) — unknown
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**Theorem (Antonelli)** (i) Let \((U, V) \mapsto (f(U, V), g(U, V))\) be Lipschitz continuous with the Lipschitz constant or time duration \(T\) being small enough. Then, (2) admits a unique adapted solution.
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2. Stochastic Optimal Control Method

- In 1992, Jin invited me to join the research.

\[ X(t) = x + \int_{0}^{t} b(X(s), Y(s), Z(s)) \, ds + \int_{0}^{t} \sigma(X(s), Y(s), Z(s)) \, dW(s) \]

\[ Y(t) = g(X(T)) + \int_{T}^{t} \hat{b}(X(s), Y(s), Z(s)) \, ds + \int_{T}^{t} \hat{\sigma}(X(s), Y(s), Z(s)) \, dW(s) \]

(FBSDE (3)) No restriction on the size of \( T \) and the Lipschitz constant.

(FBSDE (3)) is a two-point boundary value problem for SDEs.
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• FBSDE (3) is a two-point boundary value problem for SDEs.
For ODE case: **shooting method**. Consider:

\[
\begin{align*}
\dot{x}(t) &= b(x(t), y(t)), \\
\dot{y}(t) &= \hat{b}(x(t), y(t)), \\
x(0) &= x_0, \quad y(T) = g(x(T)).
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with \(y_0\) being a parameter to be chosen.
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(i) For any \((x_0, y_0)\), solve (5) to get \((x(\cdot ; x_0, y_0), y(\cdot ; x_0, y_0))\), indicating the dependence on \((x_0, y_0)\).
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(i) For any \((x_0, y_0)\), solve (5) to get \((x(\cdot; x_0, y_0), y(\cdot; x_0, y_0))\), indicating the dependence on \((x_0, y_0)\).

(ii) Select a “bullet” \(y_0\) so that the “target” is hit:

\[y(T; x_0, y_0) = g(x(T; x_0, y_0)).\]
For FBSDE (3), consider

\[
\begin{cases}
X(r) = x + \int_t^r b(X(s), Y(s), u(s))\, ds \\
\quad + \int_t^r \sigma(X(s), Y(s), u(s))\, dW(s),
\end{cases}
\]

\[
Y(r) = y - \int_t^r \hat{b}(X(s), Y(s), u(s))\, ds \\
\quad - \int_t^r \hat{\sigma}(X(s), Y(s), u(s))\, dW(s),
\]

(6)

with the cost functional

\[
J(t, x, y; u(\cdot)) = \mathbb{E} |Y(T) - g(X(T))|^2.
\]

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\[ J(t, x, y; u(\cdot)) = \mathbb{E} | Y(T) - g(X(T))|^2. \]  

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**Problem (C).** Find \( u(\cdot) \in \mathcal{U}[t, T] \), such that

\[ J(t, x, y; \bar{u}(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}[t, T]} J(t, x, y; u(\cdot)) \equiv V(t, x, y). \]

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\(V(\cdot, \cdot, \cdot)\) — **value function** of Problem (C).
• If FBSDE (3) admits an adapted solution \((X, Y, Z)\), then by choosing \(y = Y(0)\) and \(\bar{u}(\cdot) = Z(\cdot)\),

\[
J(0, x, Y(0); \bar{u}(\cdot)) = |Y(T) - g(X(T))|^2 = V(0, x, y) = 0.
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• If Problem (C) admits an optimal triple \((\bar{X}(\cdot), \bar{Y}(\cdot), \bar{u}(\cdot))\) for some \((0, x, y)\) with

\[
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\]

then \((\bar{X}(\cdot), \bar{Y}(\cdot), \bar{u}(\cdot))\) is an adapted solution of FBSDE (3).
• If FBSDE (3) admits an adapted solution \((X, Y, Z)\), then by choosing \(y = Y(0)\) and \(\bar{u}(\cdot) = Z(\cdot)\),

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\[ V(0, x, y) = 0, \tag{9} \]

then \((\bar{X}(\cdot), \bar{Y}(\cdot), \bar{u}(\cdot))\) is an adapted solution of FBSDE (3).

**Proposition (Ma-Y).** \textit{FBSDE (3) is globally solvable if and only if Problem (C) admits an optimal control at some \((0, x, y)\) which is a nodal point of} \(V(\cdot, \cdot, \cdot)\).
For Problem (C), $V(\cdot, \cdot, \cdot)$ satisfies the HJB equation:

\[
\begin{cases}
V_t(t, x, y) + H(t, x, y, V_x, V_y, V_{xx}, V_{xy}, V_{yy}) = 0, \\
(t, x, y) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^m, \\
V(T, x, y) = |y - g(x)|^2, \\
(x, y) \in \mathbb{R}^n \times \mathbb{R}^m,
\end{cases}
\]

where the Hamiltonian $H$ is given by the following:

\[
H(t, x, y, V_x, V_y, V_{xx}, V_{xy}, V_{yy}) = \inf_{u \in \mathbb{R}^{m \times d}} \left\{ \left\langle V_x, b(x, y, u) \right\rangle + \left\langle V_y, \hat{b}(x, y, u) \right\rangle \\
+ \frac{1}{2} \text{tr} \left[ \sigma(x, y, u)\sigma(x, y, u)^\top V_{xx} \right] \\
+ \text{tr} \left[ \sigma(x, y, u)\hat{\sigma}(x, y, u)^\top V_{xy} \right] \\
+ \frac{1}{2} \text{tr} \left[ \hat{\sigma}(x, y, u)\hat{\sigma}(x, y, u)^\top V_{yy} \right] \right\}.
\]
Consider the following FBSDE:

\[ \begin{align*}
X(t) &= x + \int_0^t b(X(s), Y(s)) \, ds + \int_0^t \sigma(X(s), Y(s)) \, dW(s), \\
Y(t) &= g(X(T)) + \int_t^T \hat{b}(X(s), Y(s)) \, ds \\
& \quad + \int_t^T \hat{\sigma}(X(s), Y(s), Z(s)) \, dW(s).
\end{align*} \] (11)

This is equivalent to the following:

\[ \begin{align*}
X(t) &= x + \int_0^t b(X(s), Y(s)) \, ds + \int_0^t \sigma(X(s), Y(s)) \, dW(s), \\
Y(t) &= \mathbb{E} \left[ g(X(T)) + \int_t^T \hat{b}(X(s), Y(s)) \, ds \mid \mathcal{F}_t \right].
\end{align*} \] (12)

This is comparable with the equation studied by Antonelli.
Theorem (Ma–Y). Under proper conditions, including

\[ \sigma(x, y)\sigma(x, y)^\top \geq \nu > 0, \quad \forall (x, y) \in \mathbb{R}^{n+m}, \]

and

\[ \hat{\sigma}(x, y, \mathbb{R}^{m\times d}) = \mathbb{R}^{m\times d}, \quad (13) \]

for any \( T > 0 \) and \( x \in \mathbb{R} \), FBSDE (11) is solvable.
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Our paper was completed in the early 1993, which was listed as \#1117 in the IMA preprint series.
3. The Four-Step Scheme

Invariant Embedding (Decoupling)

In solving LQ problem, one faces to the following:

\[
\begin{align*}
\dot{\bar{x}}(t) &= A\bar{x}(t) + Bu(t), \\
\dot{\bar{y}}(t) &= -A^\top\bar{y}(t) - Q\bar{x}(t), \\
\bar{x}(0) &= x_0, \\
\bar{y}(T) &= G\bar{x}(T),
\end{align*}
\]

with

\[
R\bar{u}(t) + B^\top\bar{x}(t) = 0.
\]

Then, we obtain the following coupled system:

\[
\begin{align*}
\dot{\bar{x}}(t) &= A\bar{x}(t) - BR^{-1}B^\top\bar{y}(t), \\
\dot{\bar{y}}(t) &= -A^\top\bar{y}(t) - Q\bar{x}(t), \\
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\end{aligned}
\]
To decouple, we set

\[ \bar{y}(t) = P(t)\bar{x}(t). \]
To decouple, we set

$$\tilde{y}(t) = P(t)\tilde{x}(t).$$

Then by assuming everything is fine, we have

$$-A^\top P(t)\tilde{x}(t) - Q\tilde{x}(t) = \dot{\tilde{y}}(t)$$

$$= \dot{P}(t)\tilde{x}(t) + P(t)[A\tilde{x}(t) - BR^{-1}B^\top P(t)\tilde{x}(t)]$$

$$= \left[\dot{P}(t) + P(t)A - P(t)BR^{-1}B^\top P(t)\right]\tilde{x}(t).$$
To decouple, we set
\[ \ddot{y}(t) = P(t)\ddot{x}(t). \]

Then by assuming everything is fine, we have
\[ -A^\top P(t)\ddot{x}(t) - Q\ddot{x}(t) = \dot{y}(t) \]
\[ = \dot{P}(t)\ddot{x}(t) + P(t)[A\ddot{x}(t) - BR^{-1}B^\top P(t)\ddot{x}(t)] \]
\[ = \left[ \dot{P}(t) + P(t)A - P(t)BR^{-1}B^\top P(t) \right]\ddot{x}(t). \]

Thus, \( P(\cdot) \) should be the solution to the **Riccati** equation:

\[
\begin{cases}
\dot{P}(t) + P(t)A + A^\top P(t) - P(t)BR^{-1}B^\top P(t) + Q = 0, \\
P(T) = G.
\end{cases}
\tag{14}
\]
Consider the following general FBSDE:

\[
\begin{align*}
    dX(t) &= b(t, X(t), Y(t), Z(t))dt \\
          &\quad + \sigma(t, X(t), Y(t))dW(t), \\
    dY(t) &= -g(t, X(t), Y(t), Z(t))dt + Z(t)dW(t), \\
    X(0) &= x, \quad Y(T) = h(X(T)).
\end{align*}
\] (15)
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    dX(t) & = b(t, X(t), Y(t), Z(t))dt \\
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Let

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Y(t) = \theta(t, X(t)).
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Consider the following general FBSDE:

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    X(0) = x, \quad Y(T) = h(X(T)).
\end{cases}
\] (15)

Let

\[ Y(t) = \theta(t, X(t)). \]

Then by Itô’s formula, for \( i = 1, 2, \cdots, m, \)

\[
- g^i(t, X(t), \theta(t, X(t)), Z(t))dt + Z^i(t)dW(t) = dY^i(t)
\]

\[
= \left[ \theta_t^i(t, X(t)) + \theta_x^i(t, X(t))b(t, X(t), \theta(t, X(t)), Z(t)) \\
\right.
\]

\[
\left. + \frac{1}{2} \text{tr} \left[ \theta_{xx}^i(t, X(t))(\sigma\sigma^\top)(t, X(t), \theta(t, X(t))) \right] dt \\
\right.
\]

\[
\left. + \theta_x^i(t, X(t))\sigma(t, X(t), \theta(t, X(t)))dW(t). \right]
\]
Hence, if $\theta(\cdot, \cdot)$ is a right choice, we should have

$$Z(t) = \theta_x(t, X(t))\sigma(t, X(t), \theta(t, X(t))),$$

which suggests:

$$z = p\sigma(t, x, \theta).$$

Here $p$ will be the row vector, generically representing $\theta_x$. 

(16)
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$$Z(t) = \theta_x(t, X(t))\sigma(t, X(t), \theta(t, X(t))),$$

which suggests:

$$z = p\sigma(t, x, \theta). \quad (16)$$

Here $p$ will be the row vector, generically representing $\theta_x$. Consequently, we should set

$$\begin{cases}
\theta^i_t(t, x) + \theta^i_x(t, x)b(t, x, \theta(t, x), \theta_x(t, x)\sigma(t, x, \theta(t, x))) \\
+ \frac{1}{2}\text{tr} \left[ \theta^i_{xx}(t, x)\sigma(t, x, \theta(t, x))\sigma(t, x, \theta(t, x))^\top \right] \\
+ g^i(t, x, \theta(t, x), \theta_x(t, x)\sigma(t, x, \theta(t, x))) = 0,
\end{cases} \quad (17)$$

$$\begin{align*}
(t, x) &\in [0, T] \times \mathbb{R}^n, \quad 1 \leq i \leq m, \\
\theta(T, x) &= g(x), \quad x \in \mathbb{R}^n.
\end{align*}$$
Let $\theta(\cdot, \cdot)$ be the classical solution. Then we solve SDE:

\[
\begin{cases}
    dX(t) = b(t, X(t), \theta(t, X(t)), \theta_x(t, X(t))\sigma(t, X(t), \theta(t, X(t))))dt \\
    + \sigma(t, X(t), \theta(t, X(t)))dW(t), \\
    X(0) = x.
\end{cases}
\]

(18)
Let \( \theta(\cdot, \cdot) \) be the classical solution. Then we solve SDE:

\[
\begin{cases}
  dX(t) = b(t, X(t), \theta(t, X(t)), \theta_x(t, X(t))\sigma(t, X(t), \theta(t, X(t)))) dt \\
  \quad + \sigma(t, X(t), \theta(t, X(t))))dW(t),
\end{cases}
\]

\[X(0) = x.\]  

(18)

Then by setting

\[
\begin{cases}
  Y(t) = \theta(t, X(t)), \\
  Z(t) = \theta_x(t, X(t))\sigma(t, X(t), \theta(t, X(t))),
\end{cases}
\]

\[t \in [0, T],\]  

(19)
Let $\theta(\cdot, \cdot)$ be the classical solution. Then we solve SDE:

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\end{cases}
\quad t \in [0, T],
$$  \hfill (19)

\textbf{DONE!}
Theorem (Ma–Protter–Y). Under suitable conditions, including

$$\sigma(t, x, y)^\top \sigma(t, x, y) \geq \delta I, \quad \forall (t, x, y) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^m, \quad (20)$$

for some $\delta > 0$. Then FBSDE (15) can be solved by the following

Four-Step Scheme:

Step 1. Set $z = p \sigma(t, x, \theta)$.

Step 2. Solve parabolic system (17) to get $\theta(\cdot, \cdot)$.

Step 3. Solve FSDE (18) to get $X(\cdot)$.

Step 4. Set

$$Y(t) = \theta(t, X(t)), \quad Z(t) = \theta_x(t, X(t)) \sigma(t, X(t), \theta(t, X(t))), \quad t \in [0, T].$$

A Punch Line: Jin named the method.

This paper was listed #1146 in the IMA preprint series, published in 1994 in PTRF, earlier than our first paper in the series.
Theorem (Ma–Protter–Y). Under suitable conditions, including
\[ \sigma(t, x, y)\sigma(t, x, y)^\top \geq \delta I, \quad \forall (t, x, y) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^m, \] (20)
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Theorem (Ma–Protter–Y). Under suitable conditions, including
\[ \sigma(t, x, y)\sigma(t, x, y)\top \geq \delta l, \quad \forall (t, x, y) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^m, \] (20)
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Step 4. Set

\[
\begin{aligned}
Y(t) &= \theta(t, X(t)), \\
Z(t) &= \theta_x(t, X(t))\sigma(t, X(t), \theta(t, X(t))),
\end{aligned}
\quad t \in [0, T],
\]
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\]

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$$
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$$

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4. Black’s Conjecture.

Black’s Conjecture:
The consol rate and the short rate should be related analytically.

The mathematical formulation of
infinite-horizon consol rate problem:

Problem (IHCR)
Find an adapted, locally square-integrable process
\((X, Y)\) such that

\[
X(t) = x + \int_0^t b(X(s), Y(s)) \, ds + \int_0^t \sigma(X(s), Y(s)) \, dW(s),
\]

\[
Y(t) = E \left[ \int_\infty^t \exp\left( -\int_s^t h(X(r)) \, dr \right) \, ds \mid F_t \right],
\]

\(t \in [0, \infty)\). (21)
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June of 1992, IMA.
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**Black’s Conjecture:** There should be an analytic relationship between the consol rate and the short rate.
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The mathematical formulation of *infinite-horizon consol rate* problem:

**Problem (IHCR)** Find an adapted, locally square-integrable process \((X, Y)\) such that

\[
\begin{cases}
X(t) = x + \int_0^t b(X(s), Y(s))ds + \int_0^t \sigma(X(s), Y(s))dW(s), \\
Y(t) = E \left[ \int_t^\infty \exp \left( - \int_t^s h(X(r))dr \right) ds \mid \mathcal{F}_t \right],
\end{cases}
\]

(21)

t \in [0, \infty).
The above system is equivalent to the following FBSDE:

\[
\begin{cases}
    dX(t) = b(X(t), Y(t))dt + \sigma(X(t), Y(t))dW(t), & t \in [0, \infty), \\
    dY(t) = [h(X(t))Y(t) - 1] dt + \langle Z(t), dW(t) \rangle, & t \in [0, \infty), \\
    X(0) = x, & Y(t) \text{ is uniformly bounded.}
\end{cases}
\]

This is an FBSDE in an infinite horizon. (22)
**Theorem (Duffie–Ma–Y).** Under proper conditions. Problem (IHCR) admits at least one nodal solution \((X(\cdot), Y(\cdot))\). In other words, there is a \(C^2\) bounded function \(\theta(\cdot)\) so that \(X(\cdot), Y(\cdot)\) solves (21) and

\[ Y(t) = \theta(X(t)). \]

Moreover, nodal solution is unique.
5. BSPDEs

\[
\begin{align*}
X(t) &= x + \int_0^t b(s, X(s), Y(s), Z(s)) ds \\
&\quad + \int_0^t \sigma(s, X(s), Y(s)) dW(s), \\
Y(t) &= h(X(T)) + \int_t^T g(s, X(s), Y(s), Z(s)) ds \\
&\quad - \int_t^T Z(s) dW(s).
\end{align*}
\]

(23)

In the above, all the involved functions are random fields.

Try to write \( Y(t) = u(t, X(t)) \), for some random field \( u(\cdot, \cdot) \).
Suppose

\[ du(t, x) = p(t, x)dt + \langle q(t, x), dW(t) \rangle, \quad (t, x) \in [0, T] \times \mathbb{R}^n, \quad (24) \]

with \( p(\cdot, \cdot) \) and \( q(\cdot, \cdot) \) undetermined. Then by a generalized Itô-Ventzell formula, we have

\[
u(t, X(t)) = u(T, X(T)) - \int_t^T \left\{ p(s, X(s)) + u_x(s, X(s))b(s, X(s), Y(s), Z(s)) \\
+ \frac{1}{2} \text{tr} \left[ u_{xx}(s, X(s))\sigma(s, X(s), Y(s))\sigma(s, X(s), Y(s))^\top \right] \\
+ \text{tr} \left[ q_x(s, X(s))\sigma(s, X(s)) \right] \right\} ds \\
- \int_t^T \left( q(s, X(s))^\top + u_x(s, X(s))\sigma(s, X(s), Y(s)) \right) dW(s).\]
We choose \((u, q)\) so that

\[
Y(t) = u(t, X(t)).
\]

Then

\[
Z(t) = q(t, X(t))^\top + u_x(t, X(t))\sigma(t, X(t), u(t, X(t))),
\]

\[
g(t, X(t), Y(t), Z(t))
\]

\[
= -\left\{p(t, X(t)) + u_x(t, X(t))b(t, X(t), Y(t), Z(s))
\right.
\]

\[
+ \frac{1}{2} \text{tr} \left[ u_{xx}(s, X(s))\sigma(s, X(s), Y(s))\sigma(s, X(s), Y(s))^\top \right]
\]

\[
+ \text{tr} \left[ q_x(s, X(s))\sigma(s, X(s)) \right]\right\}.
\]
Thus, we need to solve BSPDE:

\[
\begin{align*}
    du(t, x) + & \left\{ \frac{1}{2} \text{tr} \left[ u_{xx}(t, x)\sigma(t, x, u(t, x))\sigma(t, x, u(t, x))^{T} \right] \\
    & + u_{x}(t, x)b(t, x, u(t, x), q(t, x)^{T} + u_{x}(t, x)\sigma(t, x, u(t, x)) \right\} dt \\
    & + \text{tr} \left[ q_{x}(t, x)\sigma(t, x, u(t, x)) \right] \\
    & + g(t, x, u(t, x), q(t, x)^{T} + u_{x}(t, x)\sigma(t, x, u(t, x))) \right\} \\
    & - \langle q(t, x), dW(t) \rangle = 0, \\
    u(T, x) = & h(x).
\end{align*}
\]
If the above BSPDE admits a classical adapted solution \((u, q)\), then we can try to solve the following FSDE:

\[
\begin{cases}
    dX(t) = b(t, X(t), u(t, X(t)), \\
    q(t, X(t)) + u_x(t, X(t))\sigma(t, X(t), u(t, X(t)))) \, dt \\
    + \sigma(t, X(t), u(t, X(t)))dW(t), \\
    X(0) = x.
\end{cases}
\]

(26)

If this can also go through, we could finally set

\[
\begin{cases}
    Y(t) = u(t, X(t)), \\
    Z(t) = q(t, X(t)) + u_x(t, X(t))\sigma(t, X(t), u(t, X(t))).
\end{cases}
\]

(27)
Hence, at least formally, we still have the following Four-Step Scheme:

**Step 1.** Define

\[ z = q + p\sigma(t, x, y), \quad \forall (t, x, p, q) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^{1\times d} \times \mathbb{R}^{1\times n}. \]

**Step 2.** Solve BSPDE (25).

**Step 3.** Solve FSDE (26).

**Step 4.** Set \( Y(\cdot) \) and \( Z(\cdot) \) by (27).
This motivated us to study BSPDEs. As a first step, we consider

\[
    u(t, x) = h(x) + \int_t^T \left\{ \nabla \cdot \left[ A(s, x) u_x(s, x) \right] + u_x(s, x) a(s, x) \\
    + a_0(s, x) u(s, x) + \text{tr} \left[ B(s, x) q_x(s, x) \right] \\
    + \langle b_0(s, x), q(s, x) \rangle + g_0(s, x) \right\} ds \\
    - \int_t^T q(s, x) dW(s),
\]

(28)

**Theorem (Ma–Y).** Under proper conditions, including

\[
    [B(\partial x_i B^\top)]^\top = B(\partial x_i B^\top),
\]

(29)

a.e. \((t, x) \in [0, T] \times \mathbb{R}^n\), a.s. , \(1 \leq i \leq n\).

BSPDE (28) is well-posed.
• 2000, Ying Hu (of Université de Rennes 1, Frances) joined Jin and myself, working on one-dimensional semilinear BSPDEs.

• 2012, Ma–Yin–Zhang, established the equivalence of the well-posedness of random coefficient FBSDEs and the existence of the so-called random decoupling field, via the solution of BSPDEs.

• 2013, Du–Tang–Zhang removed the technical condition (29).

• 2013, Du–Zhang studied multi-dimensional semilinear case.

• 2014, Du–Chen studied BSPDEs, allowing quadratic growth of $(u_x, q)$ in the semilinear term.

• For the general BSPDEs, it seems to be still open for the cases that the differential operators $\mathcal{L}$ and $\mathcal{M}$ are nonlinear in $u$ and/or in $q$. 
In about 1997, Jin and I decided to write a book summarizing the updated theory of FBSDEs. The book was published in 1999.
6. Beyond

- FBSDEs with reflections.
- Numerical solutions of FBSDEs
- Weak solutions of FBSDEs
- Many, many more,...
Great Achievements, Jin!

Congratulations! Bro!
Numerical Algebra, Control and Optimization (NACO):
A journal of American Institute of Mathematical Sciences (AIMS).

Will publish a special issue dedicating Jin Ma “Stochastic Analysis, Mathematical Finance, and Related Topics”, edited by Jianfeng Zhang and Song Yao

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Everyone is welcome to submit papers!
Thank You Very Much!