A stochastic maximum principle for partially observed general mean-field control problems with only weak solution

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Outline

1. Objective of the talk
2. Preliminaries
3. Well-posedness of the state-observation dynamics
4. Stochastic Control Problem

1. Objective of the talk

We consider:

$+$ $(\Omega, \mathcal{F}, P; \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0})$ a filtered P.S. satisfying the usual hypotheses:
–$(\Omega, \mathcal{F}) := (C_T^2, \mathcal{B}(C_T^2))$, where $C_T^2 = C([0,T]; \mathbb{R}^2)$;
–$\mathbb{F}$ be the natural filtration generated by the coordinate process on Ω;

$+$ (E, d) separable complete metric space, $\mathcal{B}(E)$ Borel σ-field over (E, d);
$+$ $\mathcal{P}(E)$ the space of all probability measures over $(E, \mathcal{B}(E))$;
$+$ $\mathcal{P}_p(E)$ the space of probability measures on $(E, \mathcal{B}(E))$ with finite p-th moment, $p \geq 1$, endowed with the metric:

$$W_p(\mu,\nu) := \inf\Big\{ \Big(\int_{E\times E} \big(d(z,z')\big)^p \rho(dzdz')\Big)^{\frac{1}{p}} \,\Big|\, \rho \in \mathcal{P}_p(E\times E) \text{ with } \rho(\cdot\times E) = \mu,\ \rho(E\times\cdot) = \nu \Big\}.$$

Note: $(\mathcal{P}_p(E), W_p(\cdot,\cdot))$ is a complete metric space.

Brief state of the art

Mean-field problems:

1) Mean-Field SDEs have been intensively studied for a longer time as limit equ. for systems with a large number of particles (propagation of chaos)(Bossy, Méléard, Sznitman, Talay,...);

2) Mean-Field Games and related topics, since 2006-2007 by J.M.Lasry and P.L.Lions, Huang-Caines-Malhamé (2006);

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3) +) Mean-Field BSDEs/FBSDEs and associated nonlocal PDEs:

- Preliminary works in: Buckdahn, Dijehiche, L. Peng (2009, AOP), Buckdahn, L. Peng (2009, SPA);
- Classical solution of non-linear PDE related with the mean-field SDE: Buckdahn, L., Peng, Rainer (2017, AOP (2014, Arxiv));
- For the case with jumps: L., Hao (2016, NODEA);
- For the case with the mean-field forward and backward SDE jumps: L. (2017, SPA);
- For the case with continuous coefficients: L., Liang, Zhang (2018, JMAA)

Brief state of the art

- For derivative over Wasserstein spaces along curves of densities:

  Buckdahn, L., Liang. Arxiv. 2020.

+) Controlled mean-field forward and backward SDEs:

- For Pontryagin's maximum principle: L. (2012, Automatica);

  + with partial observations: Buckdahn, L., Ma (2017, AAP);

- For Peng's maximum principle: Buckdahn, Djehiche, L. (2011, AMO);

$\rightsquigarrow$Buckdahn, L., Ma (2016, AMO): Controlled mean-field stochastic system:

$$dX_t^v = b(t, P_{X_t^v}, X_t^v, , v_t)dt + \sigma(t, P_{X_t^v}, X_t^v, v_t)dW_t,\ t \in [0, T]...$$

$\rightsquigarrow$Buckdahn, Chen, L. (2021, SPA): Controlled mean-field stochastic system:

$$dX_t^v = b(t, P_{(X_t^v, v_t)}, X_t^v, v_t)dt + \sigma(t, P_{(X_t^v, v_t)}, X_t^v, v_t)dW_t,\ t \in [0, T]...$$

  + with partial observations:

  L., Liang, Mi (2021, arxiv)

- For Zero-sum stochastic differential games:

  L., Min (2016 (SICON))

....

1. Objective of the talk

Investigate Peng's maximum principle for a general type of mean-field stochastic control problems with partial observations. Extends:

- Buckdahn, L. and Ma (AAP, 2017)

The novelties in our work:

- The coefficients of the systems depend in a nonlinear way not only on the paths but also on the law of the conditional expectation of the state with respect to the observation process up to date;

- In spite of the use of reference probability measure, having only a weak solution of our controlled system, we need to work with the law under different probability measures depending on the solution, which makes the computations very hard and technical;

- The first order variational equation we obtain is of a new type of coupled mean-field SDE to the best of our knowledge.

- The SMP we obtain is of a new type too.

2. Preliminaries

Spaces we work with: For any sub-σ-field $\mathcal{G}$ of $\mathcal{F}$ and any subfiltration $\mathbb{G}$ of $\mathbb{F}$, $p \geq 1$, we denote

• $L^p(\mathcal{G}, P; \mathbb{R}^k)$ is the set of $\mathbb{R}^k$-valued, $\mathcal{G}$-measurable random variables ξ with $E^P[|\xi|^p] < \infty$. Here $E^P[\cdot]$ denotes the expectation w.r.t. P.

• $S^p_{\mathbb{G}}([0,T], P; \mathbb{R}^k)$ denotes the set of $\mathbb{R}^k$-valued, $\mathbb{G}$-adapted continuous stochastic processes X on $[0,T]$, with $E^P\big[\sup_{t\in[0,T]} |X_t|^p\big] < \infty$.

• $L^p_{\mathbb{G}}([0,T], P; \mathbb{R}^k)$ is the set of $\mathbb{R}^k$-valued, $\mathbb{G}$-progressively measurable stochastic processes X on $[0,T]$, with $E^P\Big[\Big(\int_0^T |X_t|^2 dt\Big)^{\frac{p}{2}}\Big] < \infty$.

Derivative of a function with respect to a probability measure

(see: Course at Institut de France by P.-L. Lions, 2013; notes by Cardaliaguet, 2013, but also: Cargaliaguet, Delarue, Lasry, Lions (Princeton University Press, 2019) for an equivalent approach)

+ Given any function $h : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$:

+ Its "lifted" function: $\widetilde{h} : L^2(\mathcal{F}; \mathbb{R}^d) \to \mathbb{R}$ defined by $\widetilde{h}(\xi) = h(P_\xi)$, $\xi \in L^2(\mathcal{F}; \mathbb{R}^d)$ (advantage: $L^2(\mathcal{F}; \mathbb{R}^d)$ is a Hilbert space);

+ Differentiablility: If for $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, there exists $\xi \in L^2(\mathcal{F}; \mathbb{R}^d)$ s.t. $\mu = P_\xi$ and $\widetilde{h}(\cdot) : L^2(\mathcal{F}; \mathbb{R}^d) \to \mathbb{R}$ is Fréchet differentiable at ξ, then $h : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is said to be differentiable at μ.

Remark 2.1. Let $\xi \in L^2(\mathcal{F};\mathbb{R}^d)$ s.t. $\widetilde{h}(\cdot) : L^2(\mathcal{F};\mathbb{R}^d) \to \mathbb{R}$ is Fréchet differentiable at ξ; there exists $D\widetilde{h}(\xi) \in L(L^2(\mathcal{F};\mathbb{R}^d),\mathbb{R}^d)$ s.t., for every $\eta \in L^2(\mathcal{F};\mathbb{R}^d)$,

$$\widetilde{h}(\xi+\eta) - \widetilde{h}(\xi) = D\widetilde{h}(\xi)(\eta) + o(|\eta|_{L^2}), \quad \text{as } |\eta|_{L^2} \to 0. \tag{2.1}$$

Due to the Riesz Representation Theorem, there exists $\theta \in L^2(\mathcal{F};\mathbb{R}^d)$ s.t.,

$$D\widetilde{h}(\xi)(\eta) = E[\theta \cdot \eta],\ \eta \in L^2(\mathcal{F};\mathbb{R}^d).$$

2. Preliminaries

As shown by P.-L. Lions (2013), there exists a Borel function $g : \mathbb{R}^d \to \mathbb{R}^d$ s.t. $\theta = g(\xi)$, P-$a.s.$, and g depends on ξ only through its law P_ξ.

Thus, we can write (2.1) as

$$h(P_{\xi+\eta}) - h(P_\xi) = E[g(\xi) \cdot \eta] + o(|\eta|_{L^2}), \;\; \eta \in L^2(\mathcal{F}; \mathbb{R}^d).$$

The function $g(\cdot)$ is called the derivative of $h : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ at $\mu(= P_\xi)$, and it is denoted by $\partial_\mu h(\mu, y) = g(y), y \in \mathbb{R}^d$. Hence, we have, for every $\eta \in L^2(\mathcal{F}; \mathbb{R}^d)$,

$$D\widetilde{h}(\xi)(\eta) = E[g(\xi) \cdot \eta] = E[\partial_\mu h(P_\xi, \xi) \cdot \eta].$$

That is, if $h : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is differentiable at μ with $\mu = P_\xi$, we also have

$$h(P_{\xi+\eta}) - h(P_\xi) = E[\partial_\mu h(P_\xi, \xi) \cdot \eta] + o(|\eta|_{L^2}), \;\; \eta \in L^2(\mathcal{F}; \mathbb{R}^d).$$

$\rightsquigarrow$ Buckdahn, L., Peng, Rainer (2017)

The dynamics of the state and the observation processes

$\bullet X$ is the state process and Y is the observation process defined on $(\Omega, \mathcal{F}, P)$:

$$\begin{cases} dX_t = \sigma(t, Y_{\cdot\wedge t}, X_t, \mu_t^{X|Y})dB_t^1, \ X_0 = x_0 \in \mathbb{R}; \\ dY_t = h(t, Y_{\cdot\wedge t}, X_t, \mu_t^{X|Y})dt + dB_t^2, \ Y_0 = 0, \ t \in [0, T], \end{cases} \tag{3.1}$$

where (B^1, B^2) is an $(\mathbb{F}, P)$-Brownian motion.

$+ \ U_t^{X|Y} := E^P[X_t \,|\, \mathcal{F}_t^Y], \ t \in [0, T]$, denotes the "filtered" state process and $\mu_t^{X|Y}$ its law under P, i.e., $\mu_t^{X|Y} := P_{U_t^{X|Y}}$.

$+ \ \mathbb{F}^Y$ is the filtration generated by process Y.

• Note: The state process X can not be observed directly but only through Y, so it is natural to consider the control u as $\mathbb{F}^Y$-adapted.

We will consider the well-posedness of (3.1) under the following Assumptions (H1).

Assumption (H1)

(i) The functions $\sigma, h : [0,T] \times C_T \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ are Borel measurable and bounded;

(ii) For all $(t,y) \in [0,T] \times C_T$, $x, x' \in \mathbb{R}$, $\gamma, \gamma' \in \mathcal{P}_2(\mathbb{R})$:

$$|\phi(t, y_{\cdot\wedge t}, x, \gamma) - \phi(t, y_{\cdot\wedge t}, x', \gamma')| \leq C\big(|x - x'| + W_1(\gamma, \gamma')\big),$$

for $\phi = \sigma, h$.

Remark 3.1. In (3.1) we have assumed that the drift coefficient $b = 0$. Indeed, the extension of our discussion to the case of a drift does not add additional difficulties.

• We use a reference probability measure argument. This allows to transform system (3.1) into the form

$$\left\{ \begin{array}{l} dX_t = \sigma(t, Y_{\cdot\wedge t}, X_t, \mu_t^{X|Y}) dB_t^1, \ X_0 = x_0 \in \mathbb{R}; \\ dL_t = h(t, Y_{\cdot\wedge t}, X_t, \mu_t^{X|Y}) L_t dY_t, \ L_0 = 1. \end{array} \right. \tag{3.2}$$

• For this we assume

+ (B^1, Y) is the coordinate process on $\Omega = C_T^2$,
$(B_t^1(\omega), Y_t(\omega)) = (\omega_1(t), \omega_2(t))$, $\omega = (\omega_1, \omega_2) \in \Omega$, $t \in [0, T]$.

+ Q is the Wiener measure over $(\Omega, \mathcal{F}) = (C_T^2, \mathcal{B}(C_T^2))$.

+ $\mathcal{F}$ is considered to be completed w.r.t. Q.

+ Denote by $\mathbb{F} = \mathbb{F}^{B^1, Y}$ the filtration generated by (B^1, Y) and augmented by all Q-null sets. In particular, (B^1, Y) is an $(\mathbb{F}, Q)$-Brownian motion.

+ Note that $P = L_T Q$ is a probability.

Theorem 3.1.

Under (H1) equation (3.2) possesses a unique strong solution.

Sketch of the proof. Given any $V \in S^2_{\mathbb{F}}([0,T],Q)$, and $K \in \mathcal{K}^2_{\mathbb{F}}([0,T],Q) := \big\{K \in S^2_{\mathbb{F}}([0,T],Q) \,\big|\, K_T \geq 0,\ E^Q[K_T] \ = 1,\ K_t = E^Q[K_T|\mathcal{F}_t],\ t \in [0,T]\big\}$.

• Putting $P := K_T Q$, and $\mu_t := P_{E^P[V_t|\mathcal{F}^Y_t]}$, $t \in [0,T]$, we consider the following SDE:

$$\begin{array}{l} d\overline{X}_t = \sigma(t, Y_{\cdot\wedge t}, \overline{X}_t, \mu_t) dB^1_t,\ \ \overline{X}_0 = x_0 \in \mathbb{R}; \\ d\overline{L}_t = h(t, Y_{\cdot\wedge t}, \overline{X}_t, \mu_t)\overline{L}_t dY_t,\ \ \overline{L}_0 = 1. \end{array} \tag{3.3}$$

• SDEs that (3.3) $\exists$ unique $(\overline{X}, \overline{L}) \in S^2_{\mathbb{F}}([0,T],Q) \times \mathcal{K}^2_{\mathbb{F}}([0,T],Q)$.

• Putting $\Phi(V,K) := (\overline{X}, \overline{L}) : S^2_{\mathbb{F}}([0,T],Q) \times \mathcal{K}^2_{\mathbb{F}}([0,T],Q) \to$ itself......

Remark 3.2. The existence of a strong solution (X,L) of SDE (3.2) implies, in particular, that of a weak solution of (3.1).

Definition 3.1.

A six-tuple $(\Omega, \mathcal{F}, \mathbb{F}, P, (B^1, B^2), (X, Y))$ is called a weak solution of (3.1) if:
i) $(\Omega, \mathcal{F}, \mathbb{F}, P)$ is a filtered P.S. satisfying the usual hypotheses;
ii) (B^1, B^2) is an $(\mathbb{F}, P)$-Brownian motion;
iii) All terms in (3.1) are well-defined, (X, Y) is an $\mathbb{F}$-adapted process and equation (3.1) holds true, for all $t \in [0, T]$, P-a.s.

- Note:

From the Girsanov theorem, we know that, given a strong solution (X, L) of (3.2) with driving $(\mathbb{F}, Q)$-Brownian motion (B^1, Y), $(\Omega, \mathcal{F}, \mathbb{F}, P, (B^1, B^2), (X, Y))$ is a weak solution of (3.1), where $P = L_T Q$ and $B_t^2 = Y_t - \int_0^t h(s, Y_{\cdot\wedge s}, X_s, \mu_s^{X|Y})ds, \ t \in [0, T]$. As a conclusion, under Assumptions (H1), the dynamic (3.1) admits at least one solution in the sense of Definition 3.1.

Remark 3.3.

Note that

$$U_t^{X|Y} := E^P[X_t \,|\, \mathcal{F}_t^Y] = \frac{E^Q[L_t X_t \,|\, \mathcal{F}_t^Y]}{E^Q[L_t \,|\, \mathcal{F}_t^Y]}, \; Q\text{-a.s.}, \; t \in [0,T].$$

Furthermore, as L_t and X_t are both $\mathcal{F}_t^{B^1,Y}$-measurable and thus independent of $\sigma\{Y_s - Y_t, s \in [t,T]\}$, we also have

$$U_t^{X|Y} = \frac{E^Q[L_t X_t \,|\, \mathcal{F}_t^Y]}{E^Q[L_t \,|\, \mathcal{F}_t^Y]} = \frac{E^Q[L_t X_t \,|\, \mathcal{F}_T^Y]}{E^Q[L_t \,|\, \mathcal{F}_T^Y]}, \; Q\text{-a.s.}, \; t \in [0,T].$$

From (3.2), it follows that

$$E^Q[L_t \,|\, \mathcal{F}_t^Y] = 1 + \int_0^t E^Q[L_s h(s, Y_{\cdot\wedge s}, X_s, \mu_s^{X|Y}) \,|\, \mathcal{F}_s^Y] dY_s, \; t \in [0,T],$$

Remark 3.3. (continued.)

From (3.2), it follows that

$$E^Q[L_t \,|\, \mathcal{F}_t^Y] = 1 + \int_0^t E^Q[L_s h(s, Y_{\cdot\wedge s}, X_s, \mu_s^{X|Y}) \,|\, \mathcal{F}_s^Y] dY_s, \;\; t \in [0,T],$$

and applying Itô's formula in (3.2) before taking conditional expectation gives that

$$E^Q[X_t L_t \,|\, \mathcal{F}_t^Y] = x_0 + \int_0^t E^Q[X_s L_s h(s, Y_{\cdot\wedge s}, X_s, \mu_s^{X|Y}) \,|\, \mathcal{F}_s^Y] dY_s, \;\; t \in [0,T].$$

Remark 3.3. (continued.)

Thus, applying Itô's formula to $U_t^{X|Y} = \dfrac{E^Q[L_t X_t \,|\, \mathcal{F}_t^Y]}{E^Q[L_t \,|\, \mathcal{F}_t^Y]}$ we deduce the so-called Fujisaki-Kallianpur-Kunita (FKK) equation: for $t \in [0,T]$, Q-a.s.,

$$\begin{aligned}
&dU_t^{X|Y} = dE^P[X_t \,|\, \mathcal{F}_t^Y] \\
=&\big\{E^P[X_t h(t, Y_{\cdot\wedge t}, X_t, \mu_t^{X|Y}) \,|\, \mathcal{F}_t^Y] \\
&\qquad - E^P[X_t \,|\, \mathcal{F}_t^Y] E^P[h(t, Y_{\cdot\wedge t}, X_t, \mu_t^{X|Y}) \,|\, \mathcal{F}_t^Y]\big\} dY_t \\
&+ \Big\{E^P[X_t \,|\, \mathcal{F}_t^Y]\big(E^P[h(t, Y_{\cdot\wedge t}, X_t, \mu_t^{X|Y}) \,|\, \mathcal{F}_t^Y]\big)^2 \\
&\qquad - E^P[X_t h(t, Y_{\cdot\wedge t}, X_t, \mu_t^{X|Y}) \,|\, \mathcal{F}_t^Y] E^P[h(t, Y_{\cdot\wedge t}, X_t, \mu_t^{X|Y}) \,|\, \mathcal{F}_t^Y]\Big\} dt.
\end{aligned} \tag{3.4}$$

Equation (3.4) shows in particular that $U^{X|Y}$ admits a continuous version with which we identify $U^{X|Y}$.

Theorem 3.2.

Under Assumption (H1), let $(\Omega^i, \mathcal{F}^i, \mathbb{F}^i, P^i, (B^{1,i}, B^{2,i}), (X^i, Y^i))$, $i = 1, 2$, be two weak solutions of (3.1). Then it holds that

$$P^1_{((B^{1,1},B^{2,1}),(X^1,Y^1))} = P^2_{((B^{1,2},B^{2,2}),(X^2,Y^2))}. \tag{3.5}$$

$\rightsquigarrow$ • L., Min. Weak solutions of mean-field stochastic differential equations and application to zero-sum stochastic differential games. SICON, 54, 1826–1858, 2017.

Let Q be the reference probability measure on $(\Omega, \mathcal{F})$, under which the coordinate process (B^1, Y) is a Brownian motion.

•Recall:

+ $\mathbb{F} := \mathbb{F}^{B^1,Y}$ is the filtration generated by (B^1, Y).

+ $\mathcal{F}$ and $\mathbb{F}$ are considered as complete under Q.

<u>**The dynamics of the controlled stochastic system**</u>:

$$\begin{cases} dX_t^u = \sigma(t, X_t^u, \mu_t^u, u_t)dB_t^1, \ X_0^u = x; \\ dL_t^u = L_t^u h(t, X_t^u, \mu_t^u, u_t)dY_t, \ L_0^u = 1, \ t \in [0,T], \end{cases} \tag{4.1}$$

where $P^u = L_T^u Q$, and $E^u[\cdot] := E^{P^u}[\cdot]$ is the expectation under P^u.

•$\mu_t^u = \mu_t^{X^u|Y} = P^u_{E^u[X_t^u \,|\, \mathcal{F}_t^Y]}$; •$u \in \mathcal{U}_{ad}$: an admissible control.

• For an arbitrary fixed nonempty subset $U \subset \mathbb{R}^k$ (the control state space) the control u runs the set of admissible controls

$$\mathcal{U}_{ad} = L^0_{\mathbb{F}^Y}([0,T], Q; U),$$

where $L^0_{\mathbb{F}^Y}([0,T], Q; U) := \left\{ v \,\middle|\, v = (v_t)_{t\in[0,T]}, U\text{-valued, } \mathbb{F}^Y\text{-adapted} \right\}$.

Cost functional:

$$J(u) := E^Q\Big[\Phi(X_T^u, \mu_T^u) + \int_0^T f(t, X_t^u, \mu_t^u, u_t)dt\Big],\ u \in \mathcal{U}_{ad}.$$

Control problem: A control $u^* \in \mathcal{U}_{ad}$ satisfying

$$J(u^*) = \inf_{v \in \mathcal{U}_{ad}} J(v)$$

is said to be optimal.

Objective: A necessary condition for the optimality of the control u.

Remark 4.1. We suppose the existence of an optimal control $u^* \in \mathcal{U}_{ad}$, we want to get Peng's stochastic maximum principle, i.e., to derive a necessary optimality condition for u.

We shall make the following standard assumptions.

Assumption (H2)

For the function $\phi := \sigma,\ h,\ f,\ \Phi$, we suppose
(i) The function $\phi : [0,T] \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ is Borel measurable, and to simplify the computations, we also suppose the boundedness;

(ii) For all $t \in [0,T]$, $\mu \in \mathcal{P}_2(\mathbb{R})$ and $v \in U$, the function $\phi(t,\cdot,\mu,v)$ is in $C_b^2(\mathbb{R})$;

(iii) For all $t \in [0,T]$, $x \in \mathbb{R}$ and $v \in U$, the function $\phi(t,x,\cdot,v)$ is differentiable on $\mathcal{P}_2(\mathbb{R})$; $\partial_\mu \phi(t,x,\mu,v;y)$ is bounded and also differentiable w.r.t. $\mu \in \mathcal{P}_2(\mathbb{R})$ and $x,\ y \in \mathbb{R}$, and the derivatives, denoted by $\partial_\mu(\partial_\mu \phi),\ \partial_x(\partial_\mu \phi)$ and $\partial_z(\partial_\mu \phi)$, respectively, are bounded.

(Continued)

Moreover, we have the following continuity conditions: For $t \in [0,T]$, $v \in U$, $\mu, \mu' \in \mathcal{P}_2(\mathbb{R})$ and $x, x', y, y', z, z' \in \mathbb{R}$,

i) $|\phi(t,x,\mu,v) - \phi(t,x,\mu',v)| \leq CW_1(\mu,\mu')$;

ii) $$|\partial_\mu(\partial_\mu\phi)(t,x,\mu,v;y,z) - \partial_\mu(\partial_\mu\phi)(t,x',\mu',v;y',z')| \leq C\big(W_1(\mu,\mu') + |x-x'| + |y-y'| + |z-z'|\big);$$

iii) $$|\psi(t,x,\mu,v;y) - \psi(t,x',\mu',v;y')| \leq C\big(W_1(\mu,\mu') + |x-x'| + |y-y'|\big),$$

$\psi = \partial_\mu\phi,\ \partial_x(\partial_\mu\phi)$and $\partial_z(\partial_\mu\phi)$, resp.

Remark 4.2. 1) Under the Assumption (H2), for all $u \in \mathcal{U}_{ad}$, (4.1) admits a unique solution $(X^u, L^u) \in S^2_{\mathbb{F}}([0,T], Q) \times S^2_{\mathbb{F}}([0,T], Q)$. Moreover, X^u, L^u, U^u are in all $S^p_{\mathbb{F}}([0,T], Q)$, for $p \geq 1$.

2) For all $p \geq 1$, we have $\mu^u_t \in \mathcal{P}_p(\mathbb{R}),\ t \in [0,T]$. Indeed,

$$\int_{\mathbb{R}} |x|^p \mu^u_t(dx) = E^u[|U^u_t|^p] < \infty.$$

Remark 4.3. In Buckdahn, L., Ma (AAP, 2017), the setting for $\phi = \sigma, f$ is

$$\phi(t,x,\gamma,u) := \int \phi(t,x,z,u)\gamma(dz),\ (t,x,\gamma,u) \in [0,T] \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \times \mathcal{U}_{ad},$$

$$h(t,x,\gamma,u) = h(t,x), \qquad \Phi(x,\gamma) = \int \Phi(x,z)\gamma(dz).$$

Moreover, the SMP studied there is the Pontryagin one.

The control state set U is not supposed to be convex, we shall consider Peng's stochastic maximum principle.

$+\ u := u^*$ - the optimal control;

$+\ v \in \mathcal{U}_{ad}$ - an arbitrary but fixed control.

<u>**Spike variational method**</u>. For $\varepsilon > 0$, let $E_\varepsilon \in \mathcal{B}([0,T])$ with $|E_\varepsilon| = \varepsilon$,

$$u^\varepsilon := u\mathbf{1}_{E_\varepsilon^c}(t) + v\mathbf{1}_{E_\varepsilon}(t),\ t \in [0,T].$$

The process $u^\varepsilon \in \mathcal{U}_{ad}$ is a so-called spike variation of the optimal control u.

Remark 4.4. Let $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{Q})$ be a copy of $(\Omega, \mathcal{F}, Q)$. Furthermore, for each $\xi \in L^0(\Omega, \mathcal{F}, Q)$, $\widetilde{\xi} \in L^0(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{Q})$ denotes an independent copy of ξ, i.e., ξ and $\widetilde{\xi}$ are independent, and $\widetilde{\xi}$ under $\widetilde{Q}$ has the same law as ξ under Q. In the same spirit we can consider another copy $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{Q})$......

Furthermore, for simplicity we also introduce the following notations: For $\phi = \sigma, h, f$ and Φ we set

$$
\begin{array}{ll}
\phi(t) := \phi(t, X_t^u, \mu_t^u, u_t), & \delta\phi(t) := \phi(t, X_t^u, \mu_t^u, v_t) - \phi(t, X_t^u, \mu_t^u, u_t), \\
\phi_x(t) := \partial_x \phi(t, X_t^u, \mu_t^u, u_t), & \phi_{xx}(t) := \partial_{xx}^2 \phi(t, X_t^u, \mu_t^u, u_t), \\
\phi_\mu(t, y) := \partial_\mu \phi(t, X_t^u, \mu_t^u, u_t; y), & \widetilde{\phi}_\mu(t) := \phi_\mu(t, \widetilde{U}_t^u) = \partial_\mu \phi(t, X_t^u, \mu_t^u, u_t; \widetilde{U}_t^u), \\
\widetilde{\phi}_\mu^*(t) := \partial_\mu \phi(t, \widetilde{X}_t^u, \mu_t^u, \widetilde{u}_t; U_t^u), & \phi_\mu^*(t, y) := \partial_\mu \phi(t, \widetilde{X}_t^u, \mu_t^u, \widetilde{u}_t; y) \\
\phi_{z\mu}(t, y) := \partial_z(\partial_\mu \phi)(t, X_t^u, \mu_t^u, u_t; y), & \widetilde{\phi}_{z\mu}(t) := \partial_z(\partial_\mu \phi)(t, X_t^u, \mu_t^u, u_t; \widetilde{U}_t^u), \\
\widetilde{\phi}_{z\mu}^*(t) := \partial_z(\partial_\mu \phi)(t, \widetilde{X}_t^u, \mu_t^u, \widetilde{u}_t; U_t^u). &
\end{array}
$$

and $(X, L) := (X^u, L^u)$, $P := L_T Q (= P^u)$, $U_t = U_t^u := E^P[X_t \,|\, \mathcal{F}_t^Y]$, $\mu_t := \mu_t^u := P_{U_t}^u$;

similarly we define $(X^\varepsilon, L^\varepsilon) := (X^{u^\varepsilon}, L^{u^\varepsilon})$, $P^\varepsilon := P^{u^\varepsilon}$, $\mu^\varepsilon := \mu^{u^\varepsilon}$ and $U_t^\varepsilon := E^{P^\varepsilon}[X_t^\varepsilon \,|\, \mathcal{F}_t^Y]$, $t \in [0, T]$.

For $\varepsilon > 0$, the state-observation dynamics is as follows:

$$\begin{cases} dX_t^\varepsilon = \sigma(t, X_t^\varepsilon, \mu_t^\varepsilon, u_t^\varepsilon) dB_t^1, \ X_0^\varepsilon = x; \\ dL_t^\varepsilon = L_t^\varepsilon h(t, X_t^\varepsilon, \mu_t^\varepsilon, u_t^\varepsilon) dY_t, \ L_0^\varepsilon = 1, \ t \in [0, T]; \\ \mu_t^\varepsilon = P^\varepsilon_{U_t^\varepsilon}, \text{ with } P^\varepsilon = L_T^\varepsilon Q, \ U_t^\varepsilon = E^{P^\varepsilon}[X_t^\varepsilon \,|\, \mathcal{F}_t^Y] = \dfrac{E^Q[L_t^\varepsilon X_t^\varepsilon \,|\, \mathcal{F}_t^Y]}{E^Q[L_t^\varepsilon \,|\, \mathcal{F}_t^Y]}. \end{cases} \tag{4.2}$$

For $\varepsilon = 0$, we put $(X^0, L^0, U^0, \mu^0, u^0, P^0) := (X, L, U, \mu, u, P)$.

- Note:

Formally, we should derive (4.2) with respect to ε at $\varepsilon = 0$, but as $\phi = \sigma, h$, is not differentiable in the control variable, we take $\delta\phi(t) = \phi(t, X_t, \mu_t, v_t) - \phi(t, X_t, \mu_t, u_t)$ instead of $\partial_\varepsilon\big[\phi(t, X_t, \mu_t, u_t^\varepsilon)\big]_{|\varepsilon=0}$.

In order to give an idea about how to handle the μ_t^ε-variable, we recall that, if $f : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ is continuously differentiable and $\varepsilon \to (X^\varepsilon, L^\varepsilon, U^\varepsilon)$ were differentiable in $\varepsilon = 0$, we have due to **Theorem 3.2** in (Buckdahn, L., Liang, 2020)

$$
\begin{aligned}
\partial_\varepsilon f(\mu_t^\varepsilon)_{|\varepsilon=0} =& \partial_\varepsilon\big[f\big((L_T^\varepsilon Q)_{U_t^\varepsilon}\big)\big]_{|\varepsilon=0} \\
=& \partial_\varepsilon\big[f\big((L_T^\varepsilon Q)_{U_t}\big)\big]_{|\varepsilon=0} + \partial_\varepsilon\big[f\big((L_T Q)_{U_t^\varepsilon}\big)\big]_{|\varepsilon=0} \\
=& E^Q\Big[\int_0^{U_t} \partial_\mu f\big((L_T Q)_{U_t}, y\big)dy \cdot \partial_\varepsilon L^\varepsilon_{T|\varepsilon=0}\Big] \\
& \qquad + E^{L_T Q}\big[\partial_\mu f\big((L_T Q)_{U_t}, U_t\big) \cdot \partial_\varepsilon U^\varepsilon_{t\,|\varepsilon=0}\big].
\end{aligned}
$$

4. Stochastic Control Problem

As $\int_0^{U_t} \partial_\mu f\big((L_T Q)_{U_t}, y\big) dy$ is $\mathcal{F}_t$-measurable and L^ε is an $(\mathbb{F}, Q)$-martingale, this would give

$$\begin{aligned}
\partial_\varepsilon f(\mu_t^\varepsilon)_{|\varepsilon=0} =& E^Q\Big[\int_0^{U_t} \partial_\mu f(\mu_t, y) dy \cdot \partial_\varepsilon L^\varepsilon_{t\,|\varepsilon=0}\Big] + E^Q\big[\partial_\mu f(\mu_t, U_t) L_t \cdot \partial_\varepsilon U^\varepsilon_{t\,|\varepsilon=0}\big] \\
&\Big(= E^Q\Big[\partial_\varepsilon\Big(L_t^\varepsilon \int_0^{U_t^\varepsilon} \partial_\mu f(\mu_t, y) dy\Big)_{|\varepsilon=0}\Big]\Big),
\end{aligned}$$

with

$$\begin{aligned}
&\partial_\varepsilon U^\varepsilon_{t\,|\varepsilon=0} \\
&= \frac{E^Q[X_t \partial_\varepsilon L^\varepsilon_{t\,|\varepsilon=0} + L_t \partial_\varepsilon X^\varepsilon_{t\,|\varepsilon=0} \,|\, \mathcal{F}_t^Y]}{E^Q[L_t \,|\, \mathcal{F}_t^Y]} - \frac{E^Q[L_t X_t \,|\, \mathcal{F}_t^Y]}{(E^Q[L_t \,|\, \mathcal{F}_t^Y])^2} E[\partial_\varepsilon L^\varepsilon_{t\,|\varepsilon=0} \,|\, \mathcal{F}_t^Y] \\
&= E^P\big[X_t \partial_\varepsilon[\ln L_t^\varepsilon]_{|\varepsilon=0} \,\big|\, \mathcal{F}_t^Y\big] + E^P\big[\partial_\varepsilon X^\varepsilon_{t\,|\varepsilon=0} \,\big|\, \mathcal{F}_t^Y\big] \\
&\qquad - E^P[X_t \,|\, \mathcal{F}_t^Y] E^P\big[\partial_\varepsilon[\ln L_t^\varepsilon]_{|\varepsilon=0} \,\big|\, \mathcal{F}_t^Y\big].
\end{aligned}$$

But the derivatives $\partial_\varepsilon X^\varepsilon{}_{|\varepsilon=0}$ and $\partial_\varepsilon L^\varepsilon{}_{|\varepsilon=0}$ don't exist. They will be replaced by the solution of **the first order variational equation** $Y^{1,\varepsilon} = (Y^{1,\varepsilon})_{t\in[0,T]}$ and $K^{1,\varepsilon} = (K^{1,\varepsilon})_{t\in[0,T]}$, respectively. Together with the classical dependence of the coefficients $\phi = \sigma, h$ on X^ε this suggests the following first order variational equations whose choice will have to be confirmed by the fact that $X_t^\varepsilon - (X_t + Y_t^{1,\varepsilon}) = O(\varepsilon)$ and $L_t^\varepsilon - (L_t + K_t^{1,\varepsilon}) = O(\varepsilon)$, uniformly in $t \in [0,T]$, in $L^2([0,T],Q)$, as $\varepsilon \searrow 0$.

The first-order variational equation: For $\varepsilon > 0$,

$$\begin{cases} dY_t^{1,\varepsilon} = \Big\{\sigma_x(t)Y_t^{1,\varepsilon} + \widetilde{E}^Q\Big[\displaystyle\int_0^{\widetilde{U}_t} \sigma_\mu(t,y)dy \cdot \widetilde{K}_t^{1,\varepsilon}\Big] + \widetilde{E}^Q\big[\widetilde{\sigma}_\mu(t)\widetilde{L}_t\widetilde{V}_t^{1,\varepsilon}\big] + \delta\sigma(t)\mathbf{1}_{E_\varepsilon}(t)\Big\}dB_t^1, \\ Y_0^{1,\varepsilon} = 0; \\ dK_t^{1,\varepsilon} = \Big\{h(t)K_t^{1,\varepsilon} + \Big(h_x(t)Y_t^{1,\varepsilon} + \widetilde{E}^Q\Big[\displaystyle\int_0^{\widetilde{U}_t} h_\mu(t,y)dy \cdot \widetilde{K}_t^{1,\varepsilon}\Big] \\ \qquad\qquad + \widetilde{E}^Q\big[\widetilde{h}_\mu(t)\widetilde{L}_t\widetilde{V}_t^{1,\varepsilon}\big] + \delta h(t)\mathbf{1}_{E_\varepsilon}(t)\Big)L_t\Big\}dY_t, \\ K_0^{1,\varepsilon} = 0; \\ V_t^{1,\varepsilon} = \dfrac{E^Q[L_tY_t^{1,\varepsilon} + X_tK_t^{1,\varepsilon} \mid \mathcal{F}_t^Y]}{E^Q[L_t \mid \mathcal{F}_t^Y]} - \dfrac{E^Q[L_tX_t \mid \mathcal{F}_t^Y]E^Q[K_t^{1,\varepsilon} \mid \mathcal{F}_t^Y]}{(E^Q[L_t \mid \mathcal{F}_t^Y])^2}, \ t \in [0,T]. \end{cases} \tag{4.3}$$

4. Stochastic Control Problem

Proposition 4.1.

Under Assumption (H2), (4.3) has a unique solution $(Y^{1,\varepsilon}, K^{1,\varepsilon})$ $\in S^2_{\mathbb{F}}([0,T],Q) \times S^2_{\mathbb{F}}([0,T],Q)$.

Moreover, $Y^{1,\varepsilon}$, $K^{1,\varepsilon}$, $V^{1,\varepsilon} \in S^p_{\mathbb{F}}([0,T],Q)$ for all $p \geq 1$.

- $V^{1,\varepsilon}_t = \theta_t(Y^{1,\varepsilon}_t, K^{1,\varepsilon}_t)$, where, for $\zeta \in S^2_{\mathbb{F}}([0,T],Q)$, $\eta \in S^2_{\mathbb{F}}([0,T],Q)$ we define

$$\theta_t(\zeta_t, \eta_t) = \frac{E^Q[L_t\zeta_t + X_t\eta_t \,|\, \mathcal{F}^Y_t]}{E^Q[L_t \,|\, \mathcal{F}^Y_t]} - \frac{E^Q[L_t X_t \,|\, \mathcal{F}^Y_t] E^Q[\eta_t \,|\, \mathcal{F}^Y_t]}{(E^Q[L_t \,|\, \mathcal{F}^Y_t])^2}.$$

Proposition 4.2.

For all $k \geq 1$, there exists $C_k \in \mathbb{R}_+$, such that,

(i) $E^Q\Big[\sup_{t\in[0,T]}\big(|X_t^\varepsilon|^{2k} + |L_t^\varepsilon|^{2k}\big)\Big] \leq C_k$;

(ii) $E^Q\Big[\sup_{t\in[0,T]}\big(|X_t^\varepsilon - X_t|^{2k} + |L_t^\varepsilon - L_t|^{2k}\big)\Big] \leq C_k\varepsilon^k,\ \varepsilon > 0$;

(iii) $E^Q\Big[\sup_{t\in[0,T]}\big(|Y_t^{1,\varepsilon}|^{2k} + |K_t^{1,\varepsilon}|^{2k}\big)\Big] \leq C_k\varepsilon^k,\ \varepsilon > 0$;

(iv) $E^Q\Big[\sup_{t\in[0,T]}\big(|X_t^\varepsilon - (X_t + Y_t^{1,\varepsilon})|^{2k} + |L_t^\varepsilon - (L_t + K_t^{1,\varepsilon})|^{2k}\big)\Big] \leq C_k\varepsilon^{2k},\ \varepsilon > 0.$

Remark: The proof of (iv) is very technical. For its proof, we introduce, in particular,

$$(X^{\varepsilon,\lambda}, L^{\varepsilon,\lambda}, U^{\varepsilon,\lambda}) := (1-\lambda)(X, L, U) + \lambda(X^{\varepsilon}, L^{\varepsilon}, U^{\varepsilon}),\ \lambda \in [0,1],$$

and we remark that, due to Theorem 3.2 in Buckdahn, L., Liang (2020), for $\mu_t^{\varepsilon,\lambda} := (L_t^{\varepsilon,\lambda} Q)_{U_t^{\varepsilon,\lambda}}$,

$$\begin{aligned}
\partial_\lambda \sigma(\mu_t^{\varepsilon,\lambda}, u_t^{\varepsilon}) =& \widetilde{E}^Q\Big[\int_0^{\widetilde{U}_t^{\varepsilon,\lambda}} \partial_\mu \sigma(\mu_t^{\varepsilon,\lambda}, u_t^{\varepsilon}; y)dy \cdot \partial_\lambda \widetilde{L}_t^{\varepsilon,\lambda}\Big] \\
&+ \widetilde{E}^Q\big[\partial_\mu \sigma(\mu_t^{\varepsilon,\lambda}, u_t^{\varepsilon}; \widetilde{U}_t^{\varepsilon,\lambda})\widetilde{L}_t^{\varepsilon,\lambda} \cdot \partial_\lambda \widetilde{U}_t^{\varepsilon,\lambda}\big] \\
=& \widetilde{E}^Q\Big[\int_0^{\widetilde{U}_t^{\varepsilon,\lambda}} \partial_\mu \sigma(\mu_t^{\varepsilon,\lambda}, u_t^{\varepsilon}; y)dy(\widetilde{L}_t^{\varepsilon} - \widetilde{L}_t)\Big] \\
&+ \widetilde{E}^Q\big[\partial_\mu \sigma(\mu_t^{\varepsilon,\lambda}, u_t^{\varepsilon}; \widetilde{U}_t^{\varepsilon,\lambda})\widetilde{L}_t^{\varepsilon,\lambda}(\widetilde{U}_t^{\varepsilon} - \widetilde{U}_t)\big].
\end{aligned}$$

....

In the proof of the above proposition we also have proven the following important estimates.

Corollary 4.1.

For all $k \geq 1$, there exists $C_k \in \mathbb{R}_+$ such that,

(i) $E^Q\big[\sup_{t\in[0,T]} |U_t^\varepsilon|^{2k}\big] \leq C_k;$

(ii) $E^Q\big[\sup_{t\in[0,T]} |U_t^\varepsilon - U_t|^{2k}\big] \leq C_k\varepsilon^k,\ \varepsilon > 0;$

(iii) $E^Q\big[\sup_{t\in[0,T]} |V_t^{1,\varepsilon}|^{2k}\big] \leq C_k\varepsilon^k,\ \varepsilon > 0;$

(iv) $E^Q\big[\sup_{t\in[0,T]} |U_t^\varepsilon - (U_t + V_t^{1,\varepsilon})|^{2k}\big] \leq C_k\varepsilon^{2k},\ \varepsilon > 0.$

Now we present a very subtle and useful estimate, whose proof applies and extends in a non-trivial way an idea first introduced in Buckdahn, Chen, L. (2021, SPA).

Proposition 4.3.

For all $\theta = (\theta^1, \theta^2) \in L^2_{\mathbb{F}}([0,T], Q; \mathbb{R}^2)$ with

$$E^Q\Big[\int_0^T \big(|\theta^1_t|^2 + |L_t\theta^2_t|^2\big)dt\Big] < +\infty,$$

and $(\theta^1_t, L_t\theta^2_t) \in L^2(\mathcal{F}_t, Q; \mathbb{R}^2)$ for all $t \in [0,T]$, there exists $\rho : [0,T] \times \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\big|E^Q[\theta^1_t Y^{1,\varepsilon}_t + \theta^2_t K^{1,\varepsilon}_t]\big| \le \rho_t(\varepsilon)\sqrt{\varepsilon},\ \varepsilon \in (0,1],\ t \in [0,T],$$

with $\rho_t(\varepsilon) \to 0\,(\varepsilon \searrow 0)$, $t \in [0,T]$, and

$$\rho_t(\varepsilon) \le CE^Q\big[|\theta^1_t|^2 + |L_t\theta^2_t|^2\big],\ \varepsilon \in (0,1],\ t \in [0,T].$$

The second-order variational equation:

$$
\begin{cases}
dY_t^{2,\varepsilon} = \Big\{\sigma_x(t)Y_t^{2,\varepsilon} + \frac{1}{2}\sigma_{xx}(t)(Y_t^{1,\varepsilon})^2 + \widetilde{E}^Q\Big[\int_0^{\widetilde{U}_t}\sigma_\mu(t,y)dy\cdot\widetilde{K}_t^{2,\varepsilon}\Big] + \widetilde{E}^Q\big[\widetilde{\sigma}_\mu(t)\widetilde{L}_t\widetilde{V}_t^{2,\varepsilon}\big] \\
+ \widetilde{E}^Q\big[\widetilde{\sigma}_\mu(t)\widetilde{V}_t^{1,\varepsilon}\widetilde{K}_t^{1,\varepsilon}\big] + \frac{1}{2}\widetilde{E}^Q\big[\widetilde{\sigma}_{z\mu}(t)\widetilde{L}_t(\widetilde{V}_t^{1,\varepsilon})^2\big] \\
+ \Big(\delta\sigma_x(t)Y_t^{1,\varepsilon} + \widetilde{E}^Q\Big[\int_0^{\widetilde{U}_t}\delta\sigma_\mu(t,y)dy\cdot\widetilde{K}_t^{1,\varepsilon}\Big] + \widetilde{E}^Q\big[\delta\widetilde{\sigma}_\mu(t)\widetilde{L}_t\widetilde{V}_t^{1,\varepsilon}\big]\Big)\mathbf{1}_{E_\varepsilon}(t)\Big\}dB_t^1, \\
dK_t^{2,\varepsilon} = \Big\{h(t)K_t^{2,\varepsilon} + h_x(t)L_tY_t^{2,\varepsilon} + h_x(t)Y_t^{1,\varepsilon}K_t^{1,\varepsilon} + \frac{1}{2}h_{xx}(t)L_t(Y_t^{1,\varepsilon})^2 \\
+ L_t\widetilde{E}^Q\Big[\int_0^{\widetilde{U}_t}h_\mu(t,y)dy\cdot\widetilde{K}_t^{2,\varepsilon}\Big] + L_t\widetilde{E}^Q\big[\widetilde{h}_\mu(t)\widetilde{L}_t\widetilde{V}_t^{2,\varepsilon}\big] \\
+ L_t\widetilde{E}^Q\big[\widetilde{h}_\mu(t)\widetilde{V}_t^{1,\varepsilon}\widetilde{K}_t^{1,\varepsilon}\big] + \frac{1}{2}L_t\widetilde{E}^Q\big[\widetilde{h}_{z\mu}(t)\widetilde{L}_t(\widetilde{V}_t^{1,\varepsilon})^2\big] + \Big(\delta h(t)K_t^{1,\varepsilon} \\
\qquad + \delta h_x(t)L_tY_t^{1,\varepsilon} + L_t\widetilde{E}^Q\Big[\int_0^{\widetilde{U}_t}\delta h_\mu(t,y)dy\cdot\widetilde{K}_t^{1,\varepsilon}\Big] + L_t\widetilde{E}^Q\big[\delta\widetilde{h}_\mu(t)\widetilde{L}_t\widetilde{V}_t^{1,\varepsilon}\big]\Big)\mathbf{1}_{E_\varepsilon}(t)\Big\}dY_t, \\
Y_0^{2,\varepsilon} = K_0^{2,\varepsilon} = 0, \\
V_t^{2,\varepsilon} = \theta_t(Y_t^{2,\varepsilon}, K_t^{2,\varepsilon}) + \frac{E^Q[K_t^{1,\varepsilon}Y_t^{1,\varepsilon}\,|\,\mathcal{F}_t^Y]}{E^Q[L_t\,|\,\mathcal{F}_t^Y]} - \frac{E^Q[K_t^{1,\varepsilon}\,|\,\mathcal{F}_t^Y]}{E^Q[L_t\,|\,\mathcal{F}_t^Y]}V_t^{1,\varepsilon}, t\in[0,T].
\end{cases}
\tag{4.4}
$$

4. Stochastic Control Problem

Lemma 4.1.

Under Assumption (H2), the equation (4.4) has a unique solution

$$(Y^{2,\varepsilon}, K^{2,\varepsilon}) \in S^2_{\mathbb{F}}([0,T],Q) \times S^2_{\mathbb{F}}([0,T],Q).$$

Moreover, $Y^{2,\varepsilon}$, $K^{2,\varepsilon}$, $\varepsilon > 0$, are bounded in $S^p_{\mathbb{F}}([0,T],Q)$, for all $p \geq 2$.

Proposition 4.4.

For all $p \geq 1$, there is a constant $C_p \in \mathbb{R}_+$ such that for $t \in [0,T]$, $\varepsilon > 0$,

$$\Big(E^Q \big[\big| \big(U_t^\varepsilon - (U_t + V_t^{1,\varepsilon} + V_t^{2,\varepsilon}) \big) - \theta_t \big(X_t^\varepsilon - (X_t + Y_t^{1,\varepsilon} + Y_t^{2,\varepsilon}), \\ L_t^\varepsilon - (L_t + K_t^{1,\varepsilon} + K_t^{2,\varepsilon}) \big) \big|^p \big] \Big)^{\frac{1}{p}} \leq C_p \varepsilon^{\frac{3}{2}}.$$

Proposition 4.5.

For all $p \geq 2$, there exists $C_p \in \mathbb{R}_+$, such that,

(i) $E^Q\big[\sup_{t\in[0,T]} \big|X_t^\varepsilon - (X_t + Y_t^{1,\varepsilon} + Y_t^{2,\varepsilon})\big|^p\big] \leq C_p \varepsilon^p \rho_p(\varepsilon)$;

(ii) $E^Q\big[\sup_{t\in[0,T]} \big|L_t^\varepsilon - (L_t + K_t^{1,\varepsilon} + K_t^{2,\varepsilon})\big|^p\big] \leq C_p \varepsilon^p \rho_p(\varepsilon)$;

(iii) $E^Q\big[\sup_{t\in[0,T]} \big|U_t^\varepsilon - (U_t + V_t^{1,\varepsilon} + V_t^{2,\varepsilon})\big|^p\big] \leq C_p \varepsilon^p \rho_p(\varepsilon)$,

with $\rho_p(\varepsilon) \to 0$, as $\varepsilon \searrow 0$. Moreover,

(iv) $E^Q\big[\sup_{t\in[0,T]} |Y_t^{2,\varepsilon}|^p + |K_t^{2,\varepsilon}|^p\big] \leq C_p \varepsilon^p,\ E^Q\big[|V_t^{2,\varepsilon}|^p\big] \leq C_p \varepsilon^p,\ \varepsilon > 0,\ t \in [0,T]$.

The first-order adjoint BSDE:

$$\begin{cases} dp_t^1 = -\alpha_t(q_t^1, q_t^2)dt + q_t^1 dB_t^1 + \check{q}_t^1 dY_t, \ t \in [0,T], \\ p_T^1 = -\Phi_x(T) - L_T \widetilde{E}^Q\big[E^P[\widetilde{\Phi}_\mu^*(T) \,|\, \mathcal{F}_T^Y]\big]; \\ dp_t^2 = -\beta_t(q_t^1, q_t^2)dt + \check{q}_t^2 dB_t^1 + q_t^2 dY_t, \ t \in [0,T], \\ p_T^2 = -(X_T - U_T)\widetilde{E}^Q\big[E^P[\widetilde{\Phi}_\mu^*(T) \,|\, \mathcal{F}_T^Y]\big] - \widetilde{E}^Q\Big[\int_0^{U_T} \Phi_\mu^*(T,y)dy\Big]. \end{cases} \tag{4.5}$$

where

$$\alpha_t^0(q_t^1, q_t^2) := \sigma_x(t)q_t^1 + L_t\widetilde{E}^Q\big[\widetilde{q}_t^1 E^P[\widetilde{\sigma}_\mu^*(t) \,|\, \mathcal{F}_t^Y]\big] + h_x(t)L_t q_t^2 + L_t\widetilde{E}^Q\big[\widetilde{q}_t^2 \widetilde{L}_t E^P[\widetilde{h}_\mu^*(t) \,|\, \mathcal{F}_t^Y]\big];$$

$$\alpha_t(q_t^1, q_t^2) := \alpha_t^0(q_t^1, q_t^2) - f_x(t) - L_t\widetilde{E}^Q\big[E^P[\widetilde{f}_\mu^*(t) \,|\, \mathcal{F}_t^Y]\big];$$

$$\begin{aligned} \beta_t^0(q_t^1, q_t^2) := &(X_t - U_t)\widetilde{E}^Q\big[\widetilde{q}_t^1 E^P[\widetilde{\sigma}_\mu^*(t) \,|\, \mathcal{F}_t^Y]\big] + \widetilde{E}^Q\big[\widetilde{q}_t^1 \int_0^{U_t} \sigma_\mu^*(t,y)dy\big] \\ &+ h(t)q_t^2 + (X_t - U_t)\widetilde{E}^Q\big[\widetilde{q}_t^2 \widetilde{L}_t E^P[\widetilde{h}_\mu^*(t) \,|\, \mathcal{F}_t^Y]\big] + \widetilde{E}^Q\big[\widetilde{q}_t^2 \widetilde{L}_t \int_0^{U_t} h_\mu^*(t,y)dy\big]; \end{aligned}$$

$$\beta_t(q_t^1, q_t^2) := \beta_t^0(q_t^1, q_t^2) - (X_t - U_t)\widetilde{E}^Q\big[E^P[\widetilde{f}_\mu^*(t) \,|\, \mathcal{F}_t^Y]\big] - \widetilde{E}^Q\big[\int_0^{U_t} f_\mu^*(t,y)dy\big], \ t \in [0,T].$$

Recall that $U_t = E^P[X_t \mid \mathcal{F}_t^Y]$. Using the definition of α_t and β_t, we get the following duality relation

$$
\begin{aligned}
& E^Q[p_T^1 Y_T^{1,\varepsilon} + p_T^2 K_T^{1,\varepsilon}] \\
& = E^Q\Big[\int_0^T \Big\{Y_t^{1,\varepsilon}\big(f_x(t) + L_t \widetilde{E}^Q\big[E^P[\widetilde{f}_\mu^*(t) \mid \mathcal{F}_t^Y]\big]\big) \\
& \qquad + K_t^{1,\varepsilon}\Big((X_t - U_t)\widetilde{E}^Q\big[E^P[\widetilde{f}_\mu^*(t) \mid \mathcal{F}_t^Y]\big] + \widetilde{E}^Q\Big[\int_0^{U_t} f_\mu^*(t,y)dy\Big]\Big) \\
& \qquad + \big(q_t^1 \delta\sigma(t) + q_t^2 L_t \delta h(t)\big)\mathbf{1}_{E_\varepsilon}(t)\Big\}dt\Big].
\end{aligned}
\tag{4.6}
$$

As the mean-field BSDE (4.5) does not have Lipschitz coefficients, to the best of our knowledge, it is new, so we need the following result.

Proposition 4.6.

Under Assumption (H2), BSDE (4.5) has a unique strong solution $\big((p^1,(q^1,\check{q}^1)),(p^2,(\check{q}^2,q^2))\big)$.

Furthermore, for any $p \geq 2$, it holds that $\big((p^1,(q^1,\check{q}^1)),\ (p^2,(\check{q}^2,q^2))\big) \in \big(S^p_{\mathbb{F}}([0,T],Q)\times(L^p_{\mathbb{F}}([0,T],Q))^2\big)\times\big(S^{2p}_{\mathbb{F}}([0,T],Q)\times(L^{2p}_{\mathbb{F}}([0,T],Q))^2\big)$.

We introduce the **Hamiltonian** H:

$$H(t,x,l,\gamma,v,q_1,q_2) := \sigma(t,x,\gamma,v)q_1 + h(t,x,\gamma,v)lq_2 - f(t,x,\gamma,v),$$

for $(t,x,l,\gamma,v,q_1,q_2) \in [0,T] \times \mathbb{R} \times \mathbb{R}_+ \times \mathcal{P}_2(\mathbb{R}) \times U \times \mathbb{R} \times \mathbb{R}$, and notations for the Hamiltonian H:

$$\begin{aligned}
\delta H(t) &:= \delta\sigma(t)q_t^1 + \delta h(t)L_t q_t^2 - \delta f(t),\\
H_{xx}(t) &:= \sigma_{xx}(t)q_t^1 + h_{xx}(t)L_t q_t^2 - f_{xx}(t),\\
H_x(t) &:= \sigma_x(t)q_t^1 + h_x(t)L_t q_t^2 - f_x(t),\\
\widetilde{H}_\mu^*(t) &:= \widetilde{\sigma}_\mu^*(t)\widetilde{q}_t^1 + \widetilde{h}_\mu^*(t)\widetilde{L}_t\widetilde{q}_t^2 - \widetilde{f}_\mu^*(t),\\
\widetilde{H}_{z\mu}^*(t) &:= \widetilde{\sigma}_{z\mu}^*(t)\widetilde{q}_t^1 + \widetilde{h}_{z\mu}^*(t)\widetilde{L}_t\widetilde{q}_t^2 - \widetilde{f}_{z\mu}^*(t),
\end{aligned}$$

where $\big((p^1,(q^1,\check{q}^1)),\,(p^2,(\check{q}^2,q^2))\big)$ is the solution of the first adjoint BSDE (4.5).

- For simplicity, now let us suppose:

$$h(t,x,\gamma,u)=h_0(t,x,\gamma)+\phi(x)h_1(t,\gamma,u),\ (t,x,\gamma,u)\in[0,T]\times\mathbb{R}\times\mathcal{P}_2(\mathbb{R})\times U,$$
$$\sigma(t,x,\gamma,u)=\sigma(t,\gamma,u),\ (t,x,\gamma,u)\in[0,T]\times\mathbb{R}\times\mathcal{P}_2(\mathbb{R})\times U, \tag{4.7}$$

The second-order adjoint equation:

$$\begin{cases} dP_t^1 = -\,H_{xx}(t)dt+Q_t^{1,1}dB_t^1+Q_t^{1,2}dY_t, \\ \ P_T^1 = -\,\Phi_{xx}(T). \end{cases} \tag{4.8}$$

Under Assumptions (H2), the classical linear BSDE (4.8) has a unique solution $(P^1,(Q^{1,1},Q^{1,2}))$ with

$$E\Big[\sup_{t\in[0,T]}|P^1(t)|^2+\int_0^T|Q^{1,1}(t)|^2+\int_0^T|Q^{1,2}(t)|^2dt\Big]<+\infty.$$

Theorem 4.1. (Peng's SMP)

Under the assumptions (H2) and (4.7), let $u \in \mathcal{U}_{ad}$ be optimal and (X, L) be the associated solution of system (4.1). Then, for all $v \in U$, it holds that for $dtdQ$-a.e. $(t, \omega) \in [0, T] \times \Omega$,

$$E^Q\Big[H\big(t,X_t, L_t, v, q_t^1, q_t^2\big) - H\big(t, X_t, L_t, u_t, q_t^1, q_t^2\big) + \frac{1}{2}P_t^1\big(\sigma(t, \mu_t, v) - \sigma(t, \mu_t, u_t)\big)^2$$
$$+M_t\big(\sigma_1(t, \mu_t, v) - \sigma_1(t, \mu_t, u_t)\big)^2 + R_t\big(h_1(t, \mu_t, v) - h_1(t, \mu_t, u_t)\big)^2 \,\big|\, \mathcal{F}_t^Y\Big] \leq 0, \tag{4.9}$$

where $\big((p^1, (q^1, \check{q}^1)), (p^2, (\check{q}^2, q^2))\big)$ and $\big(P^1, (Q^{1,1}, Q^{1,2})\big)$ are the unique solutions to (4.5) and (4.8), respectively,

Theorem 4.1. (Peng's SMP)(continued)

where,

$$h(s,t) := \int_s^t h_x(r, X_r, \mu_r, u_r) dY_r - \int_s^t (h \cdot h_x)(r, X_r, \mu_r, u_r) dr,$$

$$\begin{aligned}
M_t &:= -\widetilde{E}^Q [\widetilde{\Phi}^*_\mu(T) L_T E^P [h(t,T) \,|\, \mathcal{F}^Y_T]] \\
&\quad + \int_t^T \Big(\widetilde{E}^Q [\widetilde{H}^*_\mu(s)] + E^Q [(H_x(s) + f_x(s)) L_s^{-1} \,|\, \mathcal{F}^Y_s] \Big) L_s E^P [h(t,s) \,|\, \mathcal{F}^Y_s] ds,
\end{aligned}$$

$$\begin{aligned}
R_t &:= -E^Q \Big[E^P [(X_T - U_T)\phi(X_t) \,|\, \mathcal{F}^Y_T] \Big\{ E^Q [\widetilde{E}^Q [\Phi^*_\mu(T)] L_T \phi(X_t) \,|\, \mathcal{F}^Y_T] \\
&- \widetilde{E}^Q [\widetilde{\Phi}^*_\mu(T)] L_T E^P [\phi(X_t) \,|\, \mathcal{F}^Y_T] \Big\} + \frac{1}{2} (E^P [(X_T - U_T)\phi(X_t) \,|\, \mathcal{F}^Y_T])^2 \widetilde{E}^Q [\widetilde{\Phi}^*_{z\mu}(T)] L_T \,|\, \mathcal{F}^Y_t \Big] \\
&+ E^Q \Big[\int_t^T \Big(E^P [(X_s - U_s)\phi(X_t) \,|\, \mathcal{F}^Y_s] \Big\{ E^Q [\widetilde{E}^Q [\widetilde{H}^*_\mu(s)] L_s \phi(X_t) \,|\, \mathcal{F}^Y_s] \\
&- \widetilde{E}^Q [\widetilde{H}^*_\mu(s)] L_s E^P [\phi(X_t) \,|\, \mathcal{F}^Y_s] \Big\} + \frac{1}{2} (E^P [(X_s - U_s)\phi(X_t) \,|\, \mathcal{F}^Y_s])^2 \widetilde{E}^Q [\widetilde{H}^*_{z\mu}(s)] L_s \Big) ds \,|\, \mathcal{F}^Y_t \Big], \\
&\qquad t \in [0,T].
\end{aligned}$$

Remark 4.5.

Comparing the result with the SMP got in previous works by different authors, namely in the classical case (no mean field, no conditional expectation), the terms with $R = (R_t)_{t\in[0,T]}$ and $M = (M_t)_{t\in[0,T]}$ are new here. Note that $R = (R_t)_{t\in[0,T]}$ depends in a nonlocal way on (X, L, U). This comes from the fact that we have a mean-field control problem involving the law of the conditional expectation of the controlled state process.

Remark 4.6

The SMP for the case (the full observation) $U_t = \varphi(X_t, Y_{\cdot\wedge t})$ instead of $U_t = E^P[X_t \,|\, \mathcal{F}_t^Y]$, where $\varphi : \mathbb{R} \times C_T \to \mathbb{R}$ is a Borel measurable function differentiable w.r.t. $x \in \mathbb{R}$, and with bounded derivative φ_x.............

Sketch of the proof of Theorem 4.1.:

From the definition of the cost functional and the optimality of u, we obtain from Propositions 4.2, 4.3 and 4.4:

$$
\begin{aligned}
0 \leq & J(u^\varepsilon) - J(u) \\
= & E^Q\left[\Phi(X_T^\varepsilon, \mu_T^\varepsilon) - \Phi(X_T, \mu_T)\right] + E^Q\Big[\int_0^T \big(f(t, X_t^\varepsilon, \mu_t^\varepsilon, u_t^\varepsilon) - f(t, X_t, \mu_t, u_t)\big) dt\Big] \\
= & E^Q\Big[\Phi_x(T)(Y_T^{1,\varepsilon} + Y_T^{2,\varepsilon}) + \widetilde{E}^Q\Big[\int_0^{\widetilde{U}_T} \Phi_\mu(T, y) dy (\widetilde{K}_T^{1,\varepsilon} + \widetilde{K}_T^{2,\varepsilon})\Big] \\
& \qquad + \widetilde{E}^Q\big[\widetilde{\Phi}_\mu(T)\widetilde{L}_T(\widetilde{V}_T^{1,\varepsilon} + \widetilde{V}_T^{2,\varepsilon})\big]\Big] \\
& + E^Q\Big[\int_0^T \Big(f_x(t)(Y_t^{1,\varepsilon} + Y_t^{2,\varepsilon}) + \widetilde{E}^Q\Big[\int_0^{\widetilde{U}_t} f_\mu(t, y) dy (\widetilde{K}_t^{1,\varepsilon} + \widetilde{K}_t^{2,\varepsilon})\Big] \\
& \qquad + \widetilde{E}^Q\big[\widetilde{f}_\mu(t)\widetilde{L}_t(\widetilde{V}_t^{1,\varepsilon} + \widetilde{V}_t^{2,\varepsilon})\big]\Big) dt\Big] \\
& + E^Q\Big[\frac{1}{2}\Phi_{xx}(T)(Y_T^{1,\varepsilon})^2 + \widetilde{E}^Q\big[\widetilde{\Phi}_\mu(T)\widetilde{V}_T^{1,\varepsilon}\widetilde{K}_T^{1,\varepsilon}\big] + \frac{1}{2}\widetilde{E}^Q\big[\widetilde{\Phi}_{z\mu}(T)\widetilde{L}_T(\widetilde{V}_T^{1,\varepsilon})^2\big]\Big] \\
& + E^Q\Big[\int_0^T \Big(\frac{1}{2} f_{xx}(t)(Y_t^{1,\varepsilon})^2 + \widetilde{E}^Q\big[\widetilde{f}_\mu(t)\widetilde{V}_t^{1,\varepsilon}\widetilde{K}_t^{1,\varepsilon}\big] + \frac{1}{2}\widetilde{E}^Q\big[\widetilde{f}_{z\mu}(t)\widetilde{L}_t(\widetilde{V}_t^{1,\varepsilon})^2\big]\Big) dt\Big] \\
& + E^Q\Big[\int_0^T \delta f(t) \mathbf{1}_{E_\varepsilon}(t) dt\Big] + o(\varepsilon), \quad \text{as } \varepsilon \to 0.
\end{aligned}
$$

4. Stochastic Control Problem

We also need to calculate some key terms in the above formula, using the notations $\Gamma_t^1 := \dfrac{L_t}{E^Q[L_t \,|\, \mathcal{F}_t^Y]}$, $\Gamma_t := \dfrac{1}{E^Q[L_t \,|\, \mathcal{F}_t^Y]}\left(X_t - \dfrac{E^Q[X_t L_t \,|\, \mathcal{F}_t^Y]}{E^Q[L_t \,|\, \mathcal{F}_t^Y]}\right) = \dfrac{1}{E^Q[L_t \,|\, \mathcal{F}_t^Y]}\left(X_t - E^P[X_t \,|\, \mathcal{F}_t^Y]\right)$, we have

i) $E^Q\left[\widetilde{f}_\mu^*(t) L_t (V_t^{1,\varepsilon} + V_t^{2,\varepsilon})\right]$

$$=E^Q\left[E^Q\left[\widetilde{f}_\mu^*(t) L_t \,|\, \mathcal{F}_t^Y\right]\left(\Gamma_t^1 (Y_t^{1,\varepsilon} + Y_t^{2,\varepsilon}) + \Gamma_t (K_t^{1,\varepsilon} + K_t^{2,\varepsilon})\right)\right]$$
$$+ E^Q\left[\widetilde{f}_\mu^*(t) L_t E^P\left[H_\varepsilon(t) \,\middle|\, \mathcal{F}_t^Y\right]\right]$$
$$- E^Q\left[\widetilde{f}_\mu^*(t) L_t \int_0^t E^Q\left[\Gamma_t^1 \delta h(s) \,|\, \mathcal{F}_t^Y\right] E^Q\left[\Gamma_t L_t \delta h(s) \,|\, \mathcal{F}_t^Y\right] \mathbf{1}_{E_\varepsilon}(s) ds\right] + \varepsilon \rho_t(\varepsilon),$$

ii) $E^Q\left[\widetilde{f}_\mu^*(t) V_t^{1,\varepsilon} K_t^{1,\varepsilon}\right]$

$$=E^Q\left[\int_0^t E^Q[\Gamma_t L_t \delta h(s) \,|\, \mathcal{F}_t^Y] E^Q\left[\widetilde{f}_\mu^*(t) L_t \delta h(s) \middle| \mathcal{F}_t^Y\right] \mathbf{1}_{E_\varepsilon}(s) ds\right] + \varepsilon \rho_t(\varepsilon).$$

iii) $E^Q\left[\widetilde{f}_{z\mu}^*(t) L_t (V_t^{1,\varepsilon})^2\right]$

$$=E^Q\left[\widetilde{f}_{z\mu}^*(t) L_t \int_0^t \left(E^Q[\Gamma_t L_t \delta h(s) \,|\, \mathcal{F}_t^Y]\right)^2 \mathbf{1}_{E_\varepsilon}(s) ds\right] + \varepsilon \rho_t(\varepsilon).$$

iv) $$E^Q\left[\widetilde{\Phi}^*_\mu(T)L_T(V_T^{1,\varepsilon}+V_T^{2,\varepsilon})\right]$$

$$=E^Q\Big[E^Q[\widetilde{\Phi}^*_\mu(T)L_T\,|\,\mathcal{F}_T^Y]\big\{\Gamma_T^1(Y_T^{1,\varepsilon}+Y_T^{2,\varepsilon})+\Gamma_T(K_T^{1,\varepsilon}+K_T^{2,\varepsilon})$$

$$-\int_0^T E^Q\left[\Gamma_T^1\delta h(s)\,\middle|\,\mathcal{F}_T^Y\right]E^Q\left[\Gamma_T L_T\delta h(s)\,\middle|\,\mathcal{F}_T^Y\right]\mathbf{1}_{E_\varepsilon}(s)ds\big\}\Big]$$

$$+E^Q\left[\widetilde{\Phi}^*_\mu(T)L_T E^P\left[H_\varepsilon(T)\,\middle|\,\mathcal{F}_T^Y\right]\right]+\varepsilon\rho_T(\varepsilon).$$

v) $$E^Q\left[\widetilde{\Phi}^*_\mu(T)V_T^{1,\varepsilon}K_T^{1,\varepsilon}\right]=E^Q\Big[\int_0^T E^Q\left[\Gamma_T L_T\delta h(s)\,\middle|\,\mathcal{F}_T^Y\right]$$

$$\times E^Q\left[\widetilde{\Phi}^*_\mu(T)L_T\delta h(s)\middle|\mathcal{F}_T^Y\right]\mathbf{1}_{E_\varepsilon}(s)ds\Big]+\varepsilon\rho_T(\varepsilon).$$

vi) $$E^Q\left[\widetilde{\Phi}^*_{z\mu}(T)L_T(V_T^{1,\varepsilon})^2\right]=E^Q\left[\widetilde{\Phi}^*_{z\mu}(T)L_T\int_0^T\left(E^Q[\Gamma_T L_T\delta h(s)\middle|\mathcal{F}_T^Y]\right)^2\mathbf{1}_{E_\varepsilon}(s)ds\right]$$

$$+\varepsilon\rho_T(\varepsilon).$$

- $\rho_t(\varepsilon)\to 0(\varepsilon\searrow 0),\ \left|\rho_t(\varepsilon)\right|\le C,\ \varepsilon>0,\ t\in[0,T].$

We substitute i)-vi) in the previous inequality, and for $\phi = \Phi,\ f$; we use the notation:

$$\begin{aligned}\gamma_t^{\phi}\big(\delta h(s)\big) = {} & E^Q\big[\Gamma_t L_t \delta h(s) \,\big|\, \mathcal{F}_t^Y\big]\big\{E^Q\big[\widetilde{E}^Q[\widetilde{\phi}_{\mu}^*(t)]L_t \delta h(s) \,\big|\, \mathcal{F}_t^Y\big] \\ & - E^Q\big[\widetilde{E}^Q[\widetilde{\phi}_{\mu}^*(t)]L_t \,\big|\, \mathcal{F}_t^Y\big] E^Q\big[\Gamma_t^1 \delta h(s) \,\big|\, \mathcal{F}_t^Y\big]\big\} \\ & + \frac{1}{2}\Big(E^Q[\Gamma_t L_t \delta h(s) \,\big|\, \mathcal{F}_t^Y]\Big)^2 \widetilde{E}^Q[\widetilde{\phi}_{z\mu}^*(t)]L_t, \quad 0 \le s \le t \le T;\end{aligned} \tag{4.10}$$

Then, from above we get

$$
\begin{aligned}
0 \leq & J(u^\varepsilon) - J(u) \\
= & E^Q\Big[\big(\Phi_x(T) + E^Q\big[\widetilde{E}^Q[\widetilde{\Phi}^*_\mu(T)]L_T \,\big|\, \mathcal{F}^Y_T\big]\Gamma^1_T\big)\big(Y^{1,\varepsilon}_T + Y^{2,\varepsilon}_T\big)\Big] \\
& + E^Q\Big[\big(\widetilde{E}^Q\big[\int_0^{U_T} \Phi^*_\mu(T,y)dy\big] + E^Q\big[\widetilde{E}^Q[\widetilde{\Phi}^*_\mu(T)]L_T \,\big|\, \mathcal{F}^Y_T\big]\Gamma_T\big)\big(K^{1,\varepsilon}_T + K^{2,\varepsilon}_T\big)\Big] \\
& + \frac{1}{2}E^Q\Big[\Phi_{xx}(T)(Y^{1,\varepsilon}_T)^2\Big] + \frac{1}{2}E^Q\Big[\int_0^T f_{xx}(t)(Y^{1,\varepsilon}_t)^2 dt\Big] \\
& + E^Q\Big[\widetilde{E}^Q\big[\widetilde{\Phi}^*_\mu(T)\big]L_T E^P\big[H_\varepsilon(T) \,\big|\, \mathcal{F}^Y_T\big]\Big] \\
& + E^Q\Big[\int_0^T \widetilde{E}^Q\big[\widetilde{f}^*_\mu(t)\big]L_t E^P\big[H_\varepsilon(t) \,\big|\, \mathcal{F}^Y_t\big]dt\Big] \\
& + E^Q\Big[\int_0^T \big(f_x(t) + E^Q[\widetilde{E}^Q[\widetilde{f}^*_\mu(t)]L_t \,|\, \mathcal{F}^Y_t]\Gamma^1_t\big)\big(Y^{1,\varepsilon}_t + Y^{2,\varepsilon}_t\big)dt\Big] \\
& + E^Q\Big[\int_0^T \big(\widetilde{E}^Q\big[\int_0^{U_t} f^*_\mu(t,y)dy\big] + E^Q[\widetilde{E}^Q[\widetilde{f}^*_\mu(t)]L_t | \mathcal{F}^Y_t]\Gamma_t\big)\big(K^{1,\varepsilon}_t + K^{1,\varepsilon}_t\big)dt\Big] \\
& + E^Q\Big[\int_0^T \big\{E^Q\big[\gamma^\Phi_T(\delta h(t)) + \int_t^T \gamma^f_s(\delta h(t))ds \,\big|\, \mathcal{F}^Y_t\big] + \delta f(t)\big\}\mathbf{1}_{E_\varepsilon}(t)dt\Big] \\
& + o(\varepsilon), \quad \text{as } \varepsilon \searrow 0.
\end{aligned}
$$

Recall the duality (4.6), we have to calculate:

$$
\begin{aligned}
& E^Q\Big[p^1_T\big(Y^{1,\varepsilon}_T+Y^{2,\varepsilon}_T\big)+p^2_T\big(K^{1,\varepsilon}_T+K^{2,\varepsilon}_T\big)\Big] \\
= {} & E^Q\Big[\int_0^T \Big\{\big(Y^{1,\varepsilon}_t+Y^{2,\varepsilon}_t\big)\Big(f_x(t)+L_t\widetilde{E}^Q\big[E^P\big[\widetilde{f}^*_\mu(t)\,\big|\,\mathcal{F}^Y_t\big]\big]\Big) \\
& \quad + \big(K^{1,\varepsilon}_t+K^{2,\varepsilon}_t\big)\Big((X_t-U_t)\widetilde{E}^Q\big[E^P\big[\widetilde{f}^*_\mu(t)\,\big|\,\mathcal{F}^Y_t\big]\big]+\widetilde{E}^Q\Big[\int_0^{U_t} f^*_\mu(t,y)dy\Big]\Big)\Big\}dt\Big] \\
& + E^Q\Big[\int_0^T \frac{1}{2}h_{xx}(t)q^2_tL_t(Y^{1,\varepsilon}_t)^2dt\Big] \\
& + E^Q\Big[\int_0^T \Big(\widetilde{E}^Q\big[\widetilde{q}^1_t\widetilde{\sigma}^*_\mu(t)+\widetilde{q}^2_t\widetilde{L}_t\widetilde{h}^*_\mu(t)\big]+E^Q\big[q^2_th_x(t)\,\big|\,\mathcal{F}^Y_t\big]\Big)L_tE^P\big[H_\varepsilon(t)\,\big|\,\mathcal{F}^Y_t\big]dt\Big] \\
& + E^Q\Big[\int_0^T \Big\{\big(q^1_t\delta\sigma(t)+q^2_tL_t\delta h(t)\big)+E^Q\Big[\int_t^T\Big(E^P\big[(X_s-U_s)\delta h(t)\,\big|\,\mathcal{F}^Y_s\big] \\
& \quad \cdot\Big[E^Q\Big[\widetilde{E}^Q\big[\widetilde{q}^1_s\widetilde{\sigma}^*_\mu(s)+\widetilde{q}^2_s\widetilde{L}_s\widetilde{h}^*_\mu(s)\big]L_s\delta h(t)\,\Big|\,\mathcal{F}^Y_s\Big]-\widetilde{E}^Q\big[\widetilde{q}^1_s\widetilde{\sigma}^*_\mu(s)+\widetilde{q}^2_s\widetilde{L}_s\widetilde{h}^*_\mu(s)\big]L_sE^P\big[\delta h(t) \\
& \quad + \frac{1}{2}\widetilde{E}^Q\Big[\widetilde{q}^1_s\widetilde{\sigma}^*_{z\mu}(s)+\widetilde{q}^2_s\widetilde{L}_s\widetilde{h}^*_{z\mu}(s)\Big]L_s\Big(E^P\big[(X_s-U_s)\delta h(t)\,\big|\,\mathcal{F}^Y_s\big]\Big)^2\Big)ds\,\Big|\,\mathcal{F}^Y_t\Big]\Big\}\mathbf{1}_{E_\varepsilon}(t)dt\Big] \\
& + o(\varepsilon), \quad \text{as } \varepsilon \searrow 0. \qquad (4.12)
\end{aligned}
$$

Recall that $\delta h(t) = \phi(X_t)\delta h_1(t)$, and $\delta h_1(t)$ is $\mathcal{F}_t^Y$-measurable, and also recall the second-order adjoint BSDE (4.8).

Then, substituting the above formula, we deduce that:

$$
\begin{aligned}
0 \leq & - E^Q\Big[\int_0^T \Big(\delta H(t) + \frac{1}{2}P_t^1\big(\delta\sigma(t)\big)^2\Big)\mathbf{1}_{E_\varepsilon}(t)dt\Big] \\
& - E^Q\Big[\int_0^T M_t\big(\delta\sigma(t)\big)^2\mathbf{1}_{E_\varepsilon}(t)dt\Big] \\
& - E^Q\Big[\int_0^T R_t\big(\delta h_1(t)\big)^2\mathbf{1}_{E_\varepsilon}(t)dt\Big] + o(\varepsilon), \quad \text{as } \varepsilon \searrow 0,
\end{aligned}
$$

Thus, we have:

$$0 \leq -E^Q\Big[\int_0^T \big(\delta H(t)+\frac{1}{2}P_t^1\big(\delta\sigma(t)\big)^2+R_t\big(\delta h_1(t)\big)^2+M_t\big(\delta\sigma(t)\big)^2\big)\mathbf{1}_{E_\varepsilon}(t)dt\Big]+o(\varepsilon),$$

and, as $v \in \mathcal{U}_{ad}$ has been fixed arbitrarily, Lebesgue's differentiation theorem combined with standard arguments implies:

$$\begin{aligned} E^Q\Big[H\big(t, X_t, L_t, v_t, q_t^1, q_t^2\big) &- H\big(t, X_t, L_t, u_t, q_t^1, q_t^2\big) \\ &+\frac{1}{2}P_t^1\big|\sigma(t,\mu_t,v_t)-\sigma(t,\mu_t,u_t)\big|^2 + M_t\big|\sigma(t,\mu_t,v_t)-\sigma(t,\mu_t,u_t)\big|^2 \\ &+R_t\big|h_1(t,\mu_t,v_t)-h_1(t,\mu_t,u_t)\big|^2 \,\Big|\, \mathcal{F}_t^Y\Big] \leq 0, \quad dtdQ\text{-a.s.,} \end{aligned}$$

for all $v \in \mathcal{U}_{ad}$. (The fact that we have to take in this formula $E^Q\big[\ \cdot\ \big|\mathcal{F}_t^Y\big]$ stems from the fact the control processes are $\mathbb{F}^Y$-adapted). So, now finally we obtain our stochastic maximum principle.

Thank you very much
for your attention!