Optimal trade execution in a stochastic order book model

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Optimal trade execution

Task: From an initial position of size $x \in \mathbb{R}$, reach a target position at terminal time $T > 0$ by trading during the time interval $[0, T]$ with minimal execution costs.
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We extend the framework of [Obizhaeva, Wang JFinancMark’13] to stochastic order book parameters.
Underlying order book model

- symmetric block-shaped order book model
- zero bid-ask spread
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- symmetric block-shaped order book model
- zero bid-ask spread
- price of a share = unaffected price $S^0 + \text{deviation } D^X$, where $S^0$ is assumed to be a martingale (wlog $S^0 \equiv 0$)
- strategy $X$ (càdlàg, finite variation) describes position
Underlying order book model

- buy $\Delta X_t = X_t - X_{t-} > 0$ shares
- deviation jumps to $D^X_t = D^X_{t-} + \gamma_t \Delta X_t$
- deviation at $s > t$, if no trades in between: $D^X_s = D^X_t e^{-\int_t^s \rho_r dr}$
- deviation dynamics $dD^X_s = -\rho_s D^X_s ds + \gamma_s dX_s$
Underlying order book model

- buy $\Delta X_t = X_t - X_{t^-} > 0$ shares
- deviation jumps to $D_t^X = D_{t^-}^X + \gamma_t \Delta X_t$
- deviation at $s > t$, if no trades in between: $D_s^X = D_t^X e^{-\int_t^s \rho_r dr}$
- deviation dynamics $dD_s^X = -\rho_s D_s^X ds + \gamma_s dX_s$
- costs for the trade $\Delta X_t$: $(D_{t^-}^X + \frac{\gamma_t}{2} \Delta X_t) \Delta X_t$
- overall trading costs: $\int_{[0,T]} \left( D_{s^-}^X + \frac{\gamma_s}{2} \Delta X_s \right) dX_s$
Setting

\[ T > 0, \ x \in \mathbb{R}, \ d \in \mathbb{R} \]

\((\Omega, \mathcal{F}_T, (\mathcal{F}_s)_{s \in [0, T]}, P)\) filtered probability space with a Brownian motion \((W_s)_{s \in [0, T]}\)

\(\rho, \mu, \sigma, \lambda\) progressively measurable, \(dP \times ds|_{[0, T]}\)-a.e. bounded processes

\(\xi, \mathcal{F}_T\)-measurable, \(\zeta\) progressively measurable

resilience coefficient \(\rho = (\rho_s)_{s \in [0, T]}\)

price impact process \(\gamma = (\gamma_s)_{s \in [0, T]}:\)

\[ d\gamma_s = \gamma_s (\mu_s \, ds + \sigma_s \, dW_s), \quad \gamma_0 > 0 \]
Finite variation stochastic control problem

\( \mathcal{A}^{fv} \) set of all adapted, càdlàg, finite variation processes
\( X = (X_s)_{s \in [0-, T]} \) satisfying \( X_{0-} = x, \ X_T = \xi \), and suitable integrability conditions

deviation process \( D^X = (D_s^X)_{s \in [0-, T]} \) associated to \( X \in \mathcal{A}^{fv} \):

\[
dD_s^X = -\rho_s D_s^X \, ds + \gamma_s \, dX_s, \quad D_{0-}^X = d
\]

cost functional \( J^{fv} \), for \( X \in \mathcal{A}^{fv} \):

\[
J^{fv}(X) = E \left[ \int_{[0, T]} \left( D_{s-}^X + \frac{\gamma_s}{2} \Delta X_s \right) \, dX_s \right] + E \left[ \int_0^T \lambda_s \gamma_s (X_s - \zeta_s)^2 \, ds \right]
\]

optimal strategy: \( X^* \in \mathcal{A}^{fv} \) s.t. \( J^{fv}(X^*) = \inf_{X \in \mathcal{A}^{fv}} J^{fv}(X) \)

for this talk: \( \xi = 0, \ \lambda \equiv 0 \)
On the class of strategies

- in most of the literature, strategies have finite variation; few exceptions, e.g., [Lorenz, Schied FinancStoch’13]
- infinite variation strategies emerge in a limiting case in [Horst, Kivman arXiv’21]
- empirical evidence for trading with infinite variation in a related situation in [Carmona, Webster FinancStoch’19]
- in our setting, the price impact typically has infinite variation, and it is natural to expect the optimal strategy to react to these oscillations
- optimal strategies with infinite variation can come out in the optimization over semimartingales in [Ackermann, Kruse, Urusov FinancStoch’21]
- there are situations where an optimal strategy within the semimartingale strategies does not exist
Towards progressively measurable strategies

Aim: Extend the finite variation stochastic control problem to progressively measurable strategies and solve this extended problem.
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2. Establish a continuous extension of $J^{f\nu}$ to progressively measurable strategies

3. Reduce the extended problem to a standard LQ stochastic control problem
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4. Apply stochastic control literature to solve the LQ problem
Towards progressively measurable strategies

Aim: Extend the finite variation stochastic control problem to progressively measurable strategies and solve this extended problem

1. Rewrite deviation and cost functional to get rid of the strategy in the integrator
2. Establish a continuous extension of $J^{fv}$ to progressively measurable strategies
3. Reduce the extended problem to a standard LQ stochastic control problem
4. Apply stochastic control literature to solve the LQ problem
5. Recover the solution of the extended problem
Let $\nu_s = e^{\int_0^s \rho_r dr}$, $s \in [0, T]$. It holds for all $X \in A^{f\nu}$ that

$$D^X_s = \gamma_s X_s + \nu_s^{-1} \left( d - \gamma_0 X - \int_0^s X_r d (\nu_r \gamma_r) \right), \quad s \in [0, T],$$

and

$$\int_{[0, T]} \left( D^X_s + \frac{\gamma_s}{2} \Delta X_s \right) dX_s$$

$$= \frac{1}{2} \left( \gamma_T^{-1} (D_T^X)^2 - \int_0^T (D^X_s)^2 \nu_s^2 d (\nu_s^{-2} \gamma_s^{-1}) \right) - \frac{d^2}{2 \gamma_0}.$$
Extended problem

$\mathcal{A}^{pm}$ set of all progressively measurable processes $X = (X_s)_{s \in [0-, T]}$ satisfying $X_{0-} = x$, $X_T = 0$, and suitable integrability conditions

deviation process $D^X = (D^X_s)_{s \in [0-, T]}$ associated to $X \in \mathcal{A}^{pm}$:

$$D^X_s = \gamma_s X_s + \nu_s^{-1} \left( d - \gamma_0 x - \int_0^s X_r d(\nu_r \gamma_r) \right), \ s \in [0, T], \ D^X_{0-} = d$$

cost functional $J^{pm}$, for $X \in \mathcal{A}^{pm}$:

$$J^{pm}(X) = \frac{1}{2} E \left[ \gamma_T^{-1} (D^X_T)^2 + \int_0^T (D^X_s)^2 \gamma_s^{-1} (2 \rho_s + \mu_s - \sigma_s^2) ds \right] - \frac{d^2}{2 \gamma_0}$$

note: $\mathcal{A}^{fv} \subseteq \mathcal{A}^{pm}$ and $J^{fv}(X) = J^{pm}(X)$ for $X \in \mathcal{A}^{fv}$
Continuous extension of the cost functional

Let $d(X, Y) = \left( E[\int_0^T (D_s^X - D_s^Y)^2 \gamma_s^{-1} ds] \right)^{\frac{1}{2}}$ for $X, Y \in \mathcal{A}^{pm}$.

(i) Suppose that $X \in \mathcal{A}^{pm}$. For every sequence $(X^n)_{n \in \mathbb{N}}$ in $\mathcal{A}^{pm}$ with $\lim_{n \to \infty} d(X^n, X) = 0$ it holds that $\lim_{n \to \infty} |J^{pm}(X^n) - J^{pm}(X)| = 0$.

(ii) For any $X \in \mathcal{A}^{pm}$ there exists a sequence $(X^n)_{n \in \mathbb{N}}$ in $\mathcal{A}^{fv}$ such that $\lim_{n \to \infty} d(X^n, X) = 0$.

In particular, it holds that

$$\inf_{X \in \mathcal{A}^{fv}} J^{fv}(X) = \inf_{X \in \mathcal{A}^{pm}} J^{pm}(X).$$
Scaled hidden deviation process

It holds for $X \in \mathcal{A}^{pm}$ and $H_s^X = \gamma_s^{-\frac{1}{2}} (D_s^X - \gamma_s X_s)$, $s \in [0, T]$, that $H_0^X = \frac{d}{\sqrt{\gamma_0}} - \sqrt{\gamma_0} x$,

\[
dH_s^X = \frac{1}{2} \left( \left( \mu_s - \frac{1}{4} \sigma_s^2 \right) H_s^X - \left( 2(\rho_s + \mu_s) - \sigma_s^2 \right) \gamma_s^{-\frac{1}{2}} D_s^X \right) ds
\]

\[+ \left( \frac{1}{2} \sigma_s^2 H_s^X - \sigma_s \gamma_s^{-\frac{1}{2}} D_s^X \right) dW_s, \quad s \in [0, T],\]

and

\[J^{pm}(X) = \frac{1}{2} E \left[ (H_T^X)^2 + \int_0^T (2\rho_s + \mu_s - \sigma_s^2) (\gamma_s^{-\frac{1}{2}} D_s^X)^2 ds \right] - \frac{d^2}{2}\gamma_0.\]
Standard LQ stochastic control problem

\( \mathcal{L}^2 \) set of all progressively measurable processes \( u = (u_s)_{s \in [0, T]} \) such that \( E[\int_0^T u_s^2 \, ds] < \infty \).

State process \( H^u = (H^u_s)_{s \in [0, T]} \) associated to \( u \in \mathcal{L}^2 \):

\[
dH^u_s = \frac{1}{2} \left( \left( \mu_s - \frac{1}{4} \sigma_s^2 \right) H^u_s - \left( 2(\rho_s + \mu_s) - \sigma_s^2 \right) u_s \right) \, ds \\
+ \left( \frac{1}{2} \sigma_s H^u_s - \sigma_s u_s \right) \, dW_s, \quad s \in [0, T],
\]

\( H^u_0 = \frac{d}{\sqrt{\gamma_0} - \sqrt{\gamma_0} x} \)

cost functional \( J \), for \( u \in \mathcal{L}^2 \):

\[
J(u) = \frac{1}{2} E \left[ (H^u_T)^2 + \int_0^T (2\rho_s + \mu_s - \sigma_s^2) u_s^2 \, ds \right]
\]
Link between the problems

\[
\inf_{X \in A^{fv}} J^{fv}(X) = \inf_{X \in A^{pm}} J^{pm}(X) = \inf_{u \in L^2} J(u) - \frac{d^2}{2\gamma_0}
\]

- if \( X^* \in A^{pm} \) minimizes \( J^{pm} \) over \( A^{pm} \), then \( u^* = \gamma^{-\frac{1}{2}} D X^* \) minimizes \( J \) over \( L^2 \)

- if \( u^* \in L^2 \) minimizes \( J \) over \( L^2 \), then \( X^*_s = \gamma_s^{-\frac{1}{2}} (u^*_s - H_s u^*_s) \), \( s \in [0, T) \), \( X^*_0 = x \), \( X^*_T = 0 \), minimizes \( J^{pm} \) over \( A^{pm} \)
Assumptions:

$(\mathcal{F}_s)_{s \in [0, T]}$ is the augmented natural filtration of an $m$-dimensional Brownian motion $(W^1, \ldots, W^m)\top$, where $W^1 = W$

$2\rho + \mu - \sigma^2 \geq 0 \ dP \times ds|_{[0, T]}$-a.e.

$\exists \varepsilon > 0 \text{ s.t. } 2\rho + \mu - \sigma^2 \geq \varepsilon \ dP \times ds|_{[0, T]}$-a.e.

or $\sigma^2 \geq \varepsilon \ dP \times ds|_{[0, T]}$-a.e.
Consider the Riccati-type BSDE
\[ dK_s = - \left( - \frac{((\rho_s + \mu_s)K_s + \sigma_s L_s^1)^2}{\frac{1}{2}(2\rho_s + \mu_s - \sigma_s^2) + \sigma_s^2 K_s} + \mu_s K_s + \sigma_s L_s^1 \right) ds \]
\[ + \sum_{j=1}^{m} L_s^j dW_s^j, \quad s \in [0, T], \]
\[ K_T = \frac{1}{2}. \]

By [Kohlmann, Tang SPA’02], there exists a unique solution \((K, L)\) of this BSDE.
Consider the Riccati-type BSDE

\[
dK_s = - \left( - \frac{(\rho_s + \mu_s) K_s + \sigma_s L_s^{1}}{\frac{1}{2}(2\rho_s + \mu_s - \sigma_s^2) + \sigma_s^2 K_s} + \mu_s K_s + \sigma_s L_s^{1} \right) ds
\]

\[+ \sum_{j=1}^{m} L_s^j dW_s^j, \quad s \in [0, T],\]

\[K_T = \frac{1}{2}.\]

By [Kohlmann, Tang SPA’02], there exists a unique solution \((K, L)\) of this BSDE.

Define

\[
\theta_s = \frac{(\rho_s + \mu_s) K_s + \sigma_s L_s^{1}}{\frac{1}{2}(2\rho_s + \mu_s - \sigma_s^2) + \sigma_s^2 K_s}, \quad s \in [0, T].
\]
Solution

Let

\[ dH_s^* = H_s^* \frac{1}{2} \left( \mu_s - \frac{1}{4} \sigma_s^2 - (2(\rho_s + \mu_s) - \sigma_s^2) \theta_s \right) ds \]

\[ + H_s^* \left( \frac{1}{2} \sigma_s - \sigma_s \theta_s \right) dW_s^1, \quad s \in [0, T], \]

\[ H_0^* = \frac{d}{\sqrt{\gamma_0}} - \sqrt{\gamma_0} \chi. \]

By [Kohlmann, Tang SPA’02],

\[ u^* = \theta H^* \]

is the unique optimal control in \( L^2 \) for \( J \). It holds \( H_u^* = H^* \) and

\[ J(u^*) = K_0 H_0^*. \]
Solution

\[ X^* = (X^*_s)_{s \in [0-, \tau]} \] defined by

\[ X^*_0 = x, \quad X^*_\tau = 0, \quad X^*_s = \gamma_s^{-\frac{1}{2}} (\theta_s - 1) H^*_s, \quad s \in [0, \tau), \]

is the unique (up to \( dP \times ds|_{[0, \tau]} \)-null sets) optimal execution strategy in \( A^{pm} \) for \( J^{pm} \). The associated costs are given by

\[ J^{pm}(X^*) = \frac{K_0}{\gamma_0} (d - \gamma_0 x)^2 - \frac{d^2}{2\gamma_0}. \]
Remarks

- We can demand an $\mathcal{F}_T$-measurable terminal position $X_T = \xi$. In this case, a further, linear BSDE enters the solution.

- We can add a risk term of the form $E[\int_0^T \lambda_s \gamma_s (X_s - \zeta_s)^2 ds]$ (as in, e.g., [Bank, Voß SICON’18]) to the cost functional $J^{fv}$. In this case, we need an additional reformulation before applying [Kohlmann, Tang SPA’02]. If the target process $\zeta$ is not equivalent to 0, we have a further, linear BSDE.

- If $\xi = 0$ and $\zeta$ or $\lambda$ vanish, we could also apply [Sun, Xiong, Yong AAP’21].

- We can consider a diffusive resilience $dR_s = \rho_s ds + \eta_s dW^R_s$, where $dW^R_s = \bar{r}_s dW^1_s + \sqrt{1 - \bar{r}_s^2} dW^2_s$. 
Thank you!

Based on: