Optimal trade execution in a stochastic order book model

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joint work with Thomas Kruse and Mikhail Urusov

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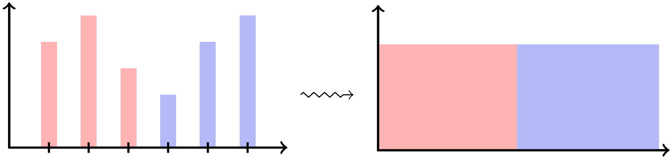
We extend the framework of [Obizhaeva,Wang JFinancMark'13] to stochastic order book parameters.

Underlying order book model



- symmetric block-shaped order book model
- zero bid-ask spread

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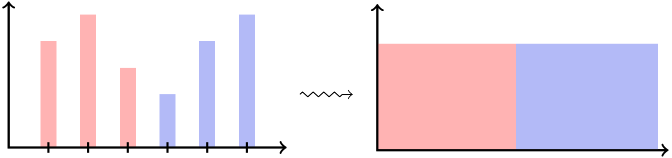
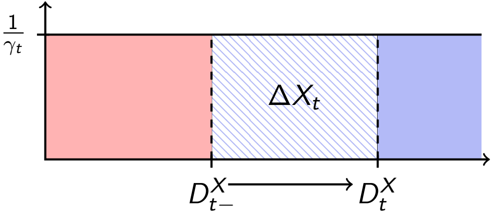


- symmetric block-shaped order book model
- zero bid-ask spread
- price of a share = unaffected price S^0 + deviation D^X, where S^0 is assumed to be a martingale (wlog $S^0 \equiv 0$)
- strategy X (càdlàg, finite variation) describes position

Underlying order book model



- buy $\Delta X_t = X_t - X_{t-} > 0$ shares
- deviation jumps to $D^X_t = D^X_{t-} + \gamma_t \Delta X_t$
- deviation at $s > t$, if no trades in between: $D^X_s = D^X_t e^{-\int_t^s \rho_r dr}$
- deviation dynamics $dD^X_s = -\rho_s D^X_s ds + \gamma_s dX_s$

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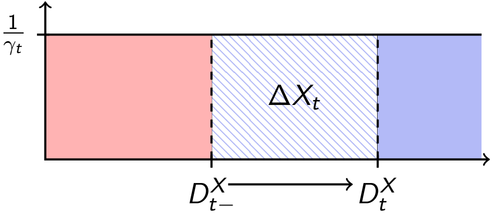

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- deviation dynamics $dD^X_s = -\rho_s D^X_s ds + \gamma_s dX_s$
- costs for the trade ΔX_t: $\left(D^X_{t-} + \frac{\gamma_t}{2}\Delta X_t\right)\Delta X_t$
- overall trading costs: $\int_{[0,T]} \left(D^X_{s-} + \frac{\gamma_s}{2}\Delta X_s\right) dX_s$

Setting

$T > 0$, $x \in \mathbb{R}$, $d \in \mathbb{R}$

$(\Omega, \mathcal{F}_T, (\mathcal{F}_s)_{s\in[0,T]}, P)$ filtered probability space with a Brownian motion $(W_s)_{s\in[0,T]}$

$\rho, \mu, \sigma, \lambda$ progressively measurable, $dP \times ds|_{[0,T]}$-a.e. bounded processes

ξ $\mathcal{F}_T$-measurable, ζ progressively measurable

resilience coefficient $\rho = (\rho_s)_{s\in[0,T]}$

price impact process $\gamma = (\gamma_s)_{s\in[0,T]}$:

$$d\gamma_s = \gamma_s\,(\mu_s\,ds + \sigma_s dW_s), \qquad \gamma_0 > 0$$

Finite variation stochastic control problem

$\mathcal{A}^{fv}$ set of all adapted, càdlàg, finite variation processes $X = (X_s)_{s\in[0-,T]}$ satisfying $X_{0-} = x$, $X_T = \xi$, and suitable integrability conditions

deviation process $D^X = (D^X_s)_{s\in[0-,T]}$ associated to $X \in \mathcal{A}^{fv}$:

$$dD^X_s = -\rho_s D^X_s ds + \gamma_s dX_s, \quad D^X_{0-} = d$$

cost functional J^{fv}, for $X \in \mathcal{A}^{fv}$:

$$J^{fv}(X) = E\left[\int_{[0,T]} \left(D^X_{s-} + \frac{\gamma_s}{2}\Delta X_s\right) dX_s\right] + E\left[\int_0^T \lambda_s \gamma_s (X_s - \zeta_s)^2 ds\right]$$

optimal strategy: $X^* \in \mathcal{A}^{fv}$ s.t. $J^{fv}(X^*) = \inf_{X\in\mathcal{A}^{fv}} J^{fv}(X)$

for this talk: $\xi = 0$, $\lambda \equiv 0$

On the class of strategies

- in most of the literature, strategies have finite variation; few exceptions, e.g., [Lorenz,Schied FinancStoch'13]
- infinite variation strategies emerge in a limiting case in [Horst,Kivman arXiv'21]
- empirical evidence for trading with infinite variation in a related situation in [Carmona,Webster FinancStoch'19]
- in our setting, the price impact typically has infinite variation, and it is natural to expect the optimal strategy to react to these oscillations
- optimal strategies with infinite variation can come out in the optimization over semimartingales in [Ackermann,Kruse,Urusov FinancStoch'21]
- there are situations where an optimal strategy within the semimartingale strategies does not exist

Towards progressively measurable strategies

Aim: Extend the finite variation stochastic control problem to progressively measurable strategies and solve this extended problem

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2. Establish a continuous extension of J^{fv} to progressively measurable strategies
3. Reduce the extended problem to a standard LQ stochastic control problem
4. Apply stochastic control literature to solve the LQ problem
5. Recover the solution of the extended problem

Alternative representations for deviation and costs

Let $\nu_s = e^{\int_0^s \rho_r dr}$, $s \in [0, T]$. It holds for all $X \in \mathcal{A}^{fv}$ that

$$D_s^X = \gamma_s X_s + \nu_s^{-1} \left(d - \gamma_0 x - \int_0^s X_r d\left(\nu_r \gamma_r\right) \right), \, s \in [0, T],$$

and

$$\int_{[0,T]} \left(D_{s-}^X + \frac{\gamma_s}{2} \Delta X_s \right) dX_s$$
$$= \frac{1}{2} \left(\gamma_T^{-1} (D_T^X)^2 - \int_0^T (D_s^X)^2 \nu_s^2 d\left(\nu_s^{-2} \gamma_s^{-1}\right) \right) - \frac{d^2}{2\gamma_0}.$$

Extended problem

$\mathcal{A}^{pm}$ set of all progressively measurable processes $X = (X_s)_{s \in [0-,T]}$ satisfying $X_{0-} = x$, $X_T = 0$, and suitable integrability conditions

deviation process $D^X = (D^X_s)_{s \in [0-,T]}$ associated to $X \in \mathcal{A}^{pm}$:

$$D^X_s = \gamma_s X_s + \nu_s^{-1} \left(d - \gamma_0 x - \int_0^s X_r d(\nu_r \gamma_r) \right), \; s \in [0,T], \; D^X_{0-} = d$$

cost functional J^{pm}, for $X \in \mathcal{A}^{pm}$:

$$J^{pm}(X) = \frac{1}{2} E \left[\gamma_T^{-1} (D^X_T)^2 + \int_0^T (D^X_s)^2 \gamma_s^{-1} (2\rho_s + \mu_s - \sigma_s^2) ds \right] - \frac{d^2}{2\gamma_0}$$

note: $\mathcal{A}^{fv} \subseteq \mathcal{A}^{pm}$ and $J^{fv}(X) = J^{pm}(X)$ for $X \in \mathcal{A}^{fv}$

Continuous extension of the cost functional

Let $\mathrm{d}(X,Y) = (E[\int_0^T (D_s^X - D_s^Y)^2 \gamma_s^{-1} ds])^{\frac{1}{2}}$ for $X, Y \in \mathcal{A}^{pm}$.

(i) Suppose that $X \in \mathcal{A}^{pm}$. For every sequence $(X^n)_{n\in\mathbb{N}}$ in $\mathcal{A}^{pm}$ with $\lim_{n\to\infty} \mathrm{d}(X^n, X) = 0$ it holds that $\lim_{n\to\infty} |J^{pm}(X^n) - J^{pm}(X)| = 0$.

(ii) For any $X \in \mathcal{A}^{pm}$ there exists a sequence $(X^n)_{n\in\mathbb{N}}$ in $\mathcal{A}^{fv}$ such that $\lim_{n\to\infty} \mathrm{d}(X^n, X) = 0$.

In particular, it holds that

$$\inf_{X\in\mathcal{A}^{fv}} J^{fv}(X) = \inf_{X\in\mathcal{A}^{pm}} J^{pm}(X).$$

Scaled hidden deviation process

It holds for $X \in \mathcal{A}^{pm}$ and $H_s^X = \gamma_s^{-\frac{1}{2}}(D_s^X - \gamma_s X_s)$, $s \in [0, T]$, that $H_0^X = \frac{d}{\sqrt{\gamma_0}} - \sqrt{\gamma_0}x$,

$$dH_s^X = \frac{1}{2}\left(\left(\mu_s - \frac{1}{4}\sigma_s^2\right) H_s^X - \left(2(\rho_s + \mu_s) - \sigma_s^2\right) \gamma_s^{-\frac{1}{2}} D_s^X\right) ds + \left(\frac{1}{2}\sigma_s H_s^X - \sigma_s \gamma_s^{-\frac{1}{2}} D_s^X\right) dW_s, \quad s \in [0, T],$$

and

$$J^{pm}(X) = \frac{1}{2}E\left[\left(H_T^X\right)^2 + \int_0^T \left(2\rho_s + \mu_s - \sigma_s^2\right) \left(\gamma_s^{-\frac{1}{2}} D_s^X\right)^2 ds\right] - \frac{d^2}{2\gamma_0}.$$

Standard LQ stochastic control problem

$\mathcal{L}^2$ set of all progressively measurable processes $u = (u_s)_{s\in[0,T]}$ such that $E[\int_0^T u_s^2 ds] < \infty$.

state process $H^u = (H_s^u)_{s\in[0,T]}$ associated to $u \in \mathcal{L}^2$:

$$dH_s^u = \frac{1}{2}\left(\left(\mu_s - \frac{1}{4}\sigma_s^2\right)H_s^u - \left(2(\rho_s + \mu_s) - \sigma_s^2\right)u_s\right)ds + \left(\frac{1}{2}\sigma_s H_s^u - \sigma_s u_s\right)dW_s, \quad s \in [0, T],$$

$$H_0^u = \frac{d}{\sqrt{\gamma_0}} - \sqrt{\gamma_0}x$$

cost functional J, for $u \in \mathcal{L}^2$:

$$J(u) = \frac{1}{2}E\left[\left(H_T^u\right)^2 + \int_0^T \left(2\rho_s + \mu_s - \sigma_s^2\right)u_s^2 ds\right]$$

Link between the problems

$$\inf_{X \in \mathcal{A}^{fv}} J^{fv}(X) = \inf_{X \in \mathcal{A}^{pm}} J^{pm}(X) = \inf_{u \in \mathcal{L}^2} J(u) - \frac{d^2}{2\gamma_0}$$

- if $X^* \in \mathcal{A}^{pm}$ minimizes J^{pm} over $\mathcal{A}^{pm}$, then $u^* = \gamma^{-\frac{1}{2}} D^{X^*}$ minimizes J over $\mathcal{L}^2$
- if $u^* \in \mathcal{L}^2$ minimizes J over $\mathcal{L}^2$, then $X_s^* = \gamma_s^{-\frac{1}{2}}(u_s^* - H_s^{u^*})$, $s \in [0, T)$, $X_{0-}^* = x$, $X_T^* = 0$, minimizes J^{pm} over $\mathcal{A}^{pm}$

Solution

Assumptions:

$(\mathcal{F}_s)_{s\in[0,T]}$ is the augmented natural filtration of an m-dimensional Brownian motion $(W^1,\dots,W^m)^\top$, where $W^1 = W$

$2\rho + \mu - \sigma^2 \geq 0 \;\; dP \times ds|_{[0,T]}$-a.e.

$\exists\, \varepsilon > 0$ s.t. $2\rho + \mu - \sigma^2 \geq \varepsilon \;\; dP \times ds|_{[0,T]}$-a.e.
or $\sigma^2 \geq \varepsilon \;\; dP \times ds|_{[0,T]}$-a.e.

Solution

Consider the Riccati-type BSDE

$$
\begin{aligned}
dK_s &= -\left(-\frac{\left((\rho_s+\mu_s)K_s+\sigma_s L_s^1\right)^2}{\frac{1}{2}(2\rho_s+\mu_s-\sigma_s^2)+\sigma_s^2 K_s}+\mu_s K_s+\sigma_s L_s^1\right)ds \\
&\quad + \sum_{j=1}^{m} L_s^j dW_s^j, \quad s\in[0,T], \\
K_T &= \frac{1}{2}.
\end{aligned}
$$

By [Kohlmann,Tang SPA'02], there exists a unique solution (K,L) of this BSDE.

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Define

$$
\theta_s = \frac{(\rho_s+\mu_s)K_s+\sigma_s L_s^1}{\frac{1}{2}(2\rho_s+\mu_s-\sigma_s^2)+\sigma_s^2 K_s}, \quad s\in[0,T].
$$

Solution

Let

$$
\begin{aligned}
dH_s^* = {} & H_s^* \frac{1}{2}\left(\mu_s - \frac{1}{4}\sigma_s^2 - \left(2(\rho_s + \mu_s) - \sigma_s^2\right)\theta_s\right) ds \\
& + H_s^*\left(\frac{1}{2}\sigma_s - \sigma_s\theta_s\right) dW_s^1, \quad s \in [0, T], \\
H_0^* = {} & \frac{d}{\sqrt{\gamma_0}} - \sqrt{\gamma_0}x.
\end{aligned}
$$

By [Kohlmann, Tang SPA'02],

$$
u^* = \theta H^*
$$

is the unique optimal control in $\mathcal{L}^2$ for J. It holds $H^{u^*} = H^*$ and

$$
J(u^*) = K_0 H_0^*.
$$

Solution

$X^* = (X_s^*)_{s\in[0-,T]}$ defined by

$$X_{0-}^* = x, \quad X_T^* = 0, \quad X_s^* = \gamma_s^{-\frac{1}{2}}\,(\theta_s - 1)\,H_s^*, \quad s \in [0, T),$$

is the unique (up to $dP \times ds|_{[0,T]}$-null sets) optimal execution strategy in $\mathcal{A}^{pm}$ for J^{pm}. The associated costs are given by

$$J^{pm}(X^*) = \frac{K_0}{\gamma_0}(d - \gamma_0 x)^2 - \frac{d^2}{2\gamma_0}.$$

Remarks

- We can demand an $\mathcal{F}_T$-measurable terminal position $X_T = \xi$. In this case, a further, linear BSDE enters the solution.
- We can add a risk term of the form $E[\int_0^T \lambda_s \gamma_s (X_s - \zeta_s)^2 ds]$ (as in, e.g., [Bank,Voß SICON'18]) to the cost functional J^{fv}. In this case, we need an additional reformulation before applying [Kohlmann,Tang SPA'02]. If the target process ζ is not equivalent to 0, we have a further, linear BSDE.
- If $\xi = 0$ and ζ or λ vanish, we could also apply [Sun,Xiong,Yong AAP'21].
- We can consider a diffusive resilience $dR_s = \rho_s ds + \eta_s dW_s^R$, where $dW_s^R = \overline{r}_s dW_s^1 + \sqrt{1 - \overline{r}_s^2} dW_s^2$.

Thank you!

Based on:

J. Ackermann, T. Kruse, and M. Urusov: **Reducing Obizhaeva-Wang type trade execution problems to LQ stochastic control problems**. Preprint, arXiv:2206.03772, 2022.