# Large ranking games

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Based on joint work with

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# Outline

1. Model
2. Two player game
3. Mean field game
4. Approximate Nash equilibrium for the *n*-player game

# Model

- symmetric game of $n$ players
- state processes $X^1, \dots, X^n$

$$dX_t^{i,a} = a_i(X_t^{1,a}, \dots, X_t^{n,a})dW_t^i,\ X_0^{i,a} = 0,$$

- $(W^1, \dots, W^n)$...Brownian motion
- $a_i : \mathbb{R}^n \to [\sigma_1, \sigma_2]$ measurable...control of player $i$
- $0 < \sigma_1 < \sigma_2$
- $\mathcal{A}_n$ set of controls available to a single player

# Model continued

- rank-based reward: player receives a reward if her state is under the best $\alpha \in (0,1)$ percent at final time $T$
- $\mu^{n,a} = \frac{1}{n}\sum_{j=1}^{n} \delta_{X_T^{j,a}}$...empirical distribution at time $T$
- $q(\mu^{n,a}, 1-\alpha)$...empirical $(1-\alpha)$-quantile at time $T$
- $$\text{reward of player } i = \begin{cases} 1, & \text{if } X_T^{i,a} > q(\mu^{n,a}, 1-\alpha), \\ 0, & \text{else.} \end{cases}$$
- player $i$ aims at maximizing
$$P(X_T^{i,a} > q(\mu^{n,a}, 1-\alpha))$$

# Motivation

- risk management: bonus if the own company is among the best performing companies
- research competition among many research and developer teams
- sports: tournament with many teams
- card games: e.g. Skat (best third shares the pot)
- political science: elections with many candidates
- biology: e.g. animal behavior

# Two player game

- $n = 2$ and $\alpha = \frac{1}{2}$
- players aim at maximizing the probability of being ahead at time $T$
- player 1: $P(X_T > Y_T) \to \max$
- player 2: $P(Y_T > X_T) \to \max$

# Two player game

- $n = 2$ and $\alpha = \frac{1}{2}$
- players aim at maximizing the probability of being ahead at time $T$
- player 1: $P(X_T > Y_T) \to \max$
- player 2: $P(Y_T > X_T) \to \max$
- zero-sum game: for player 2 equivalent $P(X_T > Y_T) \to \min$
- consider the upper value and lower value of the game
- goal: Find a tuple $(a_1^*, a_2^*)$ that are mutually best responses, i.e.

$$P(X_T^{a_1^*,a_2^*} > Y_T^{a_1^*,a_2^*}) = \sup_a P(X_T^{a,a_2^*} > Y_T^{a,a_2^*})$$
$$P(X_T^{a_1^*,a_2^*} > Y_T^{a_1^*,a_2^*}) = \inf_b P(X_T^{a_1^*,b} > Y_T^{a_1^*,b})$$

($(a_1^*, a_2^*)$ is saddle point/Nash equilibrium)

# Two player game continued

**Theorem**

*Let*

$$a_1^*(x,y) = \begin{cases} \sigma_2, & \text{if } x \le y, \\ \sigma_1, & \text{if } x > y, \end{cases}$$

*and*

$$a_2^*(x,y) = a_1^*(y,x).$$

*Then* $(a_1^*, a_2^*)$ *is a saddle point of the two player game, i.e.*

$$P(X_T^{a_1^*,a_2^*} > Y_T^{a_1^*,a_2^*}) = \sup_a P(X_T^{a,a_2^*} > Y_T^{a,a_2^*}) = \inf_b P(X_T^{a_1^*,b} > Y_T^{a_1^*,b})$$

*(and hence also a Nash equilibrium).*

# Do Nash equilibria exist?

- What happens if $n > 2$?
- **Difficulty**: payoff is discontinuous
- **Our solution**: consider mean field limit to find an approximate Nash equilibrium for large $n$

# Mean field game

- reduce problem to one generic player
- state is given by
$$dX_t = \beta_t dW_t, \ X_0 = 0$$
with $\beta : \Omega \times [0, T] \to [\sigma_1, \sigma_2]$ progr. mb.
- reward depends on the distribution of the single player's state
- classical mean field game approach:
  1. For any probability measure $\mu$ find a control $\beta^*(\mu)$ s.t.
  $$P(X_T^{\beta^*(\mu)} > q(\mu, 1-\alpha)) = \sup_{\beta} P(X_T^{\beta} > q(\mu, 1-\alpha)).$$
  2. Determine fixed point $\mu^*$ of $\mu \mapsto \mathrm{Law}(X_T^{\beta^*(\mu)})$.
- **our approach**:
  1. consider $\sup_\beta P(X_T^\beta > b)$ and find optimal control $\beta^*(b)$
  2. find fixed point of $b \mapsto q(X_T^{\beta^*(b)}, 1-\alpha)$

# Control problem

- diffusion control problem with discontinuous criterion
- McNamara (1983): optimal response is threshold control with threshold $b$, i.e.

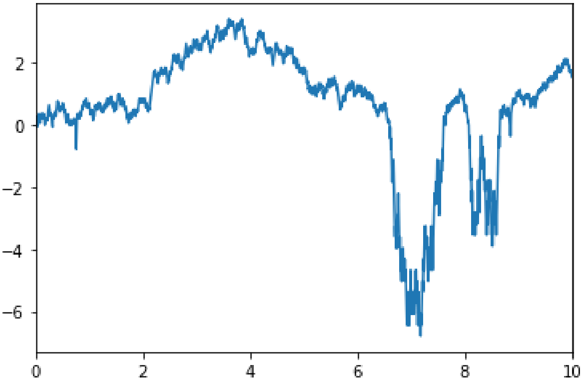
$$\sup_{\beta} P(X_T^{\beta} > b) = P(X_T^{m_b} > b)$$

  where

$$m_b(x) = \begin{cases} \sigma_2, & \text{if } x \leq b, \\ \sigma_1, & \text{if } x > b. \end{cases}$$

- $X^{m_b}$ is an oscillating Brownian motion (OBM)
- OBM has a probability density in closed form, see e.g. Keilson, Wellner (1978)

# Path of an OBM



# Mean field equilibrium

- find $b$ such that $b = q(X_T^{m_b}, 1 - \alpha)$
- equivalent to $P(X_T^{m_b} > b) = \alpha$

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**Theorem**

*The threshold strategy with threshold*

$$b^* := \begin{cases} -\sigma_2\sqrt{T}\Phi^{-1}\left(\frac{\alpha(\sigma_1+\sigma_2)}{2\sigma_2}\right), & \text{if } \alpha \leq \frac{\sigma_2}{\sigma_1+\sigma_2}, \\ \sigma_1\sqrt{T}\Phi^{-1}\left(\frac{(1-\alpha)(\sigma_1+\sigma_2)}{2\sigma_1}\right), & \text{if } \alpha > \frac{\sigma_2}{\sigma_1+\sigma_2}. \end{cases}$$

*is an equilibrium strategy for the mean field game, i.e.*

$$P(X_T^{m_{b^*}} > q(X_T^{m_{b^*}}, 1-\alpha)) = \sup_\beta P(X_T^\beta > q(X_T^{m_{b^*}}, 1-\alpha)).$$

*Moreover, it is the unique equilibrium strategy in the set of threshold strategies.*

# Dependence on parameters

$$b^* = \begin{cases} -\sigma_2\sqrt{T}\Phi^{-1}\left(\frac{\alpha(\sigma_1+\sigma_2)}{2\sigma_2}\right), & \text{if } \alpha \leq \frac{\sigma_2}{\sigma_1+\sigma_2}, \\ \sigma_1\sqrt{T}\Phi^{-1}\left(\frac{(1-\alpha)(\sigma_1+\sigma_2)}{2\sigma_1}\right), & \text{if } \alpha > \frac{\sigma_2}{\sigma_1+\sigma_2}. \end{cases}$$

- $b^* = 0$ if $\alpha = \frac{\sigma_2}{\sigma_1+\sigma_2}$
- low $\alpha$ induces riskier strategy since $b^* > 0$ and $\lim_{\alpha\downarrow 0} b^* = \infty$
- high $\alpha$ induces safer strategy since $b^* < 0$ and $\lim_{\alpha\uparrow 1} b^* = -\infty$

# The smaller the cake...

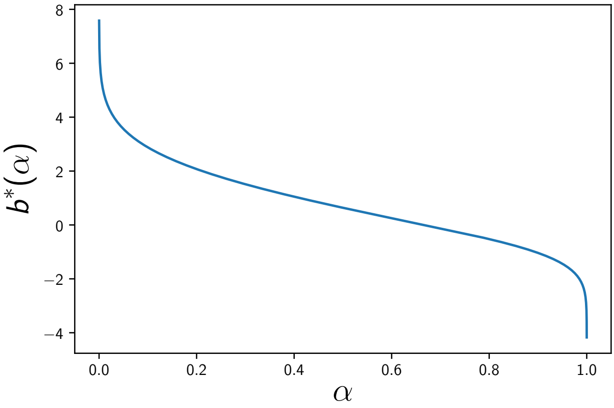


Figure: $\sigma_1 = 1$, $\sigma_2 = 2$, $T = 1$

# Comparison

| 2 players | $\infty$ players |
|---|---|
| only relative position counts | only absolute position counts |
| observability is crucial | observability is irrelevant |

# Approximate Nash equilibrium

**Definition**

Let $\varepsilon > 0$. A tuple $a = (a_1, \ldots, a_n) \in \mathcal{A}_n^n$ is called *$\varepsilon$-Nash equilibrium* of the $n$-player game if for all $i \in \{1, \ldots, n\}$, $c \in \mathcal{A}_n$ and all weak solutions $X^{i,(a_{-i},c)}$ and $X^{i,a}$ we have

$$P(X_T^{i,(a_{-i},c)} > q(\mu^{n,(a_{-i},c)}, 1-\alpha)) - P(X_T^{i,a} > q(\mu^{n,a}, 1-\alpha)) \leq \varepsilon.$$

# Mean field equilibrium yields approximate Nash equilibrium

**Theorem**

*Let* $a = (a_1, \ldots, a_n) \in \mathcal{A}_n^n$ *be the tuple of mean field equilibrium strategies, i.e.*

$$a_i(x) = \begin{cases} \sigma_2, & x_i \leq b^*, \\ \sigma_1, & x_i > b^*, \end{cases} \qquad x \in \mathbb{R}^n.$$

*There exists a sequence* $\varepsilon_n \geq 0$ *with* $\lim_n \varepsilon_n = 0$ *such that* $a$ *is an* $\varepsilon_n$*-Nash equilibrium of the n-player game. We can choose* $\varepsilon_n \in \mathcal{O}(n^{-\frac{1}{2}})$.

# Percentage of players choosing small volatility $\sigma_1$ on time horizon $[0, T]$

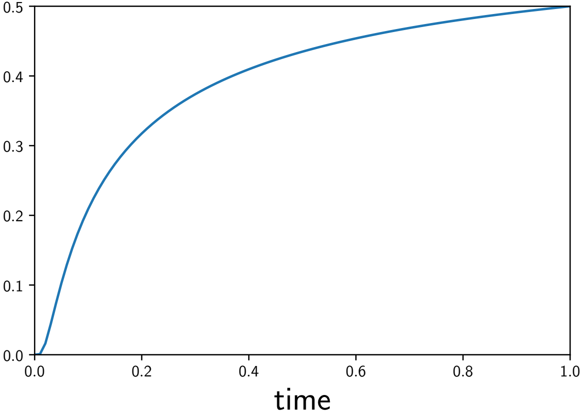


Figure: $\sigma_1 = 1$, $\sigma_2 = 2$, $T = 1$, $\alpha = \frac{1}{2} < \frac{\sigma_2}{\sigma_1+\sigma_2}$

# Percentage of players choosing small volatility $\sigma_1$ on time horizon $[0, T]$

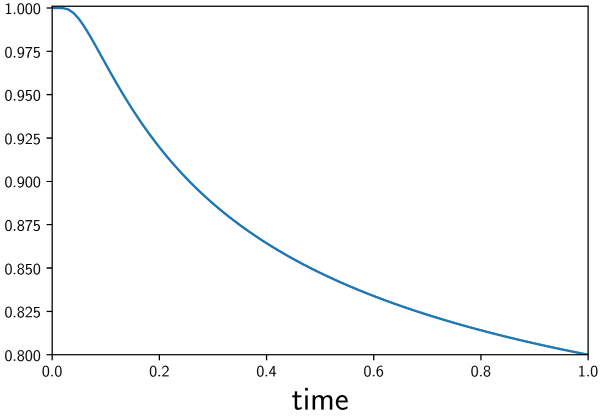


Figure: $\sigma_1 = 1$, $\sigma_2 = 2$, $T = 1$, $\alpha = 0.8 > \frac{\sigma_2}{\sigma_1+\sigma_2}$

# Conclusion

- closed form equilibria for the limiting cases $n = 2$ and $n = \infty$.
- games with $n \geq 3$ players: the larger $n$...

...the less important the relative position

...the more important the absolute position

# Thanks for listening!

# References


[1] S. Ankirchner, N. Kazi-Tani, J. Wendt, and C. Zhou. "Large ranking games with diffusion control". working paper or preprint. Nov. 2021.

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[3] J. M. McNamara. "Optimal control of the diffusion coefficient of a simple diffusion process". In: *Mathematics of Operations Research* 8.3 (1983), pp. 373–380. ISSN: 0364-765X. DOI: 10.1287/moor.8.3.373.