Path-dependent mean-field game optimal planning

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9th Colloquium on Backward Stochastic Differential Equations and Mean Field Systems

Annecy France, June 30, 2022

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MFG: Introduction

The Mean Field Game was introduced

- by Lasry and Lions (2006,2007)
- by Huang, Caines and Malhamé (2006)

to describe Nash equilibra in differential games with infinitely many players.

Features of the model:

- Players act according to the same principles, i.e.,
  - they are indistinguishable
  - they have the same optimization criteria
- Players have individually an infinitesimal influence, but their strategies take into account the mass of co-players.

Goal: introduce a macroscopic description as the number of players $N \to \infty$.

MFG: Modelling

Finitely many players

- Each player controls its state $X_t^i \in \mathbb{R}^d$ by taking an action $\alpha_t^i \in A \subset \mathbb{R}^k$

$$dX_t^i = b\Big(t, X_t^i, \alpha_t^i, \overline{\mu}^{N-1}_{X_t^{-i}}\Big)dt + \sigma\Big(t, X_t^i, \alpha_t^i, \overline{\mu}^{N-1}_{X_t^{-i}}\Big)dW_t^i,$$

  where
  - W^i are independent
  - $\overline{\mu}^{N-1}_{X_t^{-i}}$ is the empirical distribution of other players:

$$\overline{\mu}^{N-1}_{X_t^{-i}} := \frac{1}{n-1}\sum_{\substack{j=1\\ j\neq i}}^{n} \delta_{X_t^j}$$

- Each player solves the control problem

$$\sup_{\alpha^i \in \mathcal{A}} J^i(\alpha^i, \boldsymbol{\alpha}^{-i}), \quad J^i(\boldsymbol{\alpha}) = \mathbb{E}\left[\int_0^T f\Big(t, X_t^i, \alpha^i, \overline{\mu}^{N-1}_{X_t^{-i}}\Big) + g\Big(X_T^i, \overline{\mu}^{N-1}_{X_T^{-i}}\Big)\right].$$

- We look for a **Nash equilibrium**: $\widehat{\boldsymbol{\alpha}} \in \mathcal{A}^N$ ($\alpha^i \in \mathcal{A}$ for each $i = 1, \ldots, N$), s.t.

$$J^i(\widehat{\boldsymbol{\alpha}}) \geq J^i(\alpha^i, \widehat{\boldsymbol{\alpha}}^{-i}).$$

  NO player has interest to deviate unilaterally.

MFG: Modelling

As $N \to \infty$, $\overline{\mu}^{N-1}_{X_t^{-i}}$ converges to a deterministic distribution.

Nash equilibrium is described as follows (Carmona and Delarue 2017)

- The **representative player** controls its state X^α depending on the deterministic flow $\{\mu_t\}_{0\le t\le T}$:
$$dX_t^\alpha = b(t, X_t^\alpha, \alpha_t, \mu_t)dt + \sigma(t, X_t^\alpha, \alpha_t, \mu_t)dW_t.$$
- He solves the control problem
$$\sup_{\alpha\in\mathcal{A}} J^\mu(\alpha), \quad J^\mu(\alpha) = \mathbb{E}\left[\int_0^T f(t, X_t^\alpha, \alpha_t, \mu_t)dt + g(X_T^\alpha, \mu_T)\right].$$
- We look for a deterministic measure flow $\{\mu_t\}_{0\le t\le T}$, **mean field equilibrium**, such that
$$\widehat{\alpha}[\mu] \in \operatorname*{argmax}_{\alpha\in\mathcal{A}} J^\mu(\alpha) \quad \text{and} \quad \mathcal{L}\big(X_t^{\widehat{\alpha}[\mu]}\big) = \mu_t, \quad t\in[0,T].$$

MFG: PDE Approach

- The value function associated to the stochastic control problem is characterized as the solution to a **Hamilton-Jacobi-Bellman equation**

$$-\partial_t u - \sup_{a\in A}\left\{ b(t,x,a,\mu_t)\cdot\nabla u + \frac{1}{2}\sigma\sigma^\intercal(t,x,a,\mu_t) : \nabla^2 u + f(t,x,a,\mu_t)\right\} = 0,$$

with the terminal condition

$$u(T,x) = g(x,\mu_T).$$

- The optimal controlled process is given by

$$\widehat{X}_t = X_0 + \int_0^t b\big(s,\widehat{X}_s,\widehat{a}(s,\widehat{X}_s),\mu_s\big)ds + \int_0^t \sigma\big(s,\widehat{X}_s,\widehat{a}(s,\widehat{X}_s),\mu_s\big)dW_s,$$

and the flow of densities $m(t,x)$ of $\mathcal{L}(\widehat{X}_t) = \mu_t$ solves the **Fokker-Planck equation**

$$\partial_t m(t,x) - \frac{1}{2}\nabla^2 : \big[\sigma\sigma^\intercal\big(t,x,\widehat{a}(t,x),\mu_t\big)m(t,x)\big] + \nabla\cdot\big[b\big(t,x,\widehat{a}(t,x),\mu_t\big)m(t,x)\big] = 0,$$

with the initial condition $m(0,\cdot) = \text{density of } \mu_0$, where $\widehat{a}$ is the optimal feedback control.

$\Longrightarrow$ a coupled system of a (backward) HJB and a (forward) FP equation.

MFG Planning: PDE Approach

During his courses at Collège de France, Lions introduced the following planning problem for a class MFG: *Given two marginal distributions* μ_0 *and* μ_1 *on* $\mathbb{R}^d$*, find a solution* (u, m) *of the following MFG system:*

$$
\begin{aligned}
-\partial_t u - \frac{\sigma^2}{2}\Delta u - H(x, \nabla u) - F(x, m) = 0, &\qquad \text{in } (0,1) \times \mathbb{R}^d,\\
\partial_t m - \frac{\sigma^2}{2}\Delta m + \nabla \cdot \big(m \nabla_z H(x, \nabla u)\big) = 0, &\qquad \text{in } (0,1) \times \mathbb{R}^d,\\
m(0,\cdot) = \mu_0, \quad m(1,\cdot) = \mu_1, &\qquad \text{in } \mathbb{R}^d.
\end{aligned}
$$

Remark:

- Unlike the MFG formulation, the HJB equation is not complemented with a terminal condition for $u(1,\cdot)$. Instead, the Fokker-Planck equation is equipped with a terminal condition on $m(1,\cdot)$ in addition to the initial condition $m(0,\cdot)$.
- The planning problem consists in finding a good terminal condition $g := u(1,\cdot)$ for the HJB equation such the classical MFG problem has a solution satisfying the marginal constraint $m(0,\cdot) = \mu_0$ and $m(1,\cdot) = \mu_1$.
- For this reason, g is usually called the **incentive function**.

MFG Planning: PDE Approach

References:

- Lions proved an existence and uniqueness result in the quadratic Hamiltonian setting for a large class of initial and target measures.
- Various extensions have been achieved essentially allowing for Hamiltonians with quadratic growth in the gradient, and using weak solutions for the MFG equation
  - Achdou, Camilli, and Capuzzo-Dolcetta (2012)
  - Porretta (2014)
  - Graber, Mészáros, Silva, and Tonon (2019)
  - Orrieri, Porretta, and Savaré (2019)
  - Benamou, Carlier, Di Marino, and Nenna (2019)
  - etc.

Objective: extend the formulation of the planning problem to the path-dependent setting.

- HJB equation is replaced by a possibly path-dependent stochastic control problem.
- FP equation is replaced by the path-dependent SDE characterizing the dynamics of the underlying state under the optimal action induced by the control problem.
- We allow for the control of diffusion coefficient. HJB in the corresponding Markovian setting is allowed to be fully nonlinear.

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Weak formulation of the control problem

Let

- $\mathbb{P}_0 :=$ the Wiener measure on the canonical space Ω with initial distribution μ_0, i.e.,
  - $\mathbb{P}_0 \circ X_0^{-1} = \mu_0$
  - $(X_t - X_0)_{t\in[0,T]}$ is a Brownian motion independent of X_0 under $\mathbb{P}_0$.
- $\mathcal{P}(\mu_0) := \{\mathbb{P} \approx \mathbb{P}_0 : \mathbb{P} \circ X_0^{-1} = \mu_0\}$ and

$$\mathcal{P}_2(\mu_0) := \left\{\mathbb{P} \in \mathcal{P}(\mu_0) : \ln\left(\frac{d\mathbb{P}}{d\mathbb{P}_0}\right) \in \mathbb{L}^1(\mathbb{P}_0) \text{ and } \frac{d\mathbb{P}}{d\mathbb{P}_0} \in \mathbb{L}^2(\mathbb{P}_0)\right\}.$$

We have

- by the **representation theorem** that for $\mathbb{P} \in \mathcal{P}(\mu_0)$ we may find a unique process $\beta^{\mathbb{P}} \in \mathbb{H}^2_{\mathrm{loc}}(\mathbb{P}_0)$ such that

$$\frac{d\mathbb{P}}{d\mathbb{P}_0} = \mathcal{E}\left(\beta^{\mathbb{P}} \bullet X\right)_1 := \exp\left(\int_0^1 \beta_s^{\mathbb{P}} \cdot dX_s - \frac{1}{2}\int_0^1 \left|\beta_s^{\mathbb{P}}\right|^2 ds\right),$$

- by the **Girsanov theorem** that the canonical process X satisfies the dynamics

$$X_t = X_0 + \int_0^t \beta_s^{\mathbb{P}} ds + W_t^{\mathbb{P}}, \quad \mathbb{P}\text{-a.s.}.$$

- **Lemma:** For each $\mathbb{P} \in \mathcal{P}_2(\mu_0)$, we have $\beta^{\mathbb{P}} \in \mathbb{H}^2(\mathbb{P}_0)$.

Weak formulation of the control problem

Let $f : [0,T] \times \Omega \times \mathcal{P}(\mathbb{R}^d) \longrightarrow \mathbb{R}$ be such that

- $(t,\omega,\mu) \mapsto f_t(\omega,\mu)$ is $\mathbb{F}$-progressively measurable for every fixed $\mu \in \mathcal{P}(\mathbb{R}^d)$,
- for each $\mathbb{P} \in \mathcal{P}_2(\mu_0)$ and $m \in \mathbb{M}$ (flows of probability measures)

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^1 \left|f_t(m_t)\right| dt\right] < \infty.$$

Define the set of all **admissible** (path-dependent and measurable) **reward** function

$$\Xi := \left\{\xi : \Omega \to \mathbb{R} : \mathbb{E}^{\mathbb{P}}[\xi^+] < \infty,\ \forall \mathbb{P} \in \mathcal{P}_2(\mu_0)\right\}.$$

Control problem: For $\xi \in \Xi$ and $m \in \mathbb{M}$ with $m_0 = \mu_0$,

$$V_0(\xi,m) := \sup_{\mathbb{P}\in\mathcal{P}_2(\mu_0)} J(\xi,m,\mathbb{P}),$$

where

$$J(\xi,m,\mathbb{P}) := \mathbb{E}^{\mathbb{P}}\left[\xi - \int_0^1 \left(\frac{1}{2}\left|\beta_s^{\mathbb{P}}\right|^2 + f_s(m_s)\right) ds\right].$$

Mean field game and MFG planning

Definition (Mean field game)

A probability measure $\widehat{\mathbb{P}} \in \mathcal{P}_2(\mu_0)$ is a solution of the MFG with reward function $\xi \in \Xi$ if

$$V_0(\xi,m) = J(\xi,m,\widehat{\mathbb{P}}) \in \mathbb{R} \quad \text{and} \quad m_t := \widehat{\mathbb{P}} \circ X_t^{-1}, \text{ for all } t \in [0,1].$$

Denote by $\mathrm{MFG}(\xi,\mu_0)$ the collection of solutions of the MFG problem.

Our main focus here is on the following **mean field game planning problem**.

Definition (MFG planning)

An admissible reward function $\xi \in \Xi$ is a solution to the MFG planning problem with starting and target distributions $\mu_0, \mu_1 \in \mathcal{P}(\mathbb{R}^d)$ if

$$\widehat{\mathbb{P}} \circ X_1^{-1} = \mu_1, \quad \text{for some } \widehat{\mathbb{P}} \in \mathrm{MFG}(\xi,\mu_0).$$

Denote by $\mathrm{MFP}(\mu_0,\mu_1)$ the collection of solutions of the MFG planning problem.

Characterization of the solutions of mean field planning problem

Denote

$$\mathcal{P}_2(\mu_0,\mu_1) := \left\{\mathbb{P}\in\mathcal{P}_2(\mu_0) : \mathbb{P}\circ X_1^{-1} = \mu_1\right\}, \quad \mu_0,\, \mu_1 \in \mathcal{P}(\mathbb{R}^d).$$

Theorem (Ren, Tan, Touzi, Y., 2022)

For all pair of starting and target measures $(\mu_0,\mu_1)\in\mathcal{P}(\mathbb{R}^d)\times\mathcal{P}(\mathbb{R}^d)$, we have:

$$\begin{aligned}\mathrm{MFP}(\mu_0,\mu_1) &= \mathbb{L}^1(\mathcal{F}_0,\mathbb{P}_0)\\ &\quad + \left\{\int_0^1 \beta_t^{\mathbb{P}}\cdot dX_t - \int_0^1\left(\frac{1}{2}\left|\beta_t^{\mathbb{P}}\right|^2 - f_t\left(\mathbb{P}\circ X_t^{-1}\right)\right)dt \;\middle|\; \mathbb{P}\in\mathcal{P}_2(\mu_0,\mu_1)\right\}.\end{aligned}$$

Proof: "$\supseteq$": For $Y_0\in\mathbb{L}^1(\mathcal{F}_0,\mathbb{P}_0)$ and $\widehat{\mathbb{P}}\in\mathcal{P}_2(\mu_0,\mu_1)$, denote

$$\xi := Y_0 + \int_0^1 \beta_t^{\widehat{\mathbb{P}}}\cdot dX_t - \int_0^1\left(\frac{1}{2}\left|\beta_t^{\widehat{\mathbb{P}}}\right|^2 - f_t(m_t)\right)dt, \qquad m_t := \widehat{\mathbb{P}}\circ X_t^{-1}.$$

Aim: verify that $V_0(\xi,m) = J(\xi,m,\widehat{\mathbb{P}})$. This would show that $\widehat{\mathbb{P}}\in\mathrm{MFG}(\xi,\mu_0)$ and therefore $\xi\in\mathrm{MFP}(\mu_0,\mu_1)$.

Characterization of the solutions of mean field planning problem

Compute directly for all $\mathbb{P} \in \mathcal{P}_2(\mu_0)$ that

$$\begin{aligned} J(\xi,m,\mathbb{P}) &= \mathbb{E}^{\mathbb{P}}\left[\xi - \int_0^1 \left(\frac{1}{2}\left|\beta_s^{\mathbb{P}}\right|^2 + f_s\left(m_s\right)\right) ds\right] \\ &= \mathbb{E}^{\mathbb{P}_0}\left[Y_0\right] + \mathbb{E}^{\mathbb{P}}\left[\int_0^1 \beta_s^{\widehat{\mathbb{P}}} \cdot \left(dW_s^{\mathbb{P}} + \beta_s^{\mathbb{P}} ds\right) - \int_0^1 \left(\frac{1}{2}\left|\beta_s^{\widehat{\mathbb{P}}}\right|^2 + \frac{1}{2}\left|\beta_s^{\mathbb{P}}\right|^2\right) ds\right]. \end{aligned}$$

As $\beta^{\mathbb{P}} \in \mathbb{H}^2(\mathbb{P}_0)$ for each $\mathbb{P} \in \mathcal{P}_2(\mu_0)$, the stochastic integral is a true martingale under $\mathbb{P}$, i.e.,

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^1 \beta_s^{\widehat{\mathbb{P}}} \cdot dW_s^{\mathbb{P}}\right] = 0,$$

and therefore

$$J\left(\xi,m,\mathbb{P}\right) = \mathbb{E}^{\mathbb{P}_0}\left[Y_0\right] - \frac{1}{2}\mathbb{E}^{\mathbb{P}}\left[\int_0^1 \left|\beta_s^{\mathbb{P}} - \beta_s^{\widehat{\mathbb{P}}}\right|^2 ds\right],$$

so that

$$J\left(\xi,m,\mathbb{P}\right) \begin{cases} \leq \mathbb{E}^{\mathbb{P}_0}\left[Y_0\right], & \text{for all } \mathbb{P} \in \mathcal{P}_2(\mu_0), \\ = \mathbb{E}^{\mathbb{P}_0}\left[Y_0\right], & \text{for } \mathbb{P} = \widehat{\mathbb{P}}. \end{cases}$$

Characterization of the solutions of mean field planning problem

"$\subseteq$": Let $\xi \in \mathrm{MFP}(\mu_0, \mu_1)$, with a corresponding $\widehat{\mathbb{P}} \in \mathrm{MFG}(\xi, \mu_0)$, i.e.,

$$V_0(\xi, m) = J(\xi, m, \widehat{\mathbb{P}}), \quad m_t = \widehat{\mathbb{P}} \circ X_t^{-1}, \quad \widehat{\mathbb{P}} \in \mathcal{P}_2(\mu_0, \mu_1).$$

Aim: show that one can represent ξ as

$$\xi = V_0 + \int_0^1 \beta_s^{\widehat{\mathbb{P}}} \cdot dX_s - \int_0^1 \left(\frac{1}{2} \left| \beta_s^{\widehat{\mathbb{P}}} \right|^2 - f_s(m_s) \right) ds, \quad \text{for some } V_0 \in \mathbb{L}^1(\mathcal{F}_0, \mathbb{P}_0).$$

Introduce the process

$$V_t := \operatorname*{ess\,sup}_{\mathbb{P} \in \mathcal{P}_2(\mu_0)} \mathbb{E}^{\mathbb{P}} \left[\xi - \int_t^1 c_s^{\mathbb{P}} ds \,\middle|\, \mathcal{F}_t \right], \quad \text{with } c_s^{\mathbb{P}} := \frac{1}{2} \left| \beta_s^{\mathbb{P}} \right|^2 + f_s(m_s).$$

- Clearly, $\mathbb{E}^{\mathbb{P}_0}\left[V_0\right] = V_0(\xi, m) < \infty$, so that $V_0 \in \mathbb{L}^1(\mathcal{F}_0, \mathbb{P}_0)$.
- For any $\mathbb{P} \in \mathcal{P}_2(\mu_0)$, the process $\left\{ V_t - \int_0^t c_s^{\mathbb{P}} ds \right\}_{t \in [0,1]}$ is $\mathbb{P}$-supermartingale. By **Doob-Meyer** and representation theorem

$$V_t - \int_0^t c_s^{\mathbb{P}_0} ds = V_0 + \int_0^t Z_s \cdot dX_s - A_t^{\mathbb{P}_0}, \quad \mathbb{P}_0\text{-a.s.},$$

for some $Z \in \mathbb{H}^2_{\mathrm{loc}}(\mathbb{P}_0)$ and non-decreasing process $A^{\mathbb{P}_0}$ starting from zero.

Characterization of the solutions of mean field planning problem

- By the change of measure from $\mathbb{P}_0$ to $\mathbb{P}$, we have

$$V_t = \int_0^t c_s^{\mathbb{P}} ds + V_0 + \int_0^t Z_s \cdot dW_s^{\mathbb{P}} - A_t^{\mathbb{P}}, \quad \mathbb{P}\text{-a.s.},$$

with

$$A_t^{\mathbb{P}} = A_t^{\mathbb{P}_0} + \int_0^t \left(c_s^{\mathbb{P}} - c_s^{\mathbb{P}_0} - Z_s \cdot \beta_s^{\mathbb{P}}\right) ds.$$

By uniqueness of the Doob-Meyer decomposition, $A^{\mathbb{P}}$ is also non-decreasing.

- The process $\left\{V_t - \int_0^t c_s^{\widehat{\mathbb{P}}} ds\right\}_{t\in[0,1]}$ is a $\widehat{\mathbb{P}}$-martingale, i.e.,

$$0 = A_t^{\widehat{\mathbb{P}}} = A_t^{\mathbb{P}_0} + \int_0^t \left(c_s^{\widehat{\mathbb{P}}} - c_s^{\mathbb{P}_0} - Z_s \cdot \beta_s^{\widehat{\mathbb{P}}}\right) ds.$$

This implies that

$$\frac{dA_t^{\mathbb{P}}}{dt} = \left(Z_t \cdot \beta_t^{\widehat{\mathbb{P}}} - c_t^{\widehat{\mathbb{P}}}\right) - \left(Z_t \cdot \beta_t^{\mathbb{P}} - c_t^{\mathbb{P}}\right) \geq 0, \quad \text{for all } \mathbb{P} \in \mathcal{P}_2(\mu_0).$$

In particular, $\beta^{\widehat{\mathbb{P}}} = Z$ is the maximizer of $Z \cdot \beta^{\mathbb{P}} - c^{\mathbb{P}} = Z \cdot \beta^{\mathbb{P}} - \frac{1}{2}\left|\beta^{\mathbb{P}}\right|^2 - f$.

Characterization of the solutions of mean field planning problem

Therefore,

$$\xi = V_1 = V_0 + \int_0^1 Z_s \cdot dW_s^{\widehat{\mathbb{P}}} + \int_0^1 c_s^{\widehat{\mathbb{P}}} ds - 0$$
$$= V_0 + \int_0^1 \beta_s^{\widehat{\mathbb{P}}} \cdot dX_s - \int_0^1 \left(\frac{1}{2}\left|\beta_s^{\widehat{\mathbb{P}}}\right|^2 - f_s(m_s)\right) ds.$$

□

Summary: This provide a characterization of all solutions of the MFG planning problem by means of the probability measures in $\mathcal{P}_2(\mu_0, \mu_1)$.

Remark: The Hamiltonian of the control problem is

$$H(t,\omega,z,m) = \sup_{b \in U} \left\{ b \cdot z - \frac{1}{2}|b|^2 - f_t(\omega,m) \right\} = \frac{1}{2}|z|^2 - f_t(\omega,m),$$

with the optimal control

$$\widehat{b} = z = \nabla_z H(t,\omega,z,m).$$

Therefore,

$$\xi = V_0 + \int_0^1 Z_s \cdot dX_s - \int_0^1 H_s(Z_s,m_s) ds.$$

Intermezzo: stochastic optimal control and BSDE

- Consider the following stochastic control problem in weak formulation:

$$X_t = X_0 + \int_0^t \beta_s^{\mathbb{P}} ds + W_t^{\mathbb{P}}, \quad \mathbb{P}\text{-a.s.},$$

$$\sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\left[\xi - \int_0^1 c(s,\beta_s)ds\right].$$

- Hamiltonian:

$$H(t,z) := \sup_{b\in U}\left\{b\cdot z - c(t,b)\right\}.$$

- The control problem leads to the FBSDE

$$X_t = X_0 + \int_0^t \nabla_z H(s,Z_s)ds + W_t^{\widehat{\mathbb{P}}},$$

$$Y_t = \xi + \int_t^1 H(s,Z_s)ds - \int_t^1 Z_s\cdot dX_s,$$

and the optimal control:

$$\widehat{\beta}_t[Z] = \nabla_z H(t,Z_t).$$

- **Dynamic programming representation**: if we want that the agent chooses $\nabla_z H(t,Z_t)$ as optimal control, we define the reward function

$$\xi = Y_0 - \int_0^1 H(s,Z_s)ds + \int_0^1 Z_s\cdot dX_s.$$

A constructive solution to the mean field planning problem

Aim: use the characterization above to derive an explicit construction of a particular solution.

- Introduce a **reference measure**

$$\rho := \mathbb{P}_0 \circ (X_0, X_1)^{-1} \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d).$$

- Let $\pi \in \Pi(\mu_0, \mu_1)$ be some coupling measure between μ_0 and μ_1, equivalent to ρ.
- Consider the the corresponding density function $\frac{d\pi}{d\rho}$ on $\mathbb{R}^d \times \mathbb{R}^d$ and define the following positive random variable on Ω

$$\zeta := \frac{d\pi}{d\rho}(X_0, X_1).$$

Observe that

$$\mathbb{E}^{\mathbb{P}_0}[\zeta] = \mathbb{E}^{\mathbb{P}_0}\left[\frac{d\pi}{d\rho}(X_0, X_1)\right] = \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{d\pi}{d\rho}(x_0, x_1) d\rho(x_0, x_1) = 1.$$

By the martingale representation theorem, there exists a $\mathbb{F}$-progressively measurable process $\widehat{\beta}$ such that

$$M_t := \mathbb{E}^{\mathbb{P}_0}[\zeta | \mathcal{F}_t] = M_0 \mathcal{E}\left(\widehat{\beta} \bullet X\right)_t = M_0 \exp\left(\int_0^t \widehat{\beta}_s \cdot dX_s - \frac{1}{2}\int_0^t |\widehat{\beta}_s|^2 ds\right).$$

In particular, as $\pi(dx, \mathbb{R}^d) = \rho(dx, \mathbb{R}^d) = \mu_0(dx)$, we have $M_0 = 1$, $\mathbb{P}_0$-a.s.

A constructive solution to the mean field planning problem

Proposition (Ren, Tan, Touzi, Y., 2022)

Assume $\mathbb{E}^{\mathbb{P}_0}\left[|\ln\zeta| + \zeta^2\right] < \infty$. Then, the probability measure $\widehat{\mathbb{P}}$ defined by $\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}_0} = \zeta$ is an element in $\mathcal{P}_2(\mu_0,\mu_1)$.

Proof: It is clear by its definition $\widehat{\mathbb{P}} \circ X_0^{-1} = \mu_0$ and by the transformation formula

$$\widehat{\mathbb{P}} \circ X_1^{-1}(A) = \mathbb{E}^{\widehat{\mathbb{P}}}\left[1_{\{X_1 \in A\}}\right] = \mathbb{E}^{\mathbb{P}_0}\left[\frac{d\pi}{d\rho}(X_0,X_1)1_{\{X_1\in A\}}\right] = \int_{\mathbb{R}^d\times A} \frac{d\pi}{d\rho}(x_0,x_1)d\rho(x_0,x_1)$$

$$= \int_{\mathbb{R}^d\times A} d\pi(x_0,x_1) = \pi(\mathbb{R}^d, A) = \mu_1(A), \qquad \text{for } A \in \mathcal{B}(\mathbb{R}^d).$$

By the integrability assumption ζ, we have

$$\mathbb{E}^{\mathbb{P}_0}\left[\left|\ln\left(\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}_0}\right)\right| + \left(\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}_0}\right)^2\right] = \mathbb{E}^{\mathbb{P}_0}\left[|\ln\zeta| + \zeta^2\right] < \infty,$$

and therefore $\widehat{\mathbb{P}} \in \mathcal{P}_2(\mu_0, \mu_1)$. □

Solution: With $\widehat{\mathbb{P}} \in \mathcal{P}_2(\mu_0,\mu_1)$, a solution to the MFG planning problem is

$$\xi := Y_0 + \int_0^1 \beta_s^{\widehat{\mathbb{P}}} \cdot dX_s - \int_0^1 \left(\frac{1}{2}\left|\beta_s^{\widehat{\mathbb{P}}}\right|^2 - f_s(m_s)\right)ds.$$

Entropic MFG planning

Entropy of a probability $\mathbb{Q}_1$ with respect to a reference probability $\mathbb{Q}_0$:

$$\mathsf{H}(\mathbb{Q}_1|\mathbb{Q}_0) := \begin{cases} \mathbb{E}^{\mathbb{Q}_1}\left[\ln\left(\frac{d\mathbb{Q}_1}{d\mathbb{Q}_0}\right)\right] = \int_\Omega \ln\left(\frac{d\mathbb{Q}_1}{d\mathbb{Q}_0}\right) d\mathbb{Q}_1, & \text{whenever } \mathbb{Q}_1 \ll \mathbb{Q}_0, \\ \infty, & \text{otherwise.} \end{cases}$$

Proposition (Ren, Tan, Touzi, Y., 2022)

Assume $\mathbb{E}^{\mathbb{P}_0}\left[|\ln\zeta| + \zeta^2\right] < \infty$. Then, the probability measure $\widehat{\mathbb{P}}$ defined by $\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}_0} = \zeta$ is the unique minimizer of $\mathsf{H}(\cdot|\mathbb{P}_0)$ on $\mathcal{P}_\pi := \left\{\mathbb{P} \in \mathcal{P}_2(\mu_0, \mu_1) : \mathbb{P} \circ (X_0, X_1)^{-1} = \pi\right\}$.

Proof: Observe from the definition of $\widehat{\mathbb{P}}$ and Bayes formula that

$$K^{\widehat{\mathbb{P}}}(\cdot; x_0, x_1) = K^{\mathbb{P}_0}(\cdot; x_0, x_1), \quad \text{for } \pi\text{-a.e. } (x_0, x_1) \in \mathbb{R}^d \times \mathbb{R}^d.$$

Further, for any $\mathbb{P} \in \mathcal{P}_\pi$, one has $\mathbb{P}(d\omega) = K^{\mathbb{P}}(d\omega; x_0, x_1)\pi(dx_0, dx_1)$ and

$$\mathsf{H}(\mathbb{P}|\mathbb{P}_0) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \underbrace{\mathsf{H}\left(K^{\mathbb{P}}(\cdot; x_0, x_1) \middle| K^{\mathbb{P}_0}(\cdot; x_0, x_1)\right)}_{\geq 0, \quad = 0 \text{ for } \mathbb{P} = \widehat{\mathbb{P}}} \pi(dx_0, dx_1) + \mathsf{H}(\pi|\rho).$$

It follows that $\mathsf{H}(\widehat{\mathbb{P}}|\mathbb{P}_0) = \mathsf{H}(\pi|\rho) \leq \mathsf{H}(\mathbb{P}|\mathbb{P}_0)$ for all $\mathbb{P} \in \mathcal{P}_\pi$. $\square$

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Formulation of the problem

Control set: Let U be a given closed subset of $\mathbb{R}^d$, and we denote by $\mathcal{P}_2^U(\mu_0)$ the subset of all measures $\mathbb{P} \in \mathcal{P}_2(\mu_0)$ such that $\beta^{\mathbb{P}} \in U$, Leb $\otimes \mathbb{P}$-a.s.

Cost function: Let $c : [0,1] \times \Omega \times U \times \mathcal{P}(\mathbb{R}^d) \longrightarrow \mathbb{R}$ be an $\mathbb{F}$-progressively measurable map with

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^1 \left|c_s\left(\beta_s^{\mathbb{P}}, m_s\right)\right| ds\right] < \infty, \quad \text{for all } m \in \mathbb{M},\ \mathbb{P} \in \mathcal{P}_2^U(\mu_0).$$

Introduce the subset Ξ^U of all measurable reward functions $\xi \in \Xi$ such that $\mathbb{E}^{\mathbb{P}}\left[\xi^+\right] < \infty$ for all $\mathbb{P} \in \mathcal{P}_2^U(\mu_0)$.

Control problem: For all $m \in \mathbb{M}$ and $\xi \in \Xi^U$, we consider the control problem

$$V_0(\xi,m) := \sup_{\mathbb{P}\in\mathcal{P}_2^U(\mu_0)} J(\xi, m, \mathbb{P}), \quad \text{where} \quad J(\xi, m, \mathbb{P}) := \mathbb{E}^{\mathbb{P}}\left[\xi - \int_0^1 c_s\left(\beta_s^{\mathbb{P}}, m_s\right) ds\right].$$

The notions of MFG and MFG planning are defined as above, up to the substitution of $\mathcal{P}_2$ and Ξ by $\mathcal{P}_2^U$ and Ξ^U.

Characterization of the solutions of the MFG planning problem

Introduce the **Hamiltonian**

$$H_s(z,m) := H_s(\omega,z,m) := \sup_{b\in U}\big\{b\cdot z - c_s(\omega,b,m)\big\}.$$

Assumption: H satisfies the quadratic growth condition: for some $C_1,C_2>0$

$$\operatorname{ess\,inf}\min_{(s,m)\in[0,1]\times\mathcal{P}(\mathbb{R}^d)}\big|\partial_z H_s(z,m)\big| \geq C_1|z| - C_2, \quad \text{for all } z\in\mathbb{R}^d.$$

Consider the controlled McKean-Vlasov SDE (MKVSDE)

$$X_t = X_0 + \int_0^t \widehat{b}_s\big(Z_s,\mathbb{P}\circ X_s^{-1}\big)ds + W_t^{\mathbb{P}}, \quad \mathbb{P}\text{-a.s.},$$

for some measurable selection $\widehat{b}\in\partial_z H$, Leb$\otimes\mathbb{P}$-a.s. with control process $Z\in\mathbb{H}^2(\mathbb{P}_0)$.

Denote

$$\mathrm{MKV}(\mu_0,\mu_1) := \Big\{(Z,\mathbb{P})\in\mathbb{H}^2(\mathbb{P}_0)\times\mathcal{P}_2^U(\mu_0,\mu_1) : \mathbb{P} \text{ solution of MKVSDE}\Big\},$$

$$\Xi(\mu_0,\mu_1) := \mathbb{L}^1(\mathcal{F}_0,\mathbb{P}_0) + \Big\{Y_1^Z : (Z,\mathbb{P})\in\mathrm{MKV}(\mu_0,\mu_1)\Big\},$$

with

$$Y_t^Z := \int_0^t Z_s\cdot dX_s - \int_0^t H_s\big(Z_s,\mathbb{P}\circ X_s^{-1}\big)ds, \quad t\in[0,1].$$

Characterization of the solutions of the MFG planning problem

Theorem (Ren, Tan, Touzi, Y., 2022)

For all pairs of starting and target measures $(\mu_0, \mu_1) \in \mathcal{P}(\mathbb{R}^d) \times \mathcal{P}(\mathbb{R}^d)$*, we have*

$$\Xi(\mu_0, \mu_1) \subseteq \mathrm{MFP}(\mu_0, \mu_1).$$

Moreover, under the quadratic growth condition

$$\Xi(\mu_0, \mu_1) = \mathrm{MFP}(\mu_0, \mu_1).$$

Meaning:

- With $Y_0 \in \mathbb{L}^1(\mathcal{F}_0, \mathbb{P}_0)$ and $(Z, \mathbb{P}) \in \mathrm{MKV}(\mu_0, \mu_1)$,

$$\xi = Y_0 + \int_0^1 Z_s \cdot dX_s - \int_0^1 H_s\big(Z_s, \mathbb{P} \circ X_s^{-1}\big) ds$$

  is a solution of the MFG planning problem. Under technical conditions, each solution of MFG planning can be represented as above.
- It reduces the construction of a solution of the MFG planning problem to the construction of a solution of the McKean-Vlasov SDE with given starting and target marginals.

Characterization of the solutions of the MFG planning problem

Proof: "$\subseteq$": Let $\xi := Y_0 + Y_1^Z \in \Xi(\mu_0, \mu_1)$ with corresponding $(Z, \widehat{\mathbb{P}}) \in \mathrm{MKV}(\mu_0, \mu_1)$, and denote $m_s := \widehat{\mathbb{P}} \circ X_s^{-1}$. We obtain

$$J(\xi,m,\mathbb{P}) = \mathbb{E}^{\mathbb{P}_0}\big[Y_0\big] + \mathbb{E}^{\mathbb{P}}\left[\int_0^1 \Big(Z_s \cdot \beta_s^{\mathbb{P}} - c_s\big(\beta_s^{\mathbb{P}}, m_s\big) - H_s(Z_s,m_s)\Big) ds\right].$$

By the definition of the Hamiltonian H, it follows that

- $J\big(\xi,m,\mathbb{P}\big) \leq \mathbb{E}^{\mathbb{P}_0}\big[Y_0\big]$ for all $\mathbb{P} \in \mathcal{P}_2^U(\mu_0)$,
- $J(\xi,m,\widehat{\mathbb{P}}) = \mathbb{E}^{\mathbb{P}_0}\big[Y_0\big]$ as $\widehat{\mathbb{P}}$ is solution to (MKVSDE),

$$\beta_s^{\widehat{\mathbb{P}}} \in \partial_z H_s(Z_s,m_s) \Longleftrightarrow \beta_s^{\widehat{\mathbb{P}}} \text{ is an optimizer of } H.$$

This implies $V_0(\xi,m) = J(\xi,m,\widehat{\mathbb{P}})$ and therefore $\widehat{\mathbb{P}} \in \mathrm{MFG}(\xi,\mu_0)$ and $\xi \in \mathrm{MFP}(\mu_0, \mu_1)$.

"$=$": For $\xi \in \mathrm{MFP}(\mu_0, \mu_1)$, we have a $\widehat{\mathbb{P}} \in \mathrm{MFG}(\xi, \mu_0)$ such that

- $\widehat{\mathbb{P}} \circ X_1^{-1} = \mu_1$,
- $J(\xi,m,\widehat{\mathbb{P}}) = V(\xi,m)$, for $m_s := \widehat{\mathbb{P}} \circ X_s^{-1}$, $s \in [0,1]$.

Characterization of the solutions of the MFG planning problem

Define

$$V_t := \operatorname*{ess\,sup}_{\mathbb{P}\in\mathcal{P}^U(\mu_0)} \mathbb{E}^{\mathbb{P}}\left[\xi - \int_t^1 c_s\big(\beta_s^{\mathbb{P}}, m_s\big)\,ds \,\middle|\, \mathcal{F}_t\right], \quad t\in[0,1].$$

Then,

- $\mathbb{E}^{\mathbb{P}_0}\big[V_0\big] = V_0(\xi, m) \in \mathbb{R}$, so that $V_0 \in \mathbb{L}^1(\mathcal{F}_0, \mathbb{P}_0)$.
- By the martingale optimal principle, we show the existence of some $Z \in \mathbb{H}^2_{\mathrm{loc}}(\mathbb{P}_0)$ such that

$$V_t = V_0 + \int_0^t Z_s \cdot dX_s - \int_0^t \big(Z_s \cdot \beta_s^{\widehat{\mathbb{P}}} - c_s^{\widehat{\mathbb{P}}}\big)\,ds,$$

  and

$$Z_t \cdot \beta_t^{\widehat{\mathbb{P}}} - c_t^{\widehat{\mathbb{P}}} = \max_{\mathbb{P}\in\mathcal{P}_2^U(\mu_0)} \left\{Z_t \cdot \beta_t^{\mathbb{P}} - c_t^{\mathbb{P}}\right\} = H_t(Z_t, m_t).$$

- Since $\beta^{\mathbb{P}} \in \mathbb{H}^2(\mathbb{P}_0)$ by the definition of $\mathcal{P}_2^U(\mu_0)$, it follows by the quadratic growth condition that $Z \in \mathbb{H}^2(\mathbb{P}_0)$.

This concludes the proof that $(Z, \widehat{\mathbb{P}}) \in \mathrm{MKV}(\mu_0, \mu_1)$, and hence $\xi \in \Xi(\mu_0, \mu_1)$. □

Existence of solution

The theorem above reduces the construction of a solution of the MFG planning problem to the construction of a solution of the McKean-Vlasov SDE with given marginals.

Proposition (Ren, Tan, Touzi, Y., 2022)

Assumptions:

- H satisfies the quadratic growth condition and the full range condition, i.e.,

$$\partial_z H_t(\omega, \mathbb{R}^d, m) = \mathbb{R}^d, \text{ for all } (t, \omega, m) \in [0,1] \times \Omega \times \mathcal{P}(\mathbb{R}^d).$$

- There exists $\pi \in \Pi(\mu_0,\mu_1)$ equivalent to the reference measure $\rho = \mathbb{P}_0 \circ (X_0,X_1)^{-1}$ such that the density $\zeta := \frac{d\pi}{d\rho}(X_0,X_1)$ satisfies $\mathbb{E}^{\mathbb{P}_0}\left[|\ln \zeta| + \zeta^2\right] < \infty$.
- Define the measure $\widehat{\mathbb{P}}$ equivalent to $\mathbb{P}_0$ by $\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}_0} = \zeta$.
- Let Z be any measurable selection of the solutions of

$$\widehat{\beta}_s \in \partial_z H_s(Z_s, m_s), \text{ Leb} \otimes \mathbb{P}_0\text{-a.s., with } \widehat{\beta} \text{ defined by } \zeta = \mathcal{E}\left(\widehat{\beta} \bullet X\right)_1.$$

Then,

- $(Z, \widehat{\mathbb{P}}) \in \mathrm{MKV}(\mu_0, \mu_1)$, and consequently $Y_1^Z \in \mathrm{MFP}(\mu_0, \mu_1)$.
- $\widehat{\mathbb{P}}$ is the unique minimizer of $\mathrm{H}(\cdot|\mathbb{P}_0)$ on

$$\mathcal{P}_\pi := \left\{\mathbb{P} \in \mathcal{P}_2(\mu_0, \mu_1) : \mathbb{P} \circ (X_0, X_1)^{-1} = \pi\right\}.$$

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Formulation of the control problem

Let $\mathcal{P}$ denote the collection of all probability measures $\mathbb{P}$ on the canonical space Ω, under which the canonical process X is a diffusion process with the following decomposition

$$X_t = X_0 + \int_0^t \widehat{b}_s^{\mathbb{P}} ds + \int_0^t \widehat{\sigma}_s dW_s^{\mathbb{P}}, \quad t \in [0,1], \quad \mathbb{P}\text{-a.s.},$$

for some $\mathbb{P}$-Brownian motion $W^{\mathbb{P}}$.

- The quadratic variation process $\langle X \rangle$ can be defined independently of $\mathbb{P} \in \mathcal{P}$, so that $\widehat{\sigma}_t$ can be defined as the unique square root matrix of $\widehat{\sigma}_t^2$, with

$$\widehat{\sigma}_t^2 := \lim_{\varepsilon \searrow 0} \frac{\langle X \rangle_t - \langle X \rangle_{(t-\varepsilon)\vee 0}}{\varepsilon}, \quad t \in [0,1].$$

- Let U be a closed convex subset of $\mathbb{R}^d \times \mathbb{S}_+^d$, with the given two marginal μ_0 and μ_1, introduce

$$\mathcal{P}^U(\mu_0) := \Big\{ \mathbb{P} \in \mathcal{P} : \mathbb{P} \circ X_0^{-1} = \mu_0 \text{ and } \big(\widehat{b}_s^{\mathbb{P}}, \widehat{\sigma}_s^2\big) \in U, \ \text{Leb} \otimes \mathbb{P}\text{-a.e.} \Big\},$$

  and

$$\mathcal{P}^U(\mu_0, \mu_1) := \Big\{ \mathbb{P} \in \mathcal{P}^U(\mu_0) : \mathbb{P} \circ X_1^{-1} = \mu_1 \Big\}.$$

Formulation of the control problem

Introduce the following control problem in weak formulation

$$V_0(\xi, m) := \sup_{\mathbb{P}\in\mathcal{P}^U(\mu_0)} J(\xi, m, \mathbb{P}), \quad \text{with} \quad J(\xi, m, \mathbb{P}) := \mathbb{E}^{\mathbb{P}}\left[\xi - \int_0^1 c_s\big(\widehat{b}_s^{\mathbb{P}}, \widehat{\sigma}_s^2, m_s\big)\,ds\right],$$

where the reward function $\xi : \Omega \longrightarrow \mathbb{R}\cup\{-\infty\}$ is restricted to the set

$$\Xi^U := \Big\{\xi : \Omega \to \mathbb{R} : \mathbb{E}^{\mathbb{P}}\big[\xi^+\big] < \infty, \text{ for all } \mathbb{P}\in\mathcal{P}^U(\mu_0)\Big\}.$$

- For $\xi \in \Xi^U$ and $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$, we denote by

  $$\mathrm{MFG}(\xi, \mu_0) := \Big\{\widehat{\mathbb{P}} \in \mathcal{P}^U(\mu_0) : J(\xi, m, \widehat{\mathbb{P}}) = V_0(\xi, m) \text{ with } m_s = \widehat{\mathbb{P}}\circ X_s^{-1}\Big\}$$

  the set of all solutions to the MFG problem with reward function ξ.
- Given a pair (μ_0, μ_1) of starting and target marginals, we denote by

  $$\mathrm{MFP}(\mu_0, \mu_1) := \Big\{\xi \in \Xi^U : \widehat{\mathbb{P}}\circ X_1^{-1} = \mu_1 \text{ for some } \widehat{\mathbb{P}} \in \mathrm{MFG}(\xi, \mu_0)\Big\}$$

  the collection of all reward functions $\xi \in \Xi^U$ which induce some MFG solution $\widehat{\mathbb{P}}$ with marginals $\widehat{\mathbb{P}}\circ X_0^{-1} = \mu_0$, $\widehat{\mathbb{P}}\circ X_1^{-1} = \mu_1$.

Characterization of the solutions of the mean field planning problem

Define the Hamiltonian and its domain:

$$H_s(\omega, z, \gamma, m) := \sup_{(b,a)\in U} \left\{ b \cdot z + \frac{1}{2} a : \gamma - c_s(\omega, b, a, m) \right\},$$

$$D_H(s,\omega,m) := \left\{ (z, \gamma) \in \mathbb{R}^d \times \mathbb{S}^d : H_s(\omega, z, \gamma, m) < \infty \right\}.$$

Given $\mathbb{F}$-progressively measurable processes (Z, Γ) on Ω taking value in $\mathbb{R}^d \times \mathbb{S}^d$, we introduce the McKean-Vlasov SDE

$$X_t = X_0 + \int_0^t \overline{b}_s\big(Z_s, \Gamma_s, \widehat{\mathbb{P}} \circ X_s^{-1}\big) ds + \int_0^t \overline{\sigma}_s\big(Z_s, \Gamma_s, \widehat{\mathbb{P}} \circ X_s^{-1}\big) dW_s^{\widehat{\mathbb{P}}}, \quad \widehat{\mathbb{P}}\text{-a.s.}$$

for some measurable selection $\left(\overline{b}_s, \frac{1}{2}\overline{\sigma}_s^2\right)(z, \gamma, m) \in \partial_{(z,\gamma)} H_s(z, \gamma, m)$.

Let $\mathrm{MKV}(\mu_0, \mu_1)$ be the collection of all triples $(Z, \Gamma, \widehat{\mathbb{P}})$ such that

- $(Z, \Gamma) \in D_H(\cdot, \widehat{m}_\cdot)$ with $\widehat{m}_s := \widehat{\mathbb{P}} \circ X_s^{-1}$,
- $\widehat{\mathbb{P}} \in \mathcal{P}^U(\mu_0, \mu_1)$ is a (weak) solution of the McKean-Vlasov SDE,
- $Z \in \mathcal{H}^2(\mu_0) := \bigcap_{\mathbb{P}\in\mathcal{P}^U(\mu_0)} \mathbb{H}^2(\mathbb{P})$, where $\mathbb{H}^2(\mathbb{P})$ denotes the collection of all $\mathbb{F}$-prog. measurable processes $Z : [0,1] \times \Omega \to \mathbb{R}^d$ such that $\mathbb{E}^{\mathbb{P}}\big[\int_0^1 |\widehat{\sigma}_s Z_s|^2 ds\big] < \infty$.

Characterization of the solutions of the mean field planning problem

Introduce for all $(Z,\Gamma,\widehat{\mathbb{P}}) \in \mathrm{MKV}(\mu_0,\mu_1)$ the $\mathcal{F}_1$-measurable random variable

$$Y_1^{Z,\Gamma,\widehat{\mathbb{P}}} := \int_0^1 Z_s \cdot dX_s + \int_0^1 \left(\frac{1}{2}\Gamma_s : \widehat{\sigma}_s^2 - H_s\big(Z_s,\Gamma_s,\widehat{\mathbb{P}} \circ X_s^{-1}\big) \right) ds.$$

Define $\mathcal{L}_0^1(\mu_0) := \bigcap_{\mathbb{P}\in\mathcal{P}^U(\mu_0)} \mathbb{L}^1\big(\mathcal{F}_0^+,\mathbb{P}\big)$, and

$$\Xi(\mu_0,\mu_1) := \mathcal{L}_0^1(\mu_0) + \left\{ Y_1^{Z,\Gamma,\widehat{\mathbb{P}}} : (Z,\Gamma,\widehat{\mathbb{P}}) \in \mathrm{MKV}(\mu_0,\mu_1) \right\}.$$

Theorem (Ren, Tan, Touzi, Y., 2022)

- *We have*
$$\Xi(\mu_0,\mu_1) \subseteq \mathrm{MFP}(\mu_0,\mu_1).$$
- *For* $\xi \in \mathrm{MFP}(\mu_0,\mu_1)$, $\widehat{m}_s = \widehat{\mathbb{P}} \circ X_s^{-1}$ *for some* $\widehat{\mathbb{P}} \in \mathrm{MFG}(\xi,\mu_0)$, *under technical conditions, we may find* $Y_0 + Y_1^{Z,\Gamma,\widehat{\mathbb{P}}} \in \Xi(\mu_0,\mu_1)$, *such that*
$$\operatorname*{argmax}_{\mathbb{P}\in\mathcal{P}^U(\mu_0)} J\big(\xi,\widehat{m},\mathbb{P}\big) = \operatorname*{argmax}_{\mathbb{P}\in\mathcal{P}^U(\mu_0)} J\Big(Y_0 + Y_1^{Z,\Gamma,\widehat{\mathbb{P}}},\widehat{m},\mathbb{P}\Big).$$

Existence of solutions to the planning problem

Given μ_0,μ_1, find Z,Γ so that

- There is a solution to the McKean-Vlasov SDE

$$X_t = X_0 + \int_0^t \overline{b}_s\big(Z_s, \Gamma_s, \widehat{\mathbb{P}} \circ X_s^{-1}\big) ds + \int_0^t \overline{\sigma}_s\big(Z_s, \Gamma_s, \widehat{\mathbb{P}} \circ X_s^{-1}\big) dW_s^{\widehat{\mathbb{P}}}, \quad \widehat{\mathbb{P}}\text{-a.s.}$$

  for some measurable selection $\Big(\overline{b}_s, \frac{1}{2}\overline{\sigma}_s^2\Big)(z, \gamma, m) \in \partial_{(z,\gamma)} H_s(z, \gamma, m)$.

- $\widehat{\mathbb{P}} \circ X_0^{-1} = \mu_0$ and $\widehat{\mathbb{P}} \circ X_1^{-1} = \mu_1$.

Then, define

$$\xi := Y_1^{Y_0, Z, \Gamma, \widehat{\mathbb{P}}} := Y_0 + \int_0^1 Z_s \cdot dX_s + \int_0^1 \left(\frac{1}{2} \Gamma_s : d\langle X \rangle_s - H_s\big(Z_s, \Gamma_s, \widehat{\mathbb{P}} \circ X_s^{-1}\big) \right) ds.$$

Optimal transport along controlled McKean-Vlasov dynamic

One can consider an optimal MFG planning problem, by choosing an optimal solution ξ in the class $\Xi(\mu_0, \mu_1)$ w.r.t. some criteria. The problem can be reduced to an **optimal transport problem along controlled McKean-Vlasov dynamic**: for some reward function Ψ, one solves

$$\sup_{(Z,\Gamma,\widehat{\mathbb{P}})\in \mathrm{MKV}(\mu_0,\mu_1)} \Psi\big(Z, \Gamma, \widehat{\mathbb{P}}\big).$$

Recall that $\mathrm{MKV}(\mu_0, \mu_1)$ is the set of all $(Z, \Gamma, \widehat{\mathbb{P}})$ such that, with a version of subgradient $\big(\overline{b}_s, \frac{1}{2}\overline{\sigma}_s^2\big)(z, \gamma, m) \in \partial_{(z,\gamma)} H_s(z, \gamma, m)$, $\widehat{\mathbb{P}}$ is weak solution to the McKean-Vlasov equation:

$$X_t = X_0 + \int_0^t \overline{b}_s\big(Z_s, \Gamma_s, \widehat{\mathbb{P}} \circ X_s^{-1}\big) ds + \int_0^t \overline{\sigma}_s\big(Z_s, \Gamma_s, \widehat{\mathbb{P}} \circ X_s^{-1}\big) dW_s^{\widehat{\mathbb{P}}}, \quad \widehat{\mathbb{P}}\text{-a.s.},$$

under the marginal constraints:

$$\widehat{\mathbb{P}} \circ X_0^{-1} = \mu_0 \quad \text{and} \quad \widehat{\mathbb{P}} \circ X_1^{-1} = \mu_1.$$

This is an extension of the optimal semimartingale transport (Mikami and Thieullen, Tan and Touzi, etc.)

Summary

- **PDE approach**: a coupled system

  Fokker-Planck equation with initial and terminal conditions for m,

  Hamilton-Jacobi-Bellman equation without ternimal condition for u.

  Existence & uniqueness of u, m, especially $u|_{t=1} = g$.
- **Problem**: Given μ_0 and μ_1, we are looking for a terminal reward $\xi = g(X)$, such that there exists a mean-field equilibrium $\widehat{m}$ with $\widehat{m}_0 = \mu_0$ and $\widehat{m}_1 = \mu_1$.
- **Probabilistic approach**: Reduce the construction of a solution of the MFG planning problem to
  - the construction of a solution $(Z, \Gamma, \widehat{\mathbb{P}}) \in \mathrm{MKV}(\mu_0, \mu_1)$ of the McKean-Vlasov SDE with given marginals μ_0,μ_1,
  - the dynamic programming representation $\xi = Y_0 + Y_1^{Z,\Gamma,\widehat{\mathbb{P}}}$.

Merci pour votre attention !

Thank you for your attention!