McKean–Vlasov equation with rough common noise and propagation of chaos

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Equations

MKV with common noise

$$\begin{cases} dY = b_t(Y_t,\mu_t)dt + \sigma_t(Y_t,\mu_t)dB_t + f_t(Y_t,\mu_t)dW_t, \\ \mu_t = \mathrm{Law}(Y_t|W), \quad Y_0 = \xi \end{cases}$$

rough MKV

$$\begin{cases} dY_t(\omega) = b_t(\omega,Y_t(\omega),\mu_t)dt + \sigma_t(\omega,Y_t(\omega),\mu_t)dB_t(\omega) \\ \qquad\qquad + (f_t,f_t')(\omega,Y_t(\omega),\mu_t)d\boldsymbol{X}_t \\ \mu_t = \mathrm{Law}(Y_t), \quad Y_0(\omega) = \xi(\omega). \end{cases}$$

where $\boldsymbol{X} = (X, \mathbb{X})$ is a rough path modeling $(W, \int \delta W dW)$.

Particle approximation

$$\begin{cases} dY^{N,i}_t(\omega) = b_t(\omega,Y^{N,i}_t(\omega),\mu^N_t(\omega))dt + \sigma_t(\omega,Y^{N,i}_t(\omega),\mu^N_t(\omega))dB^i_t(\omega) \\ \qquad\qquad + (f_t,f_t')(\omega,Y^{N,i}_t(\omega),\mu^N_t(\omega))d\boldsymbol{X}_t, \\ \mu^N_t(\omega) = \frac{1}{N}\sum_{j=1}^N \delta_{Y^{N,j}_t(\omega)}, \quad Y^{N,i}_0(\omega) = \xi^i(\omega), \end{cases}$$

Previous works

- MKV with common Brownian noise: Coghi–Flandoli'16 ($\sigma = 0$, $f = f(y)$), Carmona–Delarue'2 volumes
- Rough MKV: Cass–Lyons'15, Bailleul'15, Bailleul–Catellier–Delarue'20'21, Delarue–Salkeld'21+

Current work

- adopt a different method from Bailleul–Catellier–Delarue'20'21
- allow progressive measurable coefficients: *required* for controlled MKV dynamics, mean field games with common noise
- allow sharp regularity conditions

Rough MKV – Assumption

$$\begin{cases} dY_t(\omega) = b_t(\omega, Y_t(\omega), \mu_t)dt + \sigma_t(\omega, Y_t(\omega), \mu_t)dB_t(\omega) \\ \qquad\qquad + (f_t, f_t')(\omega, Y_t(\omega), \mu_t)d\boldsymbol{X}_t \\ \mu_t = \mathrm{Law}(Y_t), \quad Y_0(\omega) = \xi(\omega). \end{cases}$$

- $\xi \in \mathcal{L}_q(\mathcal{F}_0)$, $\boldsymbol{X} = (X, \mathbb{X})$: α-Hölder rough path, $\alpha \in (1/3, 1/2)$.
- b, σ, f, f' are defined on $\Omega \times \mathbb{R}_+ \times W \times \mathcal{P}_q(W)$, for some fixed $q \geq 0$
- b, σ are $C^1_{b,m,q}$ w.r.t. (y, μ) for some fixed $m \geq \max(2, q)$
- (f, f') is a measure-dependent stochastic controlled vector field in $\mathbf{D}^{2\alpha}_X L_{m,\infty} C^\gamma_{b,m,q}$:
  - f is $C^\gamma_{b,m,q}$, f' is $C^{\gamma-1}_{b,m,q}$ w.r.t. (y, μ) for some $\gamma > 1/\alpha$. Both are L_m-integrable w.r.t. ω in some precise sense
  - "$f' = df/dX$": [1]

$$\mathbb{E}_s R^f_{s,t}(\omega, y, \mu) = O(t-s)^{2\alpha}, \quad R^f_{s,t} = \delta f_{s,t} - f'_s \delta X_{s,t}$$

[1] Simple examples without measure dependence: (a) $f_t(y) = h(B_t, y)$, $f'_t(y) = 0$; (b) $f_t(y) = h(X_t, y)$, $f'_t(y) = D_1 h(X_t, y)$

Rough MKV – Results

[Friz–Hocquet–L. 22+]

Under the above conditions, the *roughMKV* has a unique solution which is Lipschitz continuous w.r.t. the input data $\xi, b, \sigma, f, f', \boldsymbol{X}$.
Likewise, the particle system has a unique solution.

Propagation of chaos – Assumption

Assume additionally that for some $p \in [\max(q,1), m]$ such that

- $\|\xi\|_p < \infty$
- for $g \in \{b, \sigma, f, D_1 f, D_1^2 f, f', D_1 f'\}$, there is a modulus of continuity λ_g such that

$$\sup_{t \le T} \| \sup_{y \in W} |g_t(y, \nu) - g_t(y, \bar{\nu})| \|_\infty \le \lambda_g(\mathcal{W}_p(\nu, \bar{\nu})), \quad \forall \nu, \bar{\nu} \in \mathcal{P}_p(W).$$

- For every $\xi_1, \xi_2 \in L_q(W)$ and every $\eta \in L_\infty(W)$, we have

$$\sup_{t \le T} \| \sup_{y \in W} |(D_2 \hat{f}_t(y, \xi_1)[\eta] - D_2 \hat{f}_t(y, \xi_2)[\eta]| \|_\infty \lesssim \|\xi_1 - \xi_2\|_p \|\eta\|_\infty.$$

Propagation of chaos

FHL22+

The N-particle system converges to *roughMKV* in the following senses:

(i) $\lim_{N\to\infty} \sup_{i\le N} \| \sup_{t\in[0,T]} |Y_t^{N,i} - Y_t^i| \|_m = 0$.

(ii) For every $\beta \in (0,\alpha)$,

$$\lim_{N\to\infty} \sup_{i\le N} \left(\|Y^{N,i} - Y^i\|_{C^\alpha L_m} + \|\mathbb{E}.R^{Y^{N,i}} - \mathbb{E}.R^{Y^i}\|_{C_2^{\alpha+\beta} L_m} \right) = 0.$$

(iii) With probability one, for any fixed integer $k \ge 1$, we have the convergence

$$\lim_{N\to\infty} (Y^{N,1}, \cdots, Y^{N,k}) = (Y^1, \cdots, Y^k) \quad \text{in} \quad [C([0,T];W)]^k.$$

(iv) $\lim_{N\to\infty} \|\mathcal{W}_{C([0,T],W),p}(\mu^N, \mu)\|_p = 0$ and
$\lim_{N\to\infty} \|\mathcal{W}_{C([0,T],W),m}(\mathring{\mu}^N, \mathring{\mu})\|_m = 0$.

where $\mu^N = \frac{1}{N}\sum_{i=1}^N \delta_{Y^{N,i}}$ and $\mathring{\mu}^N = \frac{1}{N}\sum_{i=1}^N \delta_{Y^{N,i} - Y_0^{N,i}}$.

Regularity w.r.t. random variables

Previously, we say "f is $C^{\gamma}_{b,m,q}$, f' is $C^{\gamma-1}_{b,m,q}$ w.r.t. (y,μ)"

Lions lift

$\mathcal{L}_q(W)$: topological space of W-valued random variables ξ such that $\mathbb{E}|\xi|^q < \infty$.
When $q \geq 1$, $\mathcal{L}_q$ becomes the Banach space L_q.
Given $g : \mathcal{P}_q(W) \to \bar{W}$, *Lions lift* of g is the function

$$\hat{g} : \mathcal{L}_q(W) \to \bar{W}, \quad \hat{g}(\xi) = g(\mathrm{Law}(\xi))$$

Advantage of $\mathcal{L}_q$ over $\mathcal{P}_q$: the former is a vector space.

Differentiability w.r.t. random variables

Definition

Take $m \geq \max(q,1)$. The space $C^\gamma_m(\mathcal{L}_q(W),\bar{W})$ is the topological subspace of the Lipschitz space $C^\gamma_b(L_m(W),\bar{W})$ containing functions g in $C^\gamma_b(L_m(W),\bar{W})$ such that for each integer $0 \leq k < \gamma$ and for every $\eta_1,\dots,\eta_k \in L_m(W)$, the map

$$L_m(W) \ni \xi \mapsto D^k g(\xi)[\eta_1,\dots,\eta_k] \in \bar{W}$$

is uniformly bounded on $L_m(W)$ and can be extended continuously to $\mathcal{L}_q(W)$ with respect to the topology of convergence in probability.

That is

$$\sup_{\xi \in L_m(W)} D^k g(\xi)[\eta_1,\dots,\eta_k] < \infty \tag{4.1}$$

and for every $\bar{\xi} \in \mathcal{L}_q(W)$, $D^k g(\bar{\xi})[\eta_1,\dots,\eta_k]$ exists as the unique limit

$$D^k g(\bar{\xi})[\eta_1,\dots,\eta_k] := \lim_{\xi \to \bar{\xi}} D^k g(\xi)[\eta_1,\dots,\eta_k] \tag{4.2}$$

where $\xi \in L_m(W)$ and $\xi \to \bar{\xi}$ in probability.

Basic properties of $C^\gamma_{m,q}$-regularity

- $C^\gamma_m(\mathcal{L}_q(W),\bar W)$ is a closed topological subspace of $C^\gamma_b(L_m(W),\bar W)$.
- $C^\gamma_m(\mathcal{L}_m(W),\bar W) = C^\gamma_b(L_m(W),\bar W)$
- (trivial) continuous inclusions

$$C^\gamma_m(\mathcal{L}_q(W),\bar W) \hookrightarrow C^\gamma_b(L_m(W),\bar W), \quad m \geq \max(q,1), \tag{4.3}$$

$$C^\gamma_b(L_q(W),\bar W) \hookrightarrow C^\gamma_m(\mathcal{L}_q(W),\bar W), \quad m \geq q \geq 1. \tag{4.4}$$

Example

Let $h:\mathbb{R}\to\mathbb{R}$ be a bounded smooth function with bounded derivatives and define the function $g: L_2(\mathbb{R})\to\mathbb{R}$ by $g(\xi)=\mathbb{E}h(\xi)$. Then

- g belongs to $C^3_3(\mathcal{L}_2(\mathbb{R}))$.
- On the other hand, it is known from Carmona–Delarue's book: there exists a compactly supported smooth function h such that g does not belong to $C^\gamma_b(L_2(\mathbb{R}))$ for any $\gamma>2$.

Being in $C^{\gamma}_{m,q}$

If g has $C^{\gamma}_{m,q}$-regularity then

- For each integer $0 \le k < \gamma$ and $\xi \in \mathcal{L}_q$, the Fréchet derivative $D^k g(\xi)$ exists as a bounded multilinear map on $[L_m]^k$

$$\sup_{\xi \in \mathcal{L}_q(W)} |D^k g(\xi)[\eta_1, \dots, \eta_k]| \le C_k \|\eta_1\|_m \dots \|\eta_k\|_m$$

- $D^{\lfloor \gamma \rfloor} g$ is Hölder continuous:

$$\begin{aligned}|D^{\lfloor \gamma \rfloor} g(\xi)[\eta_1, \dots, \eta_k] - D^{\lfloor \gamma \rfloor} g(\bar{\xi})[\eta_1, \dots, \eta_k]| \\ \le \bar{C}_\gamma \|\xi - \bar{\xi}\|_m^{\gamma - \lfloor \gamma \rfloor} \|\eta_1\|_m \dots \|\eta_{\lfloor \gamma \rfloor}\|_m.\end{aligned}$$

Law invariance of Lions lifts

Let $g : \mathcal{P}_q(W) \to \bar{W}$.

- say g is $C^{\gamma}_{m,q}$ iff $\hat{g}$ is.
- $\hat{g}, D\hat{g}, \ldots$ are law invariance:
  - $D\hat{g}(\xi)[\eta] = D\hat{g}(\bar{\xi})[\bar{\eta}]$ for every $(\bar{\xi}, \bar{\eta}) \in \mathcal{L}_q(W) \times L_m(W)$ satisfying $\mathrm{Law}(\xi, \eta) = \mathrm{Law}(\bar{\xi}, \bar{\eta})$;
  - $D\hat{g}(\xi)[\eta] = D\hat{g}(\xi)[\mathbb{E}(\eta|\mathcal{F})]$ if $\xi \in \mathcal{F}$.

Method

General strategy in a nutshell

For (classical) MKV (Sznitman'91 or Lacker's lecture note)

1. view the dependence on the marginal law as temporal dependence
2. apply stability of SDE (Itô theory)
3. this yields uniqueness of MKV as well as propagation of chaos (with help of LLN)

Solving roughMKV (similar to the one above)

1. view the dependence on the marginal law as temporal dependence
2. apply stability results of *rough SDEs* (Friz–Hocquet–L.'21)
3. yield uniqueness of *roughMKV* as well as propagation of chaos

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Solving roughMKV (similar to the one above)

1. view the dependence on the marginal law as temporal dependence
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Sounds simple! Why didn't it work before?

- lack of direct approach to hybrid rough SDE

A very brief overview of rough SDEs

$$dY_t(\omega) = b_t(\omega, Y_t(\omega))dt + \sigma_t(\omega, Y_t(\omega))dB_t(\omega) + (f_t, f_t')(\omega, Y_t(\omega))d\mathbf{X}_t.$$

A very brief overview of rough SDEs

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Previous methods use rough path theory and flow transformation to deduce to a rough DE or an Itô DE:

- rules out progressive measurable coefficients
- high cost of regularity assumption

In Friz–Hocquet–L.'21, by building around stochastic sewing lemma (L.'18), we show that

- *roughSDE* has unique solution
- solution has stochastic control structure:

$$\mathbb{E}_s R^Y_{s,t} = O(t-s)^{2\alpha}, \quad R^Y_{s,t} = \delta Y_{s,t} - f_s(Y_s)\delta X_{s,t}.$$

- solution is Lipschitz continuous w.r.t. to the input data $\xi, b, \sigma, f, f', \boldsymbol{X}$.

Solving roughMKV: A bit more detail

For each stochastic controlled rough path (η, η'), define

$$f_t^\eta(\omega, y) = \hat{f}_t(\omega, y, \eta_t) \quad \text{and} \quad (f_t^\eta)'(\omega, y) = D_2\hat{f}_t(\omega, y, \eta_t)[\eta_t'] + \hat{f}_t'(\omega, y, \eta_t).$$

roughMKV then can be written in the following way

$$\begin{cases} dY_t(\omega) = b_t^\eta(\omega, Y_t(\omega))dt + \sigma_t^\eta(\omega, Y_t(\omega))dB_t(\omega) + (f_t^\eta, (f_t^\eta)')(\omega, Y_t(\omega))d\boldsymbol{X}_t \\ (\eta(\omega), \eta'(\omega)) = (Y(\omega), \hat{f}(\omega, Y(\omega), Y)), \quad Y_0(\omega) = \xi(\omega). \end{cases} \tag{5.1}$$

Given (η, η') Eq. (5.1) is a rough SDE which has unique solution Y^η So that solving (5.1) is the same as finding fixed point of $\eta \mapsto Y^\eta$.

Particle system as roughSDE

Let g be a function on $\Omega \times \mathbb{R}_+ \times W \times \mathcal{P}_q$. *Finite projection* of g is the function $P^N g$ on $\Omega \times \mathbb{R}_+ \times W \times W^N$

$$P^N g_t(\omega, y, z) = \hat{g}_t(\omega, y, z^\vartheta), \quad \mathrm{Law}(\vartheta) = \mathrm{Unif}(1, \ldots, d)$$

- if g has $C^\gamma_{b,m,q}$-regularity then $P^N g$ has C^γ_b-regularity on $W \times W^N$
- if W^N is equipped with metric

$$d^N_m(z) := \|z^\vartheta\|_m = \left(\frac{1}{N}\sum_{i \le N} |z^i|^m\right)^{\frac{1}{m}}, \quad z = (z^1, \ldots, z^N) \in W^N.$$

then "$|P^N g|_{C^\gamma_b}$" is bounded uniformly in N.

Propagation of chaos

Particle system becomes a (big) *roughSDE*

$$\begin{cases} dY^{N,i} = P^N b_t(Y_t^{N,i}, Y_t^{[N]})dt + P^N \sigma_t(Y_t^{N,i}, Y_t^{[N]})dB_t^i \\ \qquad\qquad + (P^N f_t, (P^N f_t'))(Y_t^{N,i}, Y_t^{[N]})d\boldsymbol{X}, \\ Y_0^{N,i} = \xi^i, \quad i = 1, \dots, N, \\ Y^{[N]} = (Y^{N,i})_{i=1}^N. \end{cases} \tag{5.2}$$

We then compare this with i.i.d. *roughMKV*

$$\begin{cases} dY_t^i = b_t(Y_t^i, \mu_t)dt + \sigma_t(Y_t^i, \mu_t)dB_t^i + (f_t, f_t')(Y_t^i, \mu_t)d\boldsymbol{X}_t , \\ \mu_t = \mathrm{Law}(Y_t^i), \quad Y_0^i = \xi^i. \end{cases} \tag{5.3}$$

By treating the dependence on $Y_t^{[N]}$ and μ_t as temporal dependence, we are able to apply stability results of rough SDEs.
Then by LLN and a Gronwall-argument, this yields convergence of the particle system to the i.i.d. roughMKV.

Propagation of chaos

Particle system becomes a (big) *roughSDE*

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Thank you!

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