Coupled FBSDEs with measurable coefficients and its application to parabolic PDEs[1] – a simple approach

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[1] Joint work with Yunxi Xu

FBSDE formulation
Does a solution exists for the following equation when b, σ, g, f, h have discontinuities in x?

$$\begin{aligned} dX_t &= \tilde{g}(t, X_t, Y_t, Z_t)dt + \sigma(t, X_t)dB_t; \quad && X_0 = x \\ dY_t &= -f(t, X_t, Y_t, Z_t)dt + Z_t dB_t; && Y_T = h(X_T) \end{aligned}$$

PDE formulation
Does a solution exists for the following PDE?

$$\begin{aligned} \mathcal{L}u + (\nabla u \tilde{g})(t, x, u, \nabla u \sigma) + f(t, x, u, \nabla u \sigma) &= 0; \\ u(T, x) &= h(x) \\ \mathcal{L}u = \partial_t + \frac{1}{2}\sum_{i,j} a_{ij}(t, x)\partial_{x_i x_j} u & \end{aligned}$$

where $a = \sigma\sigma^\intercal$.

- $\tilde{g}(x, y, z)$ does not depend on (y, z) and all coefficients are smooth with bounded derivatives:
  Pardoux and Peng (1990) and Pardoux and Peng (1992)
- $\tilde{g}(x, y, z)$ does not depend on (y, z) and $f(t, x, y, z)$ is Lipschitz in (y, z):
  El Karoui et al. (1997)
- $\tilde{g}(x, y, z)$ does not depend on (y, z) and $f(t, x, y, z)$ is continuous in (y, z):
  Hamadène et al. (1997)
- All coefficients are Lipschitz with respect to (x, y, z) and $\sigma\sigma^\intercal$ is uniformly non-degenerate:
  Delarue (2002)
- σ is a constant, h bounded, and $(f, \tilde{g})$ has linear growth in (y, z):
  Luo et al. (2022)
- PDE literature when $f(t, x, y, z), g(t, x, y, z)$ are linear with respect to y, z:
  Kim and Krylov (2007) and so on.

$$dX_t = (b(t,X_t) + \sigma(t,X_t)g(t,X_t,Y_t,Z_t))\,dt + \sigma(t,X_t)dB_t; \quad X_0 = x$$
$$dY_t = -f(t,X_t,Y_t,Z_t)dt + Z_t dB_t; \quad Y_T = h(X_T$$

- $\sigma\sigma^\intercal$ is uniformly non-degenerate, that is, there exists a constant $\varepsilon > 0$ such that

$$\varepsilon^{-1}|x'|^2 \leq (x')^\intercal(\sigma\sigma^\intercal)(t,x)x' \leq \varepsilon|x'|^2$$

  for all $x' \in \mathbb{R}^m$ and $(t,x) \in [0,T] \times \mathbb{R}^m$.
- There exists a positive constant κ such that,

$$|b(t,0)| + \sup_{|x-x'|\leq 1} |b(t,x) - b(t,x')| \leq \kappa$$

  for all $t \in [0,T], x, x' \in \mathbb{R}^m$.

We first solve the following decoupled FBSDE

$$
\begin{aligned}
dX_t &= b(t, X_t)dt + \sigma(t, X_t)dW_t; & X_0 &= x \\
dY_t &= -\left(f(t, X_t, Y_t, Z_t) + Z_t g(t, X_t, Y_t, Z_t)\right)dt + Z_t dW_t; & Y_T &= h(X_T)
\end{aligned}
$$

then use Girsanov transform $dB_t = dW_t - g(t, X_t, Y_t, Z_t)dt$ to obtain the original FBSDE.

We need to check...

- When does the decoupled FBSDE has a solution (when the coefficients are not continuous in x)?
- When can we perform the Girsanov transform?
- $(X, Y, Z) \in \mathbb{F}^B$?

- Meta Theorem A: If the forward SDE is well-posed, then its solution is a strong Markov process.
- Meta Theorem B: Assuming the BSDE is well-posed, if the "x" component of the BSDE is a strong Markov process, then there are measurable functions u, d such that the solution of BSDE is $(u(t, X_t), d(t, X_t))$: see Çinlar et al. (1980), El Karoui et al. (1997), and Hamadène et al. (1997).

Remark We don't need the continuity of $b(t,x), \sigma(t,x), f(t,x,y,z), g(t,x,y,z)$ with respect to x to guarantee the well-posedness of the decoupled system.

Lemma

Assume the decoupled FBSDE is well-posed with solution $(X, Y = u(\cdot, X.), Z = d(\cdot, X.))$ *and the Girsanov theorem works. If*

$$dP_t = (b(t,P_t) + \sigma(t,P_t)g(t,P_t,u(t,P_t),d(t,P_t)))\,dt + \sigma(t,P_t)dB_t$$

satisfies pathwise uniqueness, then $(X,Y,Z) \in \mathbb{F}^B$.

The coupled FBSDE has a strong solution if there are nonnegative constants C and r, a strictly increasing function $\theta : \mathbb{R} \to \mathbb{R}_+$, and a nondecreasing function $\rho_r : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\rho_r \equiv C$ for $r > 0$ such that either Condition (A) or Condition (B) hold.

Condition (A)

- b and σ satisfies one of the followings:
  - (F1) $|b(t,x)| \leq C$ and $\sigma(t,x)$ is locally Lipschitz with respect to x.
  - (F2) $|b(t,x)| \leq C$, $m = n = 1$, and either
    - (i) $\int \frac{du}{\theta(u)} = \infty$ and $|\sigma(t,x) - \sigma(t,y)|^2 \leq \theta(|x-y|)$, or
    - (ii) θ is bounded and $|\sigma(t,x) - \sigma(t,y)|^2 \leq |\theta(x) - \theta(y)|$.
  - (F3) $\sigma(t,x)$ is a constant matrix.

- f, g, h and $\bar{f} = f + zg$ satisfy one of the followings:
  - (B1) $|h(x)| \leq C(1 + |x|^r)$, $\bar{f}(t, x, y, z)$ is continuous in (y, z), and
  $$|f(t, x, y, z)| \leq C(1 + |x|^r + |y| + |z|)$$
  $$|g(t, x, y, z)| \leq \rho_r(|y|).$$
  - (B2) $|h(x)| \leq C$, $\bar{f}^i(t, x, y, z) = \tilde{f}^i(t, x, z^i) + \hat{f}^i(t, x, y, z)$ such that
  $$|\hat{f}(t, x, y, z)| \leq C(1 + |y|), |\tilde{f}(t, x, z)| \leq C|z|^2, |g(t, x, y, z)| \leq \rho_r(|y|)$$
  $$\left|\hat{f}(s, x, y, z) - \hat{f}(s, x, y', z')\right| \leq C(|y - y'| + |z - z'|)$$
  $$\left|\tilde{f}(s, x, z_1) - \tilde{f}(s, x, z_2)\right| \leq C(1 + |z_1| + |z_2|)|z_1 - z_2|.$$
  - (B3) $d = 1$, $|h(x)| \leq C$, $\bar{f}(t, x, y, z)$ is continuous with respect to (y, z), and
  $$|f(t, x, y, z)| \leq C(1 + |y| + |z|^2)$$
  $$|g(t, x, y, z)| \leq C(1 + \rho_r(|y|)).$$

Condition B

- (F3)
- $|h(x)| \leq C(1+|x|^r)$, $\bar{f}(t,x,y,z)$ is continuous in (y,z), and

$$|f^i(t,x,y,z)| \leq C(1+|x|^r+|y^i|) \text{ for all } i=1,2,...,d$$
$$|g(t,x,y,z)| \leq C(1+|x|+\rho_r(|y|)).$$

Remark While we require the continuity of $f+zg$ in (y,z), the continuity of (f,g) with respect to (y,z) is nonnecessary.
Remark (F3)+(B1) "generalizes" Hamadène et al. (1997) and Luo et al. (2022).

For each set of conditions, verify/prove

- Wellposedness of decoupled FBSDE
- Meta Theorem A and Meta Theorem B
- Wellposedness of SDE $F_0 = x$ and

$$dF_t = (b(t, F_t) + \sigma(t, F_t)g(t, F_t, u(t, F_t), d(t, F_t)))\, dt + \sigma(t, F_t)dB_t$$

- Girsanov transforms to couple/decouple FBSDE.

These verifications were done by weaving numerous results on SDE and FBSDE including Aronson (1967); Le Gall (1984); Hamadène et al. (1997); Kobylanski (2000); Gyöngy and Martínez (2001); Mu and Wu (2015); Hu and Tang (2016); Menoukeu-Pamen and Mohammed (2019); Menozzi et al. (2021)

Moreover, the solution is unique if

$$dY_t = -f(t, I_t, Y_t, Z_t)dt + Z_t dW_t; Y_T = h(I_T)$$

has a unique strong solution for any Itô process I. For example,

(U1) $f(t, x, y, z)$ is globally Lipschitz continuous with respect to (y, z) and $|f(t, x, 0, 0)|$ is uniformly bounded.

(U2) $d = 1$, $|h(x)| \leq C$, $f(t, x, y, z)$ is differentiable with respect to (y, z), and for any $M, \varepsilon > 0$, there exist $l_M, l_\varepsilon \in L^1([0, T]; \mathbb{R}_+), k_M \in L^2([0, T]; \mathbb{R}_+)$, and $C_M > 0$ such that f satisfies

$$|f(t, x, y, z)| \leq l_M(t) + C_M|z|^2$$
$$|\partial_z f(t, x, y, z)| \leq k_M(t) + C_M|z|$$
$$|\partial_y f(t, x, y, z)| \leq l_\varepsilon(t) + \varepsilon|z|^2$$

for all $(t, x, y, z) \in [0, T] \times \mathbb{R}^m \times [-M, M] \times \mathbb{R}^{1\times n}$.

(U3) (B2) holds for $f(t, x, y, z)$ instead of $\bar{f}(t, x, y, z)$

If we have two strong solution for coupled FBSDE, they become two weak solutions for decoupled FBSDE via Girsanov transform. Since the decoupled FBSDE has pathwise uniqueness unde our conditions, these two weak solutions should coincide pathwise.

Assume that there exist $C > 0, r \geq \frac{1}{2}$ and $\varepsilon > 0$ satisfying the following conditions:

- $\sigma : [0,T] \times \mathbb{R} \to \mathbb{R}^{1\times n}$ is a continuous function satisfying

$$|\sigma(t,x) - \sigma(t,y)| \leq C\sqrt{x-y}, \qquad \varepsilon \leq |\sigma(t,x)| \leq C$$

- $b : [0,T] \times \mathbb{R} \to \mathbb{R}, h : \mathbb{R} \to \mathbb{R}^d$ are measurable and

$$|b(t,x)| \leq C, |h(x)| \leq C(1 + |x|^r).$$

- $f : [0,T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d\times n} \to \mathbb{R}^d$ and $g : [0,T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d\times n} \to \mathbb{R}^n$ are measurable functions such that, for $\bar{f}(t,x,y,z) = f(t,x,y,z) + zg(t,x,y,z)$, we have
    - $\bar{f}(t,x,\cdot,\cdot)$ is continuous for each $(t,x) \in [0,T] \times \mathbb{R}$.
    - $|f(t,x,y,z)| \leq C(1 + |x|^r + |y| + |z|)$ and $|g(t,x,y,z)| \leq C$.

Then, there exists a unique strong solution.

Proof.

Zvonkin (1974) and Hamadène et al. (1997).

The u satisfying $Y_t = u(t, X_t)$ is a "weak" solution of

$$\mathcal{L}u + (\nabla u \tilde{g})(t, x, u, \nabla u \sigma) + f(t, x, u, \nabla u \sigma) = 0;$$
$$\mathcal{L}u = \partial_t + \frac{1}{2}\sum_{i,j} a_{ij}(t, x)\partial_{x_i x_j} u$$

where $\tilde{g} = b + \sigma g$.

The function u is required to be in the domain of $\mathcal{L}$ so that $u(t, X_t)$ becomes an Itô process. This regularity is strictly weaker than the Sobolev differentiability. The necessary and sufficient condition is given in Mania and Tevzadae (2001).

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