Robust maximization of recursive utility with penalization on the model

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ANNECY-May 27th, 2022

$\hookrightarrow$ Joint work with P. Beissner and C. Zhou.

Starting point

In Finance (see El-Karoui, Peng, Quenez 2001) consider the wealth process

$$\begin{cases} X_s^{t,x,u} = b(s, X_s^{t,x,u}, u_s)ds + \Sigma(s, X_s^{t,x,u}, u_s)dB_s, \quad s \in [t, T] \\ X_t^{t,x,u} = x \end{cases}$$

and the recursive utility process

$$\begin{cases} Y_s^{t,x,u} = \int_s^T f(r, X_r^{t,x,u}, Y_r^{t,x,u}, Z_r^{t,x,u}, u_r)dr + \int_s^T Z_r^{t,x,u}\, dB_r \\ \qquad\quad = \mathbb{E}[\int_s^T f(r, X_r^{t,x,u}, Y_r^{t,x,u}, Z_r^{t,x,u}, u_r)dr|\mathcal{F}_s] \end{cases}$$

where u $(= (c, \pi))$ is a control.
A classical optimization problem is to evaluate the value function

$$\tilde{V}(t, x) = \sup_u Y_t^{t,x,u}$$

and prove a DPP and that it satisfies an HJBI equation.
(See for example Buckdahn-Li 2008).

Our aim

- Consider the case where the volatility (of B) is unknown
- Introduce (in f) a penalization w.r.t. the model

References

- D., Kervarec: maximization of utility of the terminal wealth using duality argument.
- Matoussi, Possamaï, Zhou 2015: approach using 2BSDE.
- Hu, Ji 2017: case of the G-Brownian motion.
- Guo, Langrené, Loeper, Ning (2020): By approximation but not so rigorous.

The setting

- $\Omega = \{\omega \in C([0,T];\mathbb{R}) : \omega_0 = 0\}$ to be the canonical space of continuous sample paths starting at zero,
- B the canonical process, i.e. $B_t(\omega) = \omega_t$ for all $\omega \in \Omega$
- $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ the "standard" filtration
- $\mathbb{P}_0$ the Wiener measure on Ω under which B is a Brownian motion.

Sets of Priors

We denote by $\mathcal{O}$ the set of $\mathbb{F}$-progressively measurable processes $\alpha : \Omega \times [0, T] \to \Theta$ such that:

$$\int_0^T |\alpha_t| \, \mathrm{d}t < +\infty \quad \mathbb{P}_0\text{-a.s.},$$

where $\Theta \subset \mathbb{S}_d^{>0}$ is a convex compact set of $\mathbb{S}_d^{>0}$.
For each process $\alpha \in \mathcal{O}$ we define the process (X_t^α) by

$$X_t^\alpha = \int_0^t \alpha_s^{1/2} \mathrm{d}B_s, \tag{1}$$

and denotes $\mathbb{P}^\alpha = \mathbb{P}_0 \circ (X^\alpha)^{-1} \in \mathbb{M}_1$ the law of X^α.
We consider the set of probability measures

$$\mathcal{P} = \{\mathbb{P}^\alpha ; \, \alpha \in \mathcal{O}\}.$$

The set $\mathcal{P}$ contains mutually singular probability measures and may therefore live on disjoint domains.
As usual, we say that a property hods $\mathcal{P}$-quasi-surely if it holds $\mathbb{P}$-a.s. for all $\mathbb{P}$ in $\mathcal{P}$.

Pathwise definition of the quadratic variation

By karandikar (1995), there is some $\mathbb{F}$-progressively measurable non-decreasing process on Ω denoted by $\langle B \rangle = (\langle B \rangle_t)_{0 \le t \le T}$ which coincides with the quadratic variation of B under each semi-martingale measure $\mathbb{P}$. In particular, this provides a pathwise definition of $\langle B \rangle$ and its density a_t,

$$\langle B \rangle_t := B_t B_t' - 2 \int_0^t B_s dB_s' \quad \text{and} \quad \mathrm{a}_t := \overline{\lim_{\varepsilon \downarrow 0}} \frac{1}{\varepsilon}(\langle B \rangle_t - \langle B \rangle_{t-\varepsilon}).$$

where B' denotes the transposed of B, and the $\overline{\lim}$ is component-wise defined.

Key properties of elements in $\mathcal{P}$

Following Soner, Touzi, Zhang (2011) we know that each element $\mathbb{P} \in \mathcal{P}$ satisfies

1. The Martingale Representation Theorem
2. The Blumenthal zero-one law.

w.r.t. the filtration $(\mathcal{F}_t^{+,\mathbb{P}})_t$.

Conditioning

We now introduce, for $(t, w) \in [0, T] \times \Omega$, the set $\mathcal{P}(t, w)$ of all probabilities $\mathbb{P}$ such that:

- $\mathbb{P}$-a.s. $B_s = B_s(w)\ \forall s \in [0, t]$,
- there exists a process (α^t) adapted to the filtration $(\mathcal{F}^t_{s^+})_{s\in[t,T]}$ and taking values in Θ such that $(B_s - B_t)_{s\in[t,T]}$ has the same law as the process $(\int_t^s (\alpha^t_s)^{1/2} dB_s)_{s\in[t,T]}$ under $\mathbb{P}_0$.

An important property

In order to apply the results of Possamaï, Tan, Zhou (2018) we need to establish the following

Proposition

The graph of $\mathcal{P}$:

$$[[\mathcal{P}]] = \{(t, w, \mathbb{P});\ (t, w, \mathbb{P}) \in [0, T] \times \Omega \times \mathcal{P}(t, w)\}$$

is analytic in $[0, T] \times \Omega \times \mathbb{M}_1$.

Functional spaces

- $\mathbb{H}^2(t, T)$ denotes the set of all $\mathbb{F}_+^{\mathcal{P}}$-progressive measurable $\mathbb{R}^d$-valued processes $\mathcal{Z}$, which are defined ds-a.e. on $[t, T]$ with

$$\|\mathcal{Z}\|^2_{\mathbb{H}(t,T)} := \sup_{\mathbb{P}\in\mathcal{P}} E^{\mathbb{P}}\left[\int_t^T \left\|\mathrm{a}_s^{1/2}\mathcal{Z}_s\right\|^2 \mathrm{d}s\right] < \infty.$$

- $\mathbb{D}^2(t, T)$ denotes the set of all $\mathbb{F}_+^{\mathcal{P}}$-progressive measurable $\mathbb{R}$-valued processes u, which are defined ds-a.e. on $[t, T]$ with

$$\|u\|^2_{\mathbb{D}(t,T)} := \sup_{\mathbb{P}\in\mathcal{P}} E^{\mathbb{P}}\left[\int_t^T \left\|u_s\right\|^2 \mathrm{d}s\right] < \infty.$$

- $\mathbb{I}^2$ denotes the set of all $\mathbb{F}_+^{\mathcal{P}_0}$-progressive measurable processes $\mathcal{K}$ with $\mathcal{P}$-q.s. right-continuous, left-limited and non-decreasing paths on $[0, T]$ with $K_0 = 0$, $\mathcal{P}$-q.s. and

$$\|\mathcal{K}\|^2_{\mathbb{I}} := \sup_{\mathbb{P}\in\mathcal{P}} E^{\mathbb{P}}[|\mathcal{K}_T|^2] < +\infty.$$

- $\mathbb{S}^2(t, T)$ denotes the set of all $\mathbb{F}_+^{\mathcal{P}}$-progressively measurable $\mathbb{R}$-valued processes $\mathcal{V}$ with $\mathcal{P}$-q.s. continuous paths on $[t, T]$ with

$$\|\mathcal{V}\|_{\mathbb{S}(t,T)}^2 := \sup_{\mathbb{P}\in\mathcal{P}} E^{\mathbb{P}}\left[\sup_{t\le s\le T} |\mathcal{V}_s|^2\right] < +\infty.$$

As usual, if $t = 0$ we omit the interval $([t, T])$ is the above notations so that for example $\mathbb{S}_{\mathcal{P}}^2$ denotes the set of all $\mathbb{F}_+^{\mathcal{P}}$-progressively measurable $\mathbb{R}$-valued processes $\mathcal{V}$ with $\mathcal{P}$-q.s. continuous paths on $[0, T]$ with

$$\|\mathcal{V}\|_{\mathbb{S}}^2 := \sup_{\mathbb{P}\in\mathcal{P}} E^{\mathbb{P}}\left[\sup_{0\le s\le T} |\mathcal{V}_s|^2\right] < +\infty.$$

Set of controls

Definition

Let $K \subset \mathbb{R}^m$ be a convex compact set. A process $u \in (\mathbb{D}^2)^m$ is said to be admissible *if it takes values in K $\mathcal{P}$-quasi-surely. We denote by $\mathcal{U}$ the set of admissible processes on $[0, T]$ and $\mathcal{U}[t, T]$ the set of admissible processes on $[t, T]$ i.e. the set of processes u such that $s \mapsto u_s \times \mathbf{1}_{[t,T]}(s)$ belongs to $\mathcal{U}$.*
Let us consider $\mathcal{U}^a[t, T]$ the set of admissible controls, u, such that for all $s \in [t, T]$ u_s is $\mathcal{F}_s^t$ -measurable where $\mathcal{F}_s^t = \sigma(B_u - B_t;\ u \in [t, s])$.

The wealth process

For $t \in [0, T]$, $u \in \mathcal{U}[t, T]$ and $x \in \mathbb{R}^n$ we consider the following forward SDE (the wealth process) defined quasi-surely:

$$\begin{cases} d\mathcal{X}_s^{t,x,u} = b(s, \mathcal{X}_s^{t,x,u}, u_s)ds + \Sigma(s, \mathcal{X}_s^{t,x,u}, u_s)dB_s, \quad s \in [t, T] \\ \mathcal{X}_t^{t,x,u} = x. \end{cases}$$

And 2BSDE: $\forall s \in [t, T]$

$$\mathcal{Y}_s^{t,x,u} = \int_s^T f(r, \mathcal{X}_r^{t,x,u}, \mathcal{Y}_r^{t,x,u}, \mathcal{Z}_r^{t,x,u}, u_r, a_r)dr + \int_s^T \mathcal{Z}_r^{t,x,u} dB_r - \mathcal{K}_T^{t,x,r} + \mathcal{K}_s^{t,x,r}$$

Assumptions

Here, $b : [0, T] \times \mathbb{R}^n \times K \longrightarrow \mathbb{R}^n$, $\Sigma : [0, T] \times \mathbb{R}^n \times K \longrightarrow \mathbb{R}^{n\times d}$, $f : [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times K \times S_d^{\geq 0} \longrightarrow \mathbb{R}$ satisfy the following set of assumptions:

Assumptions (R):

There exists a constant $C \geq 0$ such that for all (x, y, z, u, a), (x', y', z', u', a') in $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times K \times \mathbb{S}_d^{\geq 0}$ and $s \in [0, T]$:

$$\begin{aligned}
|b(s,x,u) - b(s,x',u')|| &\leq C\left(|x - x'| + |u - u'|\right)\\
||\Sigma(s,x,u) - \Sigma(s,x',u')| &\leq C\left(|x - x'| + |u - u'|\right)\\
|f(s,x,y,z,u,a) - f(s,x',y',z',u',a')| &\leq C\left(|x - x'| + |y - y'| + \right.\\
&\quad \left.|z - z'| + |u - u'| + |a - a'|\right)
\end{aligned}$$

The 2BSDE

The 2BSDE: $\forall s \in [t, T]$

$$\mathcal{Y}_s^{t,x,u} = \int_s^T f(r, \mathcal{X}_r^{t,x,u}, \mathcal{Y}_r^{t,x,u}, \mathcal{Z}_r^{t,x,u}, u_r, a_r)dr + \int_s^T \mathcal{Z}_r^{t,x,u}\, dB_r - \mathcal{K}_T^{t,x,r} + \mathcal{K}_s^{t,x,r}$$

admits a unique solution $(\mathcal{Y}, \mathcal{Z}, \mathcal{K}) \in \mathbb{S}^2(t,T) \times \mathbb{H}^2(t,T) \times \mathbb{I}^2$.
Moreover, let $w \in \Omega$,we introduce the following BSDE on $[t, T]$ for each $\mathbb{P} \in \mathcal{P}(t, w)$:

$$Y_s^{\mathbb{P},t,x,u} = \int_s^T f(r, \mathcal{X}_r^{t,x,u}, Y_r^{\mathbb{P},t,x,u}, Z_r^{\mathbb{P},t,x,u}, u_r, a_r)dr + \int_s^T Z_r^{\mathbb{P},t,x,u}\, dB_r \ \mathbb{P}\text{-a.s.}$$

then

$$\mathcal{Y}_t^{t,x,u}(w) = \operatorname*{ess\,sup}_{\mathbb{P} \in \mathcal{P}(t,w)} Y_t^{\mathbb{P},t,x,u}.$$

The value function

We now introduce the value function of our control problem:

$$V(t,x) = \inf_{u\in\mathcal{U}} \mathcal{Y}_t^{t,x,u}(w) = \inf_{u\in\mathcal{U}} \sup_{\mathbb{P}\in\mathcal{P}(t,w)} Y_t^{\mathbb{P},t,x,u}$$

$$= \inf_{u\in\mathcal{U}} \sup_{\mathbb{P}\in\mathcal{P}} E_t^P[\int_t^T f(r, \mathcal{X}_r^{t,x,u}, Y_r^{\mathbb{P},t,x,u}, Z_r^{\mathbb{P},t,x,u}, u_r, a_r)dr].$$

Theorem

$V(t,x)$ is deterministic.

Why "penalization on the model"?

Let σ^{REF} be a **deterministic** reference volatility model of the investor. And we define $\mathrm{a}^{\mathrm{REF}} := (\sigma^{\mathrm{REF}})^2$.
For some control process (consumption) $= (u_t)$, the investor's robust control continuation utility $\mathcal{U}_t^c$ under volatility uncertainty is given by

$$\mathcal{U}_t^c = \inf_{\mathrm{a}\in\mathcal{O}} E_t^{\mathbb{P}^{\mathrm{a}}}\Big[\int_t^T e^{-\beta(s-t)}\Big(v(u_s) + \mathrm{d}\big(\mathrm{a}_s, \mathrm{a}_s^{\mathrm{REF}}\big)\Big)\mathrm{d}s\Big]$$

Examples

- quadratic penalty term: $\mathrm{d}\big(\mathrm{a}_t, \mathrm{a}^{\mathrm{REF}}\big) = \frac{1}{2\Xi}\big(\mathrm{a}_t - \mathrm{a}^{\mathrm{REF}}\big)^2$.
- Hellinger distance: $\mathrm{d}^{\mathrm{Hel}}(\sigma, \sigma^{\mathrm{REF}}) = \frac{1}{2}\left(\frac{\sigma}{\sigma^{\mathrm{REF}}} - 1\right)^2 (\sigma\sigma^{\mathrm{REF}})^2$.
- Relative Entropy: $\mathrm{d}^{\mathrm{Ent}}(\sigma, \sigma^{\mathrm{REF}}) = \ln\left((1 + |\sigma - \sigma^{\mathrm{REF}}|)^2\right)\sigma^4$.

The DPP satisfied by the 2BSDE

Let $x \in \mathbb{R}^n$, $t \in [0, T]$, $\delta \in [0, T-t]$, $u \in \mathcal{U}$ and $\eta \in \mathbb{L}^{2,\kappa}$ we introduce the following *backward semigroup*

$$\mathbb{G}^{t,x,u}_{t,t+\delta}[\eta] = \tilde{\mathcal{Y}}^{t,x,u}_t(\eta),$$

where $\tilde{\mathcal{Y}}$ is the solution of the following 2BSDE on $[t, t+\delta]$:

$$\begin{aligned}\tilde{\mathcal{Y}}^{t,x,u}_s(\eta) &= \eta + \int_s^{t+\delta} f(r, \mathcal{X}^{t,x,u}_r, \tilde{\mathcal{Y}}^{t,x,u}_r(\eta), \mathcal{Z}^{t,x,u}_r, u_r, a_r)dr \\ &\quad + \int_s^{t+\delta} \mathcal{Z}^{t,x,u}_r \, dB_r - \tilde{\mathcal{K}}^{t,x,r}_{t+\delta} + \tilde{\mathcal{K}}^{t,x,r}_s\end{aligned}$$

Theorem

V satisfies the following DPP:

$$\begin{aligned}V(t,x) &= \operatorname*{ess\,inf}_{u \in \mathcal{U}([t,t+\delta])} \mathbb{G}^{t,x,u}_{t,t+\delta}[V(t+\delta, \mathcal{X}^{t,x,u}_{t+\delta})] \\ &= \inf_{u \in \mathcal{U}^a([t,t+\delta])} \mathbb{G}^{t,x,u}_{t,t+\delta}[V(t+\delta, \mathcal{X}^{t,x,u}_{t+\delta})]\end{aligned}$$

The HJB equation satisfied by V

Theorem

The value function, V, is the unique viscosity solution of the following second-order differential equation:

$$\begin{cases} \partial_t V(t,x) + \inf_{u \in K} \sup_{a \in \Theta} H(t,x,V(t,x),\partial_x V(t,x),\partial^2_{xx} V(t,x),u,a) = 0 \\ V(T,x) = 0 \end{cases}$$

where

$$\begin{aligned} H(t,x,v,p,A,u,a) &= f(t,x,v,\Sigma^T(t,x,u)p,u,a) + \langle p, b(t,x,u)\rangle + \\ &\quad \frac{1}{2}\langle A \cdot \Sigma_i(t,x,u), \Sigma_j(t,x,u)\rangle a_i a_j \end{aligned}$$

The case where σ^{ref} is stochastic (Heston model?)

We now consider the case where the reference volatility is stochastic. For simplicity, we are assuming that the matrix σ^{ref} is diagonal, The diagonal coefficient are represented by the vector

$$\bar{\sigma} = \left(\sigma^{ref}_{1,1}, \cdots, \sigma^{ref}_{d,d}\right).$$

Remark

In the general case where σ^{ref} is not necessary diagonal, one has to introduce $\bar{\sigma}$ taking values in $\mathbb{R}^{\frac{d\times(d+1)}{2}}$ obtained by stacking the columns of the lower triangular matrix extracted from σ^{ref}:

$$\bar{\sigma} = \left(\sigma^{ref}_{1,1}, \cdots, \sigma^{ref}_{d,1}, \sigma^{ref}_{2,2}, \cdots, \sigma^{ref}_{d,2}, \cdots, \sigma^{ref}_{d,d}\right).$$

We assume that $\bar{\sigma}$ satisfies the following G-SDE:

$$d\bar{\sigma}_s = \alpha(s, \bar{\sigma}_s)ds + \rho(s, \bar{\sigma}_s)dB_s,$$

with initial condition $\bar{\sigma}_0$ which is fixed and deterministic.

Here, $\alpha : [0, T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$, $\rho : [0, T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times d}$ satisfy Assumptions (R'):

(1) There exists a constant $C \geq 0$ such that for all (y, y') in $\mathbb{R}^d \times \mathbb{R}^d$ and $s \in [0, T]$:

$$|\alpha(s,y) - \alpha(s,y')| + |\rho(s,y) - \rho(s,y')| \leq C\left(|y - y'|\right)$$

(2) α, ρ are continuous in s

In order to apply previous results we consider the flow generated by $\bar{\sigma}$ hence for any $t \in [0, T]$ and $y \in \mathbb{R}^d$ we consider $\bar{\sigma}^{t,y}$, solution of the following G-SDE:

$$\left\{ \begin{array}{rcl} d\bar{\sigma}_s^{t,y} & = & \alpha(s, \bar{\sigma}_s^{t,y})ds + \rho(s, \bar{\sigma}_s^{t,y})dB_s, \quad s \in [t, T] \\ \bar{\sigma}_t^{t,y} & = & y. \end{array} \right.$$

The control problem

We introduce the 2BSDE: $\forall s \in [t, T]$

$$\mathcal{Y}_s^{t,x,y,u} = \int_s^T f(r, \mathcal{X}_r^{t,x,u}, \bar{\sigma}_r^{t,y}, \mathcal{Y}_r^{t,x,y,u}, \mathcal{Z}_r^{t,x,y,u}, u_r, a_r)dr + \int_s^T \mathcal{Z}_r^{t,x,y,u}\, dB_r - \mathcal{K}_T^{t,x,y,u} + \mathcal{K}_s^{t,x,y,u}$$

Here, $f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times K \times S_d^{\geq 0} \longrightarrow \mathbb{R}$ satisfies usual assumptions
The value function of our control problem now is:

$$V(t, x, y) = \inf_{u \in \mathcal{U}} \mathcal{Y}_t^{t,x,y,u}(w).$$

Remark

Let us remark that here the "true" value function is $V'(0, x) = V(0, x, \bar{\sigma}_0)$.

The trick

We introduce the process

$$\forall 0 \leq t \leq s \leq T,\ \tilde{\mathcal{X}}_s^{t,x,y,u} = (\mathcal{X}_s^{t,x,u}, \bar{\sigma}_s^{t,y})$$

that we (artificially) consider as the "wealth" process of our control problem.
We remark that $\tilde{\mathcal{X}}^{t,x,y,u}$ takes values in $\mathbb{R}^n \times \mathbb{R}^d$ and satisfies the following G-SDE:

$$\begin{cases} d\tilde{\mathcal{X}}_s^{t,x,y,u} &= \tilde{b}(s, \tilde{\mathcal{X}}_s^{t,x,y,u}, u_s)ds + \tilde{\sigma}(s, \tilde{\mathcal{X}}_s^{t,x,u}, u_s)dB_s, \quad s \in [t, T] \\ \tilde{\mathcal{X}}_t^{t,x,y,u} &= (x, y). \end{cases}$$

where $\tilde{b} : [0, T] \times \mathbb{R}^n \times \mathbb{R}^d \times K \longrightarrow \mathbb{R}^{n+d}$ and
$\tilde{\sigma} : [0, T] \times \mathbb{R}^n \times \mathbb{R}^d \times K \longrightarrow \mathbb{R}^{n+d} \times \mathbb{R}^d$ are defined in the following way:
for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^d$, $t \in [0, T]$ and $u \in K$:

$$\tilde{b}(t, x, y, u) = \begin{pmatrix} b(t, x, u) \\ \alpha(t, y) \end{pmatrix} \text{ and } \tilde{\sigma}(t, x, y, u) = \begin{pmatrix} \Sigma(t, x, u) \\ \rho(t, y) \end{pmatrix}.$$

Notations

We introduce the following notation for the Hessian matrix of any real-valued function F defined on $\mathbb{R}^n \times \mathbb{R}^d$:

$$D^2_{x,y}F(x,y) = (D_{i,j}(x,y))_{1\leq i,j\leq d+n}$$

where for all $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_d)$

$$D_{i,j}(x,y) = \begin{cases} \partial_{x_i}\partial_{x_j}F(x,y) & \text{if } 1 \leq i,j \leq n \\ \partial_{y_{i-n}}\partial_{y_{j-n}}F(x,y) & \text{if } n+1 \leq i,j \leq d+1 \\ \partial_{x_i}\partial_{y_{j-n}}F(x,y) & \text{if } 1 \leq i \leq n \text{ and } n+1 \leq j \leq n+d \\ \partial_{x_j}\partial_{y_{i-n}}F(x,y) & \text{if } n+1 \leq i \leq n+d \text{ and } 1 \leq j \leq d \end{cases}$$

The HJB equation

Theorem

The value function, V, is the unique viscosity solution of the following second-order differential equation:

$$\begin{cases} \partial_t V(t,x,y) \\ + \inf_{u \in K} \sup_{a \in \Theta} H(t,x,y,V(t,x,y),\partial_x V(),\partial_y V(),D^2_{x,y}V(),u,a) = 0 \\ V(T,x,y) = 0 \end{cases}$$

where for all
$(t,x,y,v,p,q,A,u,a) \in$
$[0,T] \times \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^{(n+d)\times(n+d)} \times K \times \mathbb{S}_d^{\geq 0}$:

$$\begin{aligned} H(t,x,y,v,p,q,A,u,a) &= f\left(t,x,y,v,\Sigma^T(t,x,u)p + \rho^T(t,y)q,u,a\right) \\ &\quad + \langle p, b(t,x,u)\rangle + \langle q, \alpha(t,y)\rangle \\ &\quad + \frac{1}{2}\sum_{i,j=1}^{d} \langle A \cdot \tilde{\sigma}_i(t,x,u), \tilde{\sigma}_j(t,x,u)\rangle_{\mathbb{R}^{n+d}} a_i a_j \end{aligned}$$

Example

If the case where $n = d = 1$ then the hamiltonian is:

$$
\begin{aligned}
&H(t,x,y,V(t,x,y),\partial_x V(t,x,y),\partial_y V(t,x,y),D^2_{x,y}V(t,x,y),u,a) \quad = \\
&f\left(t,x,y,V(t,x,y),\sigma(t,x,u)\partial_x V(t,x,y)+\rho(t,y)\partial_y V(t,x,y),u,a\right) \\
&+\partial_x V(t,x,y)b(t,x,u)+\partial_y V(t,x,y)\alpha(t,y) \\
&+\frac{a^2}{2}\left(\Sigma^2(t,x,u)\partial_x^2 V(t,x,y)+\rho^2(t,y)\partial_y^2 V(t,x,y)\right. \\
&\qquad\qquad\qquad\qquad \left.+2\sigma(t,x,u)\rho(t,x)\partial_x\partial_y V(t,x,y)\right)
\end{aligned}
$$