# Mean field approximations via log-concavity, and a non-asymptotic perspective on mean field control

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# High-dimensional stochastic control

Players $i = 1, \ldots, n$ have state processes $X = (X^1, \ldots, X^n)$,

$$dX_t^i = \alpha_i(t, X_t)dt + dW_t^i, \quad X_0^i = 0$$

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**Collectively optimize:**

$$V = \sup_\alpha J(\alpha) = \sup_\alpha \mathbb{E}\left[g(X_T) - \frac{1}{2n}\sum_{i=1}^n \int_0^T |\alpha_i(t, X_t))|^2 dt\right]$$

Here $g : \mathbb{R}^n \to \mathbb{R}$ is arbitrary, say bounded from above.

# The usual case

**"Mean field control" case**: $g$ takes the form

$$g(x) = G(L_n(x)), \qquad L_n(x) := \frac{1}{n}\sum_{i=1}^{n} \delta_{x_i}, \qquad G : \mathcal{P}(\mathbb{R}) \to \mathbb{R}.$$

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**Mean field limit** as $n \to \infty$,

$$V \;\to\; \overline{V} := \sup_{\alpha}\; G(\mathrm{Law}(\overline{X}_T)) - \frac{1}{2}\mathbb{E}\int_0^T |\overline{\alpha}(t,\overline{X}_t)|^2 dt,$$

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These approximate optimizers are distributed/decentralized!

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**Guiding example:** Heterogeneous interactions:

$$g(x) = \frac{1}{n}\sum_{i=1}^{n} U(x_i) + \frac{1}{n}\sum_{1\le i<j\le n} J_{ij} K(x_i - x_j)$$

**Ex A:** Usual case is $J_{ij} = 1/n$
**Ex B:** $J =$ scaled adjacency matrix of a graph

Can anything be done?

# The distributed optimal control problem

**Recall:**

$$V = \sup_{\alpha} J(\alpha) = \sup_{\alpha} \mathbb{E}\left[g(X_T) - \frac{1}{2n}\sum_{i=1}^{n}\int_0^T |\alpha_i(t, X_t))|^2 dt\right]$$

**Define:**

$$V_{\mathrm{dstr}} = \sup_{\alpha \ \mathrm{dstr}} J(\alpha)$$

where sup is over controls of the form $\alpha_i(t, X_t) = \tilde{\alpha}_i(t, X_t^i)$.

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Related (independent) idea: Seguret-Alasseur-Bonnans-De Paola-Oudjane-Trovato

# Comparing the distributed and original problems

**Theorem**

*Let $g : \mathbb{R}^n \to \mathbb{R}$ be $C^2$ concave, $|g(x)| \leq c_1 e^{c_2 |x|^2}$, $c_2 < 1/2T$. Then*

$$0 \leq V - V_{\mathrm{dstr}} \leq nT^2 \sum_{1 \leq i < j \leq n} \|\partial_{ij} g\|_\infty^2 .$$

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**Sanity check 2:** $g(x) = G(L_n(x))$, $G : \mathcal{P}(\mathbb{R}) \to \mathbb{R}$ smooth,

$$\rightsquigarrow \partial_{ij} g(x) = \frac{1}{n^2} D_m^2 G(L_n(x), x_i, x_j), \quad \text{for } i \ne j$$

$$\rightsquigarrow \mathrm{RHS} \le \frac{T^2}{2n} \|D_m^2 G\|_\infty^2$$

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**Heterogeneous interactions:** $U, K$ concave, $K$ even, $J_{ij} \ge 0$,

$$g(x) = \frac{1}{n} \sum_{i=1}^{n} U(x_i) + \frac{1}{n} \sum_{1 \le i < j \le n} J_{ij} K(x_i - x_j)$$

$$\rightsquigarrow \|\partial_{ij} g\|_\infty = \frac{1}{n} J_{ij} \|K''\|_\infty$$

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**Key condition:** $\mathrm{tr}(J^2) = o(n)$.

# The bigger picture: a static problem

**Proof step 0:** With $f = ng$, $\gamma_T = N_n(0, TI)$,

$$nV = \sup_{\mu \in \mathcal{P}(\mathbb{R}^n)} \left( \langle \mu, f \rangle - H(\mu \,|\, \gamma_T) \right) \overset{(*)}{=} \log \int_{\mathbb{R}^n} e^f \, d\gamma_T$$

$$nV_{\mathrm{dstr}} = \sup_{\mu \in \mathcal{P}_{\mathrm{prod}}(\mathbb{R}^n)} \left( \langle \mu, f \rangle - H(\mu \,|\, \gamma_T) \right)$$

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**Static problem:** When is Gibbs variational formula $(*)$ "nearly" saturated by product measures?

cf. nonlinear large deviations theory, Chatterjee-Dembo '16,
also Basak-Mukherjee '17, Eldan '18, Austin '19, Augeri '20...

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**Easy identity:** Let $P(dx) = (1/Z)e^{f(x)}\gamma_T(dx)$. Then

$$n(V - V_{\mathrm{dstr}}) = \inf \left\{ H(\mu \,|\, P) : \mu \in \mathcal{P}_{\mathrm{prod}}(\mathbb{R}^n) \right\}$$

# Proof outline

$$n(V - V_{\mathrm{dstr}}) = \inf\left\{ H(\mu \,|\, P) : \mu \in \mathcal{P}_{\mathrm{prod}}(\mathbb{R}^n) \right\}$$

**Step 1:** First-order condition for an optimizer $\mu^*$:

$$\frac{d\mu^*}{d\gamma_T} = (1/Z') \exp \sum_{i=1}^{n} \mathbb{E}_{\mu^*}[f(X) \,|\, X_i]$$

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Concavity of $f$ $\Rightarrow$ uniqueness of optimizer!

Proof by displacement convexity a la McCann '97.

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**Step 2:** $\mu^*$ is log-concave: $\nabla^2 \log \frac{d\mu^*}{d\gamma_T} \preceq 0$

# Proof outline

**Step 3, the main calculation:**

$P$ log-concave $\Rightarrow$ log-Sobolev inequality:

$$n(V - V_{\mathrm{dstr}}) = H(\mu^* \mid P) \leq \frac{T}{2} \mathbb{E}_{\mu^*} \left| \nabla \log \frac{d\mu^*}{dP} \right|^2$$
$$= \ldots = \frac{T}{2} \sum_{i=1}^{n} \mathbb{E}_{\mu^*} \mathrm{Var}_{\mu^*}(\partial_i f(X) \mid X_i)$$

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$\mu^*$ log-concave $\Rightarrow$ Poincaré inequality:

$$\leq \mathrm{Var}_{\mu^*}(\partial_i f(X) \mid X_i) \leq T \sum_{j \neq i} \mathbb{E}_{\mu^*}\left[|\partial_{ij} f(X)|^2 \mid X_i\right]$$

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Combine with tower property:

$$n(V - V_{\mathrm{dstr}}) \leq T^2 \sum_{1 \leq i < j \leq n} \mathbb{E}_{\mu^*}|\partial_{ij} f(X)|^2$$

# Approximate independence

More can be said using $\mu^*$ about $P(dx) = (1/Z)e^{f(x)}\gamma_T(dx)$:

- Empirical measure is similar under $P$ and $\mu^*$:

$$\mathbb{E}_P\left[\left(\frac{1}{n}\sum_{i=1}^n \varphi(X_i) - \frac{1}{n}\sum_{i=1}^n \mathbb{E}_{\mu^*}[\varphi(X_i)]\right)^2\right] \le \frac{T}{n}(1+\sqrt{2RHS})^2$$

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- $\frac{1}{n}\sum_{i=1}^n \mathcal{W}_2^2(P_i, \mu_i^*) \le (2T/n)RHS$

# Back to the control problem

The optimal controls for $V$ and $V_{\mathrm{dstr}}$ can be characterized in terms of $P(dx) = (1/Z)e^{f(x)}\gamma_T(dx)$ and $\mu^*$:

- $V$: optimal $X = (X^1, \ldots, X^n)$ is Brownian bridge $0 \to P$
- $V_{\mathrm{dstr}}$: optimal $X = (X^1, \ldots, X^n)$ is Brownian bridge $0 \to \mu^*$:

**Brownian bridge** $0 \to Q \ll \gamma_T$**:** The process $X$ with $X_T \sim Q$, and $(X|X_T = x) \sim$ (Brownian bridge from 0 to $x$ over $[0, T]$).

Corresponding control:

$$\alpha_i(t,x) = \partial_i \log \mathbb{E}[(dQ/d\gamma_T)(W_T) \,|\, W_t = x]$$