BSDE2022: Summer school on propagation of chaos, lecture 1

Sunday, June 26, 2022 7:43 AM

Qualitive / Quantitative

Surveys/refs:

- Sznitman '89/91
- Jabin-Wang '18
- Chaintron-Diez '21-22

I. Kac's chaos (molecular chaos) (asymp. indep.)

Let $P^n \in \mathcal{P}(E^n)$, E a Polish space.

$(X_1^n, \dots, X_n^n) \sim P^n$

Assume P^n exchangeable: $(X_1^n, \dots, X_n^n) \overset{d}{=} (X_{\sigma(1)}^n, \dots, X_{\sigma(n)}^n)$

for all permutations σ

Def: For $1 \le k \le n$, let $P_k^n = \mathrm{Law}(X_1^n, \dots, X_k^n)$.

Def: Let $\mu \in \mathcal{P}(E)$. We say $(P^n)_{n \in \mathbb{N}}$ is μ-chaotic if $P_k^n \Rightarrow \mu^{\otimes k}$ as $n \to \infty$, for each $k \in \mathbb{N}$.

Note: $\Rightarrow$ weak convergence, $\varphi \in C_b(E^k)$

$$\langle P_k^n, \varphi \rangle = \int_{E^k} \varphi \, dP_k^n \longrightarrow \langle \mu^{\otimes k}, \varphi \rangle$$

$$= \int \varphi(x_1, \dots, x_k) \, \mu(dx_1) \dots \mu(dx_k)$$

Theorem (Sznitman-Tanaka):

Let μ and P^n be as above. Then

(local) ① P^n is μ-chaotic

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iff

(global) ② $P^n\left(x \in E^n : \left|\langle L_n(x), \varphi\rangle - \langle \mu, \varphi\rangle\right| > \varepsilon\right) \to 0$

for all $\varepsilon > 0$, $\varphi \in C_b(E)$, where $L_n : E^n \to \mathcal{P}(E)$

$$L_n(x_1, \dots, x_n) = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}$$

$\boxed{\underline{\text{OR}}: \quad P^n \circ L_n^{-1} \Rightarrow \delta_\mu \text{ in } \mathcal{P}(\mathcal{P}(E))}$

$\boxed{\underline{\text{OR}}: \quad L_n \to \mu \text{ in probability}}$

Proof:

$1 \Rightarrow 2$ Let $\varphi \in C_b(E)$.

$$\mathbb{E}_{P^n}\left(\langle L_n(x^n), \varphi\rangle - \langle \mu, \varphi\rangle\right)^2$$

$$= \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^n \left(\varphi(X_i^n) - \langle \mu, \varphi\rangle\right)\right)^2$$

$$= \frac{1}{n^2}\sum_{i=1}^n \mathbb{E}\left(\varphi(X_i^n) - \langle \mu, \varphi\rangle\right)^2 \quad O\left(\tfrac{1}{n}\right)$$

$$+ \frac{1}{n^2}\sum_{i=1}^n \sum_{j \neq i} \mathbb{E}\left[\left(\varphi(X_i^n) - \langle \mu, \varphi\rangle\right)\left(\varphi(X_j^n) - \langle \mu, \varphi\rangle\right)\right]$$

$$\underset{(\text{exch})}{\leq} \int_{E^2} \left(\varphi(x_1) - \langle \mu, \varphi\rangle\right)\left(\varphi(x_2) - \langle \mu, \varphi\rangle\right) P_2^n(dx_1, dx_2)$$

$$\xrightarrow[(\mu\text{-chaotic})]{(k=2)} \int_{E^2} \quad '' \quad '' \quad \mu(dx_1)\mu(dx_2)$$

$$= 0.$$

1 $\Leftarrow$ 2 $\qquad \varphi \in C_b(E^k)$

$$\to \langle P^n_k - \mu^{\otimes k}, \varphi\rangle = \underbrace{\langle P^n_k, \varphi\rangle - \int \langle L_n^{\otimes k}, \varphi\rangle dP^n}_{\text{a bit of work}}$$

$$+ \underbrace{\int \langle L_n^{\otimes k}, \varphi\rangle dP^n - \langle \mu^{\otimes k}, \varphi\rangle}_{\to 0 \text{ by asmp}}$$

a bit of work

Diaconis-Freedman

$P^n_k \approx$ law of k particles drawn w/o replacement

$\int L_n^{\otimes k} dP^n \approx$ law of k particles drawn with replacement

... rest an exercise

II. Interacting Diffusions

- Particles $i=1,\dots,n$ living in $\mathbb{R}^d$

$$dX^i_t = b(X^i_t, \mu^n_t)dt + dW^i_t, \quad \mu^n_t = \frac{1}{n}\sum_{j=1}^n \delta_{X^j_t}$$

where $b : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d$,

$W^1, \dots, W^n$ independent Brownian.

Note:
- constant σ
- no common noise

- Let $P^n_t = \text{Law}(X^1_t, \dots, X^n_t)$, $t \ge 0$.

Ex: linear/scalar/pairwise interactions

$$b(x, m) = \int_{\mathbb{R}^d} a(x,y)\, m(dy)$$

$$\rightsquigarrow b(X^i_t, \mu^n_t) = \frac{1}{n}\sum_{j=1}^n a(X^i_t, X^j_t)$$

Mean field limit $n \to \infty$? $\qquad$ (McKean 60s)

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McKean-Vlasov equation

$$dY_t = b(Y_t, \mu_t)\,dt + dW_t, \qquad \mu_t = \mathrm{Law}(Y_t)$$

"Propagation of chaos" refers to the implication

$$\left[P_0^n \text{ is } \mu_0\text{-chaotic}\right] \Rightarrow \left[P_t^n \text{ is } \mu_t\text{-chaotic } \forall t>0\right]$$

$\left(P_0^n = \mu^{\otimes n} \text{ special case} \right)$

Laboratory example of Sznitman:

Define $W_p(\mu,\nu) = \inf\left\{ \left(\mathbb{E}|X-Y|^p\right)^{\frac{1}{p}} : X \sim \mu,\ Y \sim \nu \right\}$
for $\mu, \nu \in \mathcal{P}^p(\mathbb{R}^d)$ i.e. $\int |x|^p d\mu(x) < \infty$.

Assume b is Lipschitz:

$$|b(x,m) - b(x',m')| \le L\left(|x-x'| + W_1(m,m')\right)$$

Assume initial distribution $\lambda \in \mathcal{P}^1(\mathbb{R}^d)$.

Thm 1: There exists a unique (strong) solution of
$$dY_t = b(Y_t,\mu_t)\,dt + dW_t, \qquad Y_0 \sim \lambda.$$

Proof sketch: Fix μ_t^1, μ_t^2, $i=1,2$, $t \ge 0$, $\mu_0^1 = \mu_0^2 = \lambda$.

Solve $dY_t^i = b(Y_t^i, \mu_t^i)\,dt + dW_t$, $Y_0^1 = Y_0^2 \sim \lambda$.

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Let $\bar{\mu}_t^i = \mathrm{Law}(Y_t^i)$.

$$|Y_t^1 - Y_t^2| \le \int_0^t |b(Y_s^1, \mu_s^1) - b(Y_s^2, \mu_s^2)|\, ds$$

$$\le L \int_0^t \left(|Y_s^1 - Y_s^2| + W_1(\mu_s^1, \mu_s^2) \right) ds$$

$$\Rightarrow \quad |Y_t^1 - Y_t^2| \le L e^{Lt} \int_0^t W_1(\mu_s^1, \mu_s^2)\, ds$$

$$\Rightarrow \quad W_1(\bar{\mu}_t^1, \bar{\mu}_t^2) \le \mathbb{E}|Y_t^1 - Y_t^2| \le c \int_0^t W_1(\mu_s^1, \mu_s^2)\, ds$$

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Theorem 2: Assume X_0^i are iid λ.

Solve $dY_t^i = b(Y_t^i, \mu_t)dt + dW_t^i$, $\quad Y_0^i = X_0^i$.

$\mu_t = \mathrm{Law}(Y_t^i)$,

where W^i are same Brownian motions driving X^i's.

Then

1. $\mathbb{E}|X_t^i - Y_t^i| \to 0$ as $n \to \infty$, $\forall i$ fixed.
2. $W_1(\mu_t^n, \mu_t) \to 0$ a.s. and L^1 as $n \to \infty$.

Proof: $t \in [0, T]$

$$|X_t^i - Y_t^i| \le L \int_0^t \left(|X_s^i - Y_s^i| + W_1(\mu_s^n, \mu_s) \right) ds$$

$$\to \quad |X_t^i - Y_t^i| \le L e^{LT} \int_0^t W_1(\mu_s^n, \mu_s)\, ds$$

$$\le L e^{LT} \int_0^t \left[W_1(\mu_s^n, \nu_s^n) + W_1(\nu_s^n, \mu_s) \right] ds$$

$$\leq L e^{LT} \int_0^t \left[W_1(\mu^n_s, \nu^n_s) + W_1(\nu^n_s, \mu_s) \right] ds$$

where $\nu^n_s = \frac{1}{n} \sum_{i=1}^n \delta_{Y^i_s}$. $\quad \left(\nu^n_s \xrightarrow{a.s.} \mu_s \right)$

Then

$$W_1(\mu^n_t, \nu^n_t) \leq \frac{1}{n} \sum_{i=1}^n |X^i_t - Y^i_t|$$

$$\left(\frac{1}{n} \sum_{i=1}^n \delta_{X_i} \quad \frac{1}{n} \sum_{i=1}^n \delta_{Y_i} \searrow \swarrow \frac{1}{n} \sum_{i=1}^n \delta_{(X_i, Y_i)} \right)$$

$$\leq L e^{LT} \int_0^t W_1(\mu^n_s, \nu^n_s) + W_1(\nu^n_s, \mu_s)\, ds$$

$$\Rightarrow \quad W_1(\mu^n_t, \nu^n_t) \leq C \int_0^t W_1(\nu^n_s, \mu_s)\, ds.$$

$\to 0$ a.s. and in L^1 because Y^i_s iid $\underline{\mu_s \in \mathcal{P}^1(\mathbb{R}^d)}$ — check!

$$|X^i_t - Y^i_t| \leq \tilde{C} \int_0^t W_1(\nu^n_s, \mu_s)\, ds$$

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<u>Key idea:</u> "synchronous coupling" / "trajectorial" (Lipschitz)

cf. Durmus - Eberle - Guillin - Zimmer '20

$\rightsquigarrow$ "reflection coupling"

<u>Rates:</u>

(1) $\mathbb{E}\, W_1(\mu^n_t, \mu_t)$ and $\mathbb{E}\, |X^i_t - Y^i_t|$

controlled in terms of $\underline{\mathbb{E}\, W_1(\nu^n_t, \mu_t)}$, Y^i's iid $\to \nu^n$

cf. Fournier - Guillin '15

$$\begin{cases} \frac{1}{\sqrt{n}} & \text{in } d = 1 \\ \sqrt{\frac{\log n}{n}} & \text{in } d = 2 \\ \frac{1}{n^{1/d}} & \text{in } d \geq 3 \end{cases}$$

(2) <u>Linear case:</u> $b(x, m) = \int a(x, y)\, m(dy)$, a is L-Lipschitz

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Same argument: Y^i_s iid $\sim \mu_s$

$$\mathbb{E}|X^i_t - Y^i_t|^2 \le C\,\mathbb{E}\int_0^t \Big|\frac{1}{n}\sum_{j=1}^n \big(a(Y^i_s, Y^j_s) - \langle \mu_s, a(Y^i_s, \cdot)\rangle\big)\Big|^2 ds$$

$$= C\int_0^t \mathbb{E}\,\mathrm{Var}\Big(\frac{1}{n}\sum_{j=1}^n a(Y^i_s, Y^j_s)\,\Big|\,Y^i_s\Big)\,ds$$

$$= O\Big(\frac{1}{n}\Big)$$

$\rightarrow$ $\underline{\mathbb{E}|X^i_t - Y^i_t|} = O\big(\frac{1}{\sqrt{n}}\big)$ dimension-free

$W_1(\mathcal{L}(X^i_s), \mathcal{L}(Y^i_s)) = O(\frac{1}{n})$ $\leftarrow$ ☆ pt 2: can be improved to $O(\frac{1}{n})$! (diff. method)

III. Weak compactness (versatile)

- PDE: Nonlinear Fokker–Planck equation associated to

$$dY_t = b(Y_t, \mu_t)\,dt + dW_t \;{\color{green}+ \,dB_t}, \quad \mu_t = \mathrm{Law}(Y_t \,{\color{green}|\, (B_s)_{s\le t}})$$

Ito ... $\forall \varphi \in C_c^\infty(\mathbb{R}^d)$

"superposition principle" Figalli / Trevisan

$$\rightarrow \frac{d}{dt}\langle \mu_t, \varphi\rangle = \langle \mu_t, L_{\mu_t}\varphi\rangle \;{\color{green}+\, \sigma\langle \mu_t, \varphi\rangle\, dB_t}$$

where

$$L_m\varphi(x) = b(x,m)\cdot\nabla\varphi(x) + \frac{1}{2}\Delta\varphi(x), \quad x\in\mathbb{R}^d,\ m\in\mathcal{P}(\mathbb{R}^d).$$

Formally: $\partial_t\mu_t = L^*_{\mu_t}\mu_t = -\mathrm{div}(\mu_t\, b(\cdot,\mu_t)) + \frac{1}{2}\Delta\mu_t$

Theorem: Assume $b: \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d$ bounded + continuous.

Assume $\frac{1}{n}\sum_{i=1}^n \delta_{X^i_0} \to \lambda$. Then, $\forall T > 0$, Assume $\frac{1}{n}\sum_{i=1}^{n}\delta_{X_0^i} \to \lambda$. Then, $\forall T>0$,

$$\left\{\mathcal{L}\left((\mu_t^n)_{t\in[0,T]}\right) : n\in\mathbb{N}\right\} \subset \mathcal{P}\left(C([0,T];\mathcal{P}(\mathbb{R}^d))\right)$$

is tight, and every limit point is supported on the set S_{MV} of $(m_t)_{t\in[0,T]} \in C([0,T];\mathcal{P}(\mathbb{R}^d))$ s.t.

$$\langle m_t,\varphi\rangle = \langle\lambda,\varphi\rangle + \int_0^t \langle m_s, L_{m_s}\varphi\rangle\, ds.$$

☆ In particular, if $S_{MV} = \{\mu_\cdot\}$ is a singleton, then $\mu_\cdot^n \to \mu_\cdot$.

Ref: Oelschlager, Méléard, Gärtner 80s

Proof sketch:

1. tightness (Aldous' criterion, Sznitman)
2. Characterize limits. Let $\varphi\in C_c^\infty(\mathbb{R}^d)$.

Then
$$\varphi(X_t^i) = \varphi(X_0^i) + \int_0^t \Big(\underbrace{b(X_s^i,\mu_s^n)\cdot\nabla\varphi(X_s^i) + \tfrac{1}{2}\Delta\varphi(X_s^i)}_{\boxed{L_{\mu_s^n}\varphi(X_s^i)}}\Big)ds + \int_0^t \nabla\varphi(X_s^i)\cdot dW_s^i$$

So $\langle\mu_t^n,\varphi\rangle = \frac{1}{n}\sum_{i=1}^n \varphi(X_t^i)$

$dX_t^i = b\,dt + dW^i + \gamma dB_t$

$$= \langle\mu_0^n,\varphi\rangle + \int_0^t \langle\mu_s^n, L_{\mu_s^n}\varphi\rangle ds + \frac{1}{n}\sum_{i=1}^n\int_0^t \nabla\varphi(X_s^i)\cdot dW_s^i.$$

$+\gamma\int_0^t\langle\mu_s^n,\varphi\rangle dB_s$ → $\gamma\int_0^t\langle\mu_s,\varphi\rangle dB_s$

↪ Vanishes as $n\to\infty$!

Note
$$= \mathbb{E}\left[\left(\frac{1}{n}\sum_{i=1}^n\int_0^t \nabla\varphi(X_s^i)\cdot dW_s^i\right)^2\right]$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} \mathbb{E} \int_0^t |\nabla\varphi(X_s^i)|^2 ds = O\left(\frac{1}{n}\right).$$

Define $G_{t,\varphi} : C([0,T]; \mathcal{P}(\mathbb{R}^d)) \to \mathbb{R}$ by

$$G_{t,\varphi}(m_\cdot) = \langle m_t, \varphi\rangle - \langle \lambda, \varphi\rangle - \int_0^t \langle m_s, L_{m_s}\varphi\rangle ds.$$

Then $G_{t,\varphi}$ is continuous ✓ and

$$= \mathbb{E}\,|G_{t,\varphi}(\mu^n_\cdot)|^2 \to 0$$

$\Rightarrow \quad G_{t,\varphi}(\mu) = 0$ a.s. for any limit μ of μ^n.

$\forall t, \varphi$