

A BSDEs approach to pathwise uniqueness for stochastic evolution equations

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PLAN

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 - Regularizing properties of the transition semigroup-dampedwave equation
 - The related forward-backward system and the PDE
4. Well posedness for the stochastic damped wave equation
5. A unified BSDE approach for evolution equations

Stochastic wave equation

$$\begin{cases} \frac{\partial^2}{\partial \tau^2} y(\tau, \xi) = \frac{\partial^2}{\partial \xi^2} y(\tau, \xi) + b(\tau, \xi, y(\tau, \xi)) + \varepsilon \dot{W}(\tau, \xi), \\ y(\tau, 0) = y(\tau, 1) = 0, \\ y(0, \xi) = x_0(\xi), \\ \frac{\partial y}{\partial \tau}(0, \xi) = x_1(\xi), \quad \tau \in (0, T], \quad \xi \in [0, 1], \end{cases} \quad (1)$$

- $\dot{W}(\tau, \xi)$ space-time white noise, f: $(e_k)_{k \geq 1}$ o.n. basis in $L^2([0, 1])$,

$$W(\tau, \xi) = \sum_{k \geq 1} \beta_k(\tau) e_k(\xi);$$

- b bounded measurable, β -Hölder continuous in y , $\beta \in (2/3, 1)$.
- Without noise ($\varepsilon = 0$) equation (1) not well posed.

$$\begin{aligned} b(\xi, y) &= 56 \sqrt[4]{\sin \xi |y|^3} \cdot I_{\{|y| < 2T^8\}} + |y| \cdot I_{\{|y| < 2T^8\}} \\ &\quad + 56 \sqrt[4]{8T^{24} \sin \xi} \cdot I_{\{|y| \geq 2T^8\}} + 2T^8 I_{\{|y| \geq 2T^8\}}. \end{aligned}$$

Stochastic wave equation: Abstract reformulation

$\Lambda = -\frac{d^2}{dx^2}$ with Dirichlet boundary conditions: in $U = L^2([0, 1])$

$$\mathcal{D}(\Lambda) = H_0^1([0, 1]) \cap H^2([0, 1]), \mathcal{D}(\Lambda^{1/2}) = H_0^1([0, 1]), \mathcal{D}(\Lambda^{-1/2}) = H^{-1}([0, 1]).$$

Set $X_\tau^{0,x} = (y(\tau), \frac{dy}{d\tau}(\tau))$, y solution to (1), $x = (x_0, x_1)$.

Wave operator: $A = \begin{pmatrix} 0 & I \\ -\Lambda & 0 \end{pmatrix}$ in $H = L^2([0, 1]) \times H^{-1}([0, 1])$

Λ positive self-adjoint operator on $U \rightsquigarrow H = U \times \mathcal{D}(\Lambda^{-1/2})'$

$$dX_\tau^{0,x} = AX_\tau^{0,x}d\tau + GB(\tau, X_\tau^{0,x})d\tau + GdW_\tau, \quad \tau \in [0, T], \quad X_0^{0,x} = x \in H.$$

where

$$GdW_\tau = \begin{pmatrix} 0 \\ dW_\tau \end{pmatrix}, \quad GB(\tau, X_\tau) = \begin{pmatrix} 0 \\ B(\tau, X_\tau) \end{pmatrix}$$

Setting of the problem

- $W_A(\tau) := \int_0^\tau e^{(\tau-s)A} G dW_s$ not well defined in $K = \mathcal{D}(\Lambda^{1/2}) \times U$ even if $B = 0$

- X evolves in $H = U \times \mathcal{D}(\Lambda^{-1/2}) = L^2([0, 1]) \times H^{-1}([0, 1])$.

- $B : [0, T] \times H \rightarrow U$ Borel, bounded and α -Holder continuous, $\alpha \in (2/3, 1)$:

$$|B(t, x + h) - B(t, x)|_U \leq C|h|_H^\alpha, \quad x, h \in H, \quad t \in [0, T], \quad \alpha \in (2/3, 1).$$

$$B \in B_b([0, T]; C_b^\alpha(H, U)), \quad GB \in B_b([0, T]; C_b^\alpha(H, H)).$$

- Existence of a weak solution: by the Girsanov Theorem

We prove **pathwise uniqueness** \rightsquigarrow **strong existence** by the Yamada-Watanabe principle (see [Ondreját, Dissertationes Math. 2004]).

Overview on regularization by noise

- A.K. Zvonkin : Mat. Sb.(N.S.) (1974) [$b \in L^\infty(\mathbb{R})$: $d = 1$]
- A.J. Veretennikov: Mat. Sb.(N.S.) (1980) [$b \in L^\infty(\mathbb{R}^d)$, $d \geq 1$].

Idea of the method: ODEs

- A variant of the **Zvonkin-Veretennikov approach**: the **Ito-Tanaka trick** for SDEs (cf. Flandoli-Gubinelli-Priola 2010) :

$b : \mathbb{R} \rightarrow \mathbb{R}$ be an irregular function (it could be Hölder continuous).

$$X_t = x + \int_0^t b(X_s)ds + W_t, \quad t \geq 0, \quad x \in \mathbb{R}.$$

Write

$$X_t - x - W_t = \int_0^t b(X_s)ds.$$

Overview on regularization by noise

Let v be a “regular” solution of

$$\lambda v - Lv = b \quad \text{on } \mathbb{R}, \quad \lambda > 0,$$

$L = \frac{1}{2} \frac{d^2}{dx^2} + b(x) \cdot \frac{d}{dx}$ then by Itô's formula:

$$v(X_t) = v(x) + \int_0^t v'(X_s) dW_s + \int_0^t Lv(X_s) ds$$

and so

$$v(X_t) = v(x) + \int_0^t v'(X_s) dW_s + \int_0^t (\lambda v(X_s) - b(X_s)) ds$$

and

$$X_t + v(X_t) = x + v(x) + W_t + \int_0^t v'(X_s) dW_s + \lambda \int_0^t v(X_s) ds$$

\Rightarrow **uniqueness thank to the regularity of v**

Overview on regularization by noise

Idea of the method: non degenerate **SPDEs**

$$dX_\tau^{0,x} = AX_\tau^{0,x}d\tau + B(\tau, X_\tau^{0,x})d\tau + \sqrt{Q}dW_\tau, \quad \tau \in [0, T], \quad X_0^{0,x} = x \in \mathcal{H}.$$

Assumptions

- $A : \mathcal{D}(A) \subset \mathcal{H} \rightarrow \mathcal{H}$, $Ae_n = -\alpha_n e_n$, $(\alpha_n)_{n \geq 1}$ non-decreasing
- $B \in C([0, T]; C_b^\alpha(H, H))$, set $B_n := \langle B, e_n \rangle$.
- either $\text{Tr } r(Q) < \infty$ or

$$\sum_{n=1}^{\infty} \frac{\|B_n\|_\alpha}{\alpha_n} < \infty.$$

- $Q_t = \int_0^t e^{sA} Q e^{sA^*} ds$ and $e^{tA}(H) \in Q_t^{1/2}(H)$, $t > 0$

We mainly refer to Da Prato-Flandoli, JFA 2010.

Idea of the method

- Kolmogorov PDEs:

$$\frac{\partial U_n(t, x)}{\partial t} + \mathcal{L}_t[U_n(t, \cdot)](x) = GB_n(t, x), \quad x \in H, \quad \mathcal{U}_n(T, x) = 0, \quad n \geq 1$$

$$\mathcal{L}_t[f](x) = \frac{1}{2} \operatorname{Tr} GG^* \nabla^2 f(x) + \langle Ax, \nabla f(x) \rangle + \langle GB(t, x), \nabla U_n(t, x) \rangle.$$

- $U(t, x) = \sum_{n=1}^{\infty} U_n(t, x) e_n$ ↪ Ito-Tanaka trick: Ito formula to $\mathcal{U}(t, X_t)$

$$B(t, X_t) = d\mathcal{U}(t, X_t) - \nabla \mathcal{U}(t, X_t) \sqrt{Q} dW_t$$

- mild form for X_t :

$$X_t = e^{tA} (x - U(0, x)) + U(t, X_t) + \int_0^t A e^{(t-s)A} U(s, X_s) ds$$

$$+ \int_0^t e^{(t-s)A} \nabla U(s, X_s) dW_s - \int_0^t e^{(t-s)A} \sqrt{Q} dW_s.$$

Wave transition semigroup

Coming back to our setting...

wave equation

$$dX_\tau^{t,x} = AX_\tau^{t,x}d\tau + GB(\tau, X_\tau^{t,x})d\tau + GdW_\tau, \quad \tau \in [t, T], \quad X_t^{t,x} = x \in H.$$

Ornstein Uhlenbeck process for the wave equation

$$d\Xi_\tau^{0,x} = A\Xi_\tau^{0,x}d\tau + GdW_\tau, \quad \tau \in [0, T], \quad \Xi_0^{0,x} = x \in H.$$

Wave transition semigroups

$$P_\tau [\phi] (x) = \mathbb{E}\phi (\Xi_\tau^{0,x}), \quad \phi \in B_b(H, \mathbb{R}), \quad R_\tau [\Phi] (x) = \mathbb{E}\Phi (\Xi_\tau^{0,x}), \quad \Phi \in B_b(H, H)$$

$(R_\tau)_{\tau \geq 0}$: H -valued transition semigroup

- **Regularizing properties:** from B_b functions, to differentiable and G -differentiable functions.

$$\nabla_a^G F(x) = \nabla_{Ga} F(x) = \lim_{s \rightarrow 0} \frac{F(x + sGa) - F(x)}{s} = \nabla_{Ga} F(x), \quad a \in U, \quad x \in H.$$

Regularizing properties

$$\begin{cases} \dot{w}(t) = Aw(t) + Gu(t), \\ w(0) = k \in H, \end{cases}$$

null controllable $\Leftrightarrow \text{Im } e^{tA} \subset \text{Im } Q_t^{1/2}$

- $|Q_t^{-1/2}e^{tA}h|_H \leq \frac{c}{t^{3/2}}|h|_H, \quad h \in H;$

$$\begin{cases} \dot{w}(t) = Aw(t) + Gu(t), \\ w(0) = k \in \text{Im}(G), \end{cases}$$

null controllable $\Leftrightarrow \text{Im } e^{tA}G \subset \text{Im } Q_t^{1/2}$

- $|Q_t^{-1/2}e^{tA}Ga|_H \leq \frac{c}{t^{1/2}}|\Lambda^{-1/2}a|_U, \quad a \in U; \quad (\text{M. AMO 2005, M-Priola JDE 2017})$

Damped wave + wave transition semigroup

Lemma (first & second order regularization) $\Phi \in C_b(H, H) \quad \forall t > 0$

$$\sup_{x \in H} |\nabla_k R_t[\Phi](x)| \leq \frac{c}{t^{\frac{3}{2}}} \|\Phi\|_\infty |k|_H;$$

$$\sup_{x \in H} |\nabla_a^G R_t[\Phi](x)| \leq \frac{c}{t^{\frac{1}{2}}} \|\Phi\|_\infty |\Lambda^{-1/2} a|_U, \quad \sup_{x \in H} \|\nabla^G R_t[\Phi](x)\|_{L_2(U, H)} \leq \frac{c}{t^{\frac{1}{2}}} \|\Phi\|_\infty.$$

$$\sup_{x \in H} \|\nabla_k (\nabla^G R_t[\Phi])(x)\|_{L_2(U, H)} \leq \frac{c |k|_H}{t^2} \|\Phi\|_\infty,$$

$$\lim_{x \rightarrow 0} \sup_{|k|=1} \sup_{y \in H} \|\nabla_k (\nabla^G R_t[\Phi])(x+y) - \nabla_k (\nabla^G R_t[\Phi])(y)\|_{L_2(U, H)} = 0.$$

Lemma (interpolation)

$\Phi \in C_b^\alpha(H, H), \quad k \in H, \quad \forall t > 0$

$$\sup_{x \in H} |\nabla_k R_t[\Phi](x)|_H \leq \frac{c}{t^{\frac{3}{2}(1-\alpha)}} \|\Phi\|_\alpha |k|_H;$$

$$\sup_{x \in H} \|\nabla_k (\nabla^G R_t[\Phi])(x)\|_{L_2(U, H)} \leq \frac{c}{t^{\frac{4-3\alpha}{2}}} \|\Phi\|_\alpha |k|_H.$$

Forward-Backward system (FBSDE)

$$\begin{cases} d\Xi_\tau^{t,x} = A\Xi_\tau^{t,x}d\tau + GdW_\tau, & \tau \in [t, T], \\ \Xi_t^{t,x} = x, \\ -dY_\tau^{t,x} = -AY_\tau^{t,x} + GB(\tau, \Xi_\tau^{t,x})d\tau - Z_\tau^{t,x}B(\tau, \Xi_\tau^{t,x})d\tau - Z_\tau^{t,x}dW_\tau, & \tau \in [0, T], \\ Y_T^{t,x} = 0, \end{cases}$$

A wave operator $\sim -A$ generator of a semigroup of operators

FBSDE in mild formulation

$$Y_\tau^{t,x} = \int_\tau^T e^{-(s-\tau)A} GB(s, \Xi_s^{t,x}) ds - \int_\tau^T e^{-(s-\tau)A} Z_s^{t,x} B(s, \Xi_s^{t,x}) ds - \int_\tau^T e^{-(s-\tau)A} Z_s^{t,x} dW_s,$$

Existence and regularity results (Hu-Peng SAP 1991, Guatteri JAMSA 2007 .) \exists a unique **solution** (Y, Z) s.t.

$$\mathbb{E} \sup_{\tau \in [0, T]} |Y_\tau^{t,x}|_H^2 + \mathbb{E} \int_0^T |Z_\tau|_{L_2(U, H)}^2 \leq C \sup_{t \in [0, T], x \in H} |B(t, x)|_U.$$

If moreover $x \mapsto B(\tau, x)$, $H \rightarrow U$, differentiable, for a.a. $\tau \in [0, T]$, then $x \mapsto (Y_t^{t,x}, Z_t^{t,x})$ is also differentiable.

Definition of $v(t, x) := Y_t^{t,x}$. $\rightsquigarrow x \mapsto v(t, x)$ differentiable.

Identification of Z with $\nabla^G v$

- If $x \mapsto B(\tau, x)$, $H \rightarrow U$, differentiable,

$$\nabla_k^G v(\tau, \Xi_\tau^{t,x}) = Z_\tau^{t,x} k \text{ in } H \text{ } \mathbb{P} \text{ a.s. for a.a. } \tau \in [t, T]$$

- If $x \mapsto B(\tau, x)$, $H \rightarrow U$, Hölder continuous: by an approximation procedure (see Peszat-Zabczyk AoP 1995) on B

$$B^n(\tau, x) = \int_{\mathbb{R}^n} \rho_n(y - Q_n x) B\left(\tau \sum_{i=1}^n y_i g_i\right) dy,$$

to B^n it is associated v^n

$$\nabla_k^G v^n(\tau, \Xi_\tau^{t,x}) = Z_\tau^{n,t,x} k \text{ in } H \text{ } \mathbb{P} \text{ a.s. for a.a. } \tau \in [t, T]$$

let $n \rightarrow \infty$.

Regularity of v

$$\begin{aligned}
 v(t, x) &= \int_t^T e^{-(s-t)A} GB(s, \Xi_s^{t,x}) ds - \int_t^T e^{-(s-t)A} Z_s^{t,x} B(s, \Xi_s^{t,x}) ds + \int_t^T e^{-(s-t)A} Z_s^{t,x} dW_s \\
 &= \int_t^T e^{-(s-t)A} GB(s, \Xi_s^{t,x}) ds - \int_t^T e^{-(s-t)A} \nabla^G v(s, \Xi_s^{t,x}) B(s, \Xi_s^{t,x}) ds \\
 &\quad + \int_t^T e^{-(s-t)A} \nabla^G v(s, \Xi_s^{t,x}) dW_s,
 \end{aligned}$$

taking expectation

$$\begin{aligned}
 v(t, x) &= \mathbb{E} \int_t^T e^{-(s-t)A} GB(s, \Xi_s^{t,x}) ds - \mathbb{E} \int_t^T e^{-(s-t)A} \nabla^G v(s, \Xi_s^{t,x}) B(s, \Xi_s^{t,x}) ds \\
 &= \int_t^T R_{s-t} \left[e^{-(s-t)A} GB(s, \cdot) \right] (x) ds - \int_t^T R_{s-t} \left[e^{-(s-t)A} \nabla^G v(s, \cdot) B(s, \cdot) \right] (x) ds
 \end{aligned}$$

$$\sup_{t \in [0, T_0], x \in H} \|\nabla v(t, x)\| \leq 1/2; \quad \nabla_k (\nabla^G u(t, x)) \in B_b([0, T]; L_2(U, H)), \quad k \in H.$$

$$dX_\tau^{t,x} = AX_\tau^{t,x}d\tau + \textcolor{red}{GB}(\tau, X_\tau^{t,x})d\tau + GdW_\tau, \quad \tau \in [t, T], \quad X_t^{t,x} = x,$$

mild formulation

$$X_\tau^{t,x} = e^{(\tau-t)A}x + \int_t^\tau e^{(\tau-s)A} \textcolor{red}{GB}(s, X_s)ds + \int_t^\tau e^{(\tau-s)A} G dW_s, \quad \tau \in [t, T].$$

Set: $\tilde{Y}_\tau^{t,x} := v(\tau, X_\tau^{t,x})$, $\tilde{Z}_\tau^{t,x} := \nabla^G v(\tau, X_\tau^{t,x})$. \rightsquigarrow solution to the BSDE

$$-\tilde{dY}_\tau^{t,x} = -A\tilde{Y}_\tau^{t,x}d\tau + \textcolor{blue}{GB}(\tau, X_\tau^{t,x})d\tau - \tilde{Z}_\tau^{t,x} dW_\tau, \quad \tilde{Y}_T^{t,x} = 0.$$

mild formulation of the BSDE

$$\tilde{Y}_\tau^{t,x} = \int_\tau^T e^{-(s-\tau)A} \textcolor{red}{GB}(s, X_s^{t,x}) ds - \int_\tau^T e^{-(s-\tau)A} \tilde{Z}_s^{t,x} dW_s. \quad \tau \in [0, T].$$

Link between the FBSDEs: $v(t, x) = Y_t^{t,x} = \tilde{Y}_t^{t,x}$

Proposition (M-Priola, JDE 2017) For $t = 0$ the mild of the stochastic wave equation can be rewritten as

$$X_\tau^{0,x} = e^{\tau A}x + e^{\tau A}v(0, x) - v(\tau, X_\tau^x) + \int_0^\tau e^{(\tau-s)A} \nabla^G v(s, X_s^x) dW_s + \int_0^\tau e^{(\tau-s)A} G dW_s.$$

The “bad” term B has been removed

Proof For $\tau \in [0, T]$

$$e^{-\tau A} \tilde{Y}_\tau^{0,x} = e^{-\tau A} \int_\tau^T e^{-(s-\tau)A} GB(s, X_s^{0,x}) ds - e^{-\tau A} \int_\tau^T e^{-(s-\tau)A} \tilde{Z}_s^{0,x} dW_s.$$

$$\tau = 0 : \tilde{Y}_0^{0,x} = v(0, x) = \int_0^T e^{-sA} GB(s, X_s^{0,x}) ds - \int_0^T e^{-sA} \tilde{Z}_s^{0,x} dW_s.$$

So

$$\int_0^\tau e^{(\tau-s)A} GB(s, X_s^x) ds = e^{\tau A}v(0, x) - v(\tau, X_\tau^x) - \int_0^\tau e^{(\tau-s)A} \nabla^G v(s, X_s^x) dW_s$$

Theorem (M-Priola, JDE 2017) For the stochastic wave equation (1) path-wise uniqueness holds. $\exists c > 0$ s. t.

$$\sup_{\tau \in [0, T]} E|X_\tau^{x_1} - X_\tau^{x_2}|_H^2 \leq c|x_1 - x_2|_H^2, \quad x_1, x_2 \in H$$

Proof X^1, X^2 starting at x_1, x_2 . By the proposition

$$\begin{aligned} X_\tau^1 - X_\tau^2 &= e^{\tau A}(x_1 - x_2) + e^{\tau A}[v(0, x_1) - v(0, x_2)] \\ &\quad - [v(\tau, X_\tau^1) - v(\tau, X_\tau^2)] + \int_0^\tau e^{(\tau-s)A} [\nabla^G v(s, X_s^1) - \nabla^G v(s, X_s^2)] dW_s. \end{aligned}$$

by regularity properties of v

$$\begin{aligned} &\|e^{\tau A}(x_1 - x_2)|_H + |e^{\tau A}[v(0, x_1) - v(0, x_2)]|_H + |v(\tau, X_\tau^1) - v(\tau, X_\tau^2)|_H \\ &\leq C|x_1 - x_2|_H + \frac{1}{2}|X_\tau^1 - X_\tau^2|_H \end{aligned}$$

by Ito isometry

$$\begin{aligned} & \mathbb{E} \left| \int_0^\tau e^{(\tau-s)A} [\nabla^G v(s, X_s^1) - \nabla^G v(s, X_s^2)] dW_s \right|^2 \\ & \leq \mathbb{E} \int_0^\tau \| \nabla^G v(s, X_s^1) - \nabla^G v(s, X_s^2) \|_{L_2(U,H)}^2 ds \end{aligned}$$

$(e_k)_k$ basis in U

$$\begin{aligned} \mathbb{E} \int_0^\tau \| \nabla^G v(s, X_s^1) - \nabla^G v(s, X_s^2) \|_{L_2(U,H)}^2 ds &= \sum_{k \geq 1} \mathbb{E} \int_0^\tau | \nabla_{e_k}^G v(s, X_s^1) - \nabla_{e_k}^G v(s, X_s^2) |_H^2 ds \\ &\leq \sup_{t,x} \sup_{|k|_H=1} \| \nabla_k \nabla^G v(t, x) \|_{L_2(U,H)}^2 \int_0^\tau \mathbb{E} |X_s^1 - X_s^2|_H^2 ds. \end{aligned}$$

Gronwall Lemma: for $x_1 = x_2 \rightsquigarrow$ uniqueness.

For $x_1, x_2 \in H$, $x_1 \neq x_2 \rightsquigarrow$ Lipschitz continuous dependence on the initial data.

By the Yamada-Watanabe Theorem: there exists a strong solution to (1).

Stochastic damped wave equation

Stochastic damped wave equation

$$\begin{cases} \frac{\partial^2 y}{\partial t^2}(t) = -\Lambda y(t) - \rho \Lambda^\alpha \frac{\partial y}{\partial t}(t) + b \left(t, y(t), \frac{\partial y}{\partial t}(t) \right) + \dot{W}(t), & t \in (0, T], \\ y(0) = y_0, \\ \frac{\partial y}{\partial t}(0) = y_1, \end{cases} \quad \bullet \quad \rho > 0, \alpha \in [0, 1) \quad (2)$$

- b bounded measurable, β -Hölder continuous in y , $\beta \in (\beta_\alpha, 1)$ with

$$\beta_\alpha = \begin{cases} \frac{2}{3}, & \alpha \in [0, \frac{3}{4}], \\ 2 - \frac{1}{\alpha}, & \alpha \in [\frac{3}{4}, 1), \end{cases}$$

- Without noise ($\varepsilon = 0$) equation (2) not well posed.

Stochastic damped wave equation

Stochastic damped wave equation: Abstract reformulation

$\Lambda = -\frac{d^2}{dx^2}$ with Dirichlet boundary conditions: in $U = L^2([0, 1])$

$\mathcal{D}(\Lambda) = H_0^1([0, 1]) \cap H^2([0, 1]), \mathcal{D}(\Lambda^{1/2}) = H_0^1([0, 1]), \mathcal{D}(\Lambda^{-1/2}) = H^{-1}([0, 1]).$

Set $X_\tau^{0,x} = (y(\tau), \frac{dy}{d\tau}(\tau))$, y solution to (2), $x = (x_0, x_1)$.

Damped wave operator: $\mathcal{A}_{\alpha,\rho} := \begin{pmatrix} 0 & I \\ -\Lambda & -\rho\Lambda^\alpha \end{pmatrix}$, in $H := L^2([0, 1]) \times H^{-1}([0, 1])$

$$dX_\tau^{0,x} = \mathcal{A}_{\alpha,\rho} X_\tau^{0,x} d\tau + GB(\tau, X_\tau^{0,x}) d\tau + GdW_\tau, \quad \tau \in [0, T], \quad X_0^{0,x} = x \in H.$$

where

$$G := \begin{pmatrix} 0 \\ I \end{pmatrix}, \quad GdW_\tau = \begin{pmatrix} 0 \\ dW_\tau \end{pmatrix}, \quad GB(\tau, X_\tau) = \begin{pmatrix} 0 \\ B(\tau, X_\tau) \end{pmatrix}$$

$$\begin{cases} dX_t = \mathcal{A}_{\alpha,\rho} X_t dt + G dW_t, & t \in [0, T], \\ X_0 = (x_0^1, x_0^2)^T \in H. \end{cases}$$

Well posedness in H in **mild sense**

$$X_t = e^{t\mathcal{A}_{\alpha,\rho}} x + \int_0^t e^{(t-s)\mathcal{A}_{\alpha,\rho}} G dW_s, \quad t \in [0, T].$$

\rightsquigarrow study the **stochastic convolution**

$$W_{\mathcal{A}_{\alpha,\rho}}(t) := \int_0^t e^{(t-s)\mathcal{A}_{\alpha,\rho}} G dW_s, \quad t \in [0, T],$$

\rightsquigarrow prove that $\int_0^T \|e^{t\mathcal{A}_{\alpha,\rho}} G\|_{L_2(U,H)}^2 dt < +\infty$.: assume Λ^{-1} trace class operator $U \rightarrow U$.

Covariance operator $Q_t^{\alpha,\rho} := \int_0^t e^{s\mathcal{A}_{\alpha,\rho}} G G^* e^{s\mathcal{A}_{\alpha,\rho}^*} ds$ trace class operator

spectral decomposition of $\mathcal{A}_{\alpha,\rho}$: Chen-Russel 1981, Triaggiani 1988, Lasiecka-Triggiani 2003

Damped wave transition semigroup

Damped wave equation

$$dX_\tau^{t,x} = \mathcal{A}_{\alpha,\rho} X_\tau^{t,x} d\tau + GB(\tau, X_\tau^{t,x}) d\tau + GdW_\tau, \quad \tau \in [t, T], \quad X_t^{t,x} = x \in H.$$

Ornstein Uhlenbeck process for the damped wave equation with $\alpha \in [0, \frac{3}{4})$

$$d\Xi_\tau^{0,x} = \mathcal{A}_{\alpha,\rho} \Xi_\tau^{0,x} d\tau + GdW_\tau, \quad \tau \in [0, T], \quad \Xi_0^{0,x} = x \in H.$$

Damped wave transition semigroups

$$P_\tau [\phi] (x) = \mathbb{E} \phi (\Xi_\tau^{0,x}), \quad \phi \in B_b(H, \mathbb{R}), \quad R_\tau [\Phi] (x) = \mathbb{E} \Phi (\Xi_\tau^{0,x}), \quad \Phi \in B_b(H, H)$$

$(R_\tau)_{\tau \geq 0}$: H -valued transition semigroup

- **Regularizing properties:** from B_b functions, to differentiable and G -differentiable functions.

Regularizing properties

$$\begin{cases} \dot{w}(t) = \mathcal{A}_{\alpha,\rho} w(t) + Gu(t), \\ w(0) = k \in H, \end{cases}$$

null controllable $\Leftrightarrow \text{Im } e^{t\mathcal{A}_{\alpha,\rho}} \subset \text{Im } Q_t^{1/2}$

- $|Q_t^{-1/2} e^{t\mathcal{A}_{\alpha,\rho}} h|_H \leq \frac{c}{t^{3/2}} |h|_H, \quad h \in H;$

$$\begin{cases} \dot{w}(t) = \mathcal{A}_{\alpha,\rho} w(t) + Gu(t), \\ w(0) = k \in \text{Im}(G), \end{cases}$$

null controllable $\Leftrightarrow \text{Im } e^{t\mathcal{A}_{\alpha,\rho}} G \subset \text{Im } Q_t^{1/2}$

- $|Q_t^{-1/2} e^{t\mathcal{A}_{\alpha,\rho}} Ga|_H \leq \frac{c}{t^{1/2}} |\Lambda^{-1/2} a|_U, \quad a \in U.$ (Addona-M-Priola 2021)

$-\mathcal{A}_{\alpha,\rho}$ damped wave operator $\rightsquigarrow -\mathcal{A}_{\alpha,\rho}$ is not the generator of a semigroup of operators

take $\mathcal{A}_{\alpha,\rho}^j$ Yosida approximants of $\mathcal{A}_{\alpha,\rho}$ and then pass to the limit

Forward-Backward system (FBSDE)

$$\begin{cases} d\Xi_\tau^{t,x,j} = \mathcal{A}_{\alpha,\rho}^j \Xi_\tau^{t,x,j} d\tau + G dW_\tau, & \tau \in [t, T], \\ \Xi_t^{t,x,j} = x, \\ -dY_\tau^{t,x,j} = -\mathcal{A}_{\alpha,\rho}^j Y_\tau^{t,x,j} + GB(\tau, \Xi_\tau^{t,x,j}) d\tau - Z_\tau^{t,x,j} B(\tau, \Xi_\tau^{t,x,j}) d\tau - Z_\tau^{t,x,j} dW_\tau, & \tau \in [0, T], \\ Y_T^{t,x} = 0, \end{cases}$$

Set $v^j(t, x) := Y_t^{t,x,j}$.

$$\begin{aligned} v^j(t, x) &= \mathbb{E} \int_t^T e^{-(s-t)\mathcal{A}_{\alpha,\rho}^j} GB(s, \Xi_s^{t,x}) ds - \mathbb{E} \int_t^T e^{-(s-t)\mathcal{A}_{\alpha,\rho}^j} \nabla^G v(s, \Xi_s^{t,x}) B(s, \Xi_s^{t,x}) ds \\ &= \int_t^T R_{s-t} \left[e^{-(s-t)\mathcal{A}_{\alpha,\rho}^j} GB(s, \cdot) \right] (x) ds - \int_t^T R_{s-t} \left[e^{-(s-t)\mathcal{A}_{\alpha,\rho}^j} \nabla^G v(s, \cdot) B(s, \cdot) \right] (x) ds \end{aligned}$$

\rightsquigarrow no convergence here if we let $j \rightarrow +\infty$.

regularity of v from regularizing properties of R_t

FBSDE- damped wave equation

$$v^j(t, x) = \int_t^T R_{s-t} \left[e^{-(s-t)\mathcal{A}_{\alpha,\rho}^j} GB(s, \cdot) \right] (x) ds - \int_t^T R_{s-t} \left[e^{-(s-t)\mathcal{A}_{\alpha,\rho}^j} \nabla^G v(s, \cdot) B(s, \cdot) \right] (x) ds$$

Set $\tilde{v}^j(t, x) := e^{(T-\tau)\mathcal{A}_{\alpha,\rho}^j} Y_t^{t,x,j}$. We can prove that \tilde{v}^j satisfies

$$\tilde{v}^j(t, x) := \int_t^T R_{s-t} \left[e^{(T-s)\mathcal{A}_{\alpha,\rho}^j} G \tilde{C}(s, \cdot) \right] (x) ds + \int_t^{T_0} R_{s-t} \left[\nabla^G \tilde{v}^j(s, \cdot) \tilde{C}(s, \cdot) \right] (x) ds,$$

Now we can we let $j \rightarrow +\infty$: $\tilde{v}(t, x) = \lim_{j \rightarrow +\infty} \tilde{v}^j(t, x)$

$$\tilde{v}(t, x) := \int_t^T R_{s-t} \left[e^{(T-s)\mathcal{A}_{\alpha,\rho}} G \tilde{C}(s, \cdot) \right] (x) ds + \int_t^{T_0} R_{s-t} \left[\nabla^G \tilde{v}(s, \cdot) \tilde{C}(s, \cdot) \right] (x) ds,$$

$$dX_\tau^{t,x} = \mathcal{A}_{\alpha,\rho} X_\tau^{t,x} d\tau + \textcolor{red}{GB}(\tau, X_\tau^{t,x}) d\tau + G dW_\tau, \quad \tau \in [t, T], \quad X_t^{t,x} = x \in H.$$

mild formulation

$$X_\tau^{t,x} = e^{(\tau-t)\mathcal{A}_{\alpha,\rho}}x + \int_t^\tau e^{(\tau-s)\mathcal{A}_{\alpha,\rho}} \textcolor{red}{GB}(s, X_s) ds + \int_t^\tau e^{(\tau-s)\mathcal{A}_{\alpha,\rho}} G dW_s, \quad \tau \in [t, T].$$

By the definition of \tilde{v}^j and by BSDEs techniques

$$\begin{aligned} \int_0^\tau e^{(\tau-s)\mathcal{A}_{\alpha,\rho}^j} \textcolor{blue}{GB}(s, X_s^x) ds &= e^{\tau\mathcal{A}_{\alpha,\rho}^j} Y_0^{0,x,j} + \int_0^\tau e^{(\tau-s)\mathcal{A}_{\alpha,\rho}^j} Z_s^{0,x,j} dW_s \\ &= \tilde{v}^j(0, x) + \int_0^\tau \nabla^G \tilde{v}^j(s, X_s^x) dW_s, \quad \forall \tau \in [0, T]. \end{aligned}$$

Proposition (Addona-M-Priola) For $t = 0$ the mild form of the damped stochastic wave equation can be rewritten as

$$\begin{aligned} X_\tau^{0,x} = & e^{\tau \mathcal{A}_{\alpha,\rho}} x + \int_0^\tau \left(e^{(\tau-s)\mathcal{A}_{\alpha,\rho}} - e^{(\tau-s)\mathcal{A}_{\alpha,\rho}^j} \right) B(s, X_s^x) ds + \tilde{v}^j(0, x) \\ & + \int_0^\tau \nabla^G \tilde{v}^j(s, X_s^x) dW_s + \int_0^\tau e^{(\tau-s)\mathcal{A}_{\alpha,\rho}} G dW_s. \end{aligned}$$

The “bad” term B appears through

$$\int_0^\tau \left(e^{(\tau-s)\mathcal{A}_{\alpha,\rho}} - e^{(\tau-s)\mathcal{A}_{\alpha,\rho}^j} \right) B(s, X_s^x) ds := \delta^j(t) \rightarrow 0, \text{ as } j \rightarrow \infty$$

Theorem (Addona-M-Priola) For the stochastic damped wave equation (2) **pathwise uniqueness** holds. $\exists c > 0$ s. t.

$$\sup_{\tau \in [0, T]} E|X_\tau^{x_1} - X_\tau^{x_2}|_H^2 \leq c|x_1 - x_2|_H^2, \quad x_1, x_2 \in H$$

Proof X^1, X^2 starting at x_1, x_2 . By the proposition

$$X_\tau^1 - X_\tau^2 = e^{\tau A_{\alpha,\rho}}(x_1 - x_2) + e^{\tau A_{\alpha,\rho}}[\tilde{v}^j(0, x_1) - \tilde{v}^j(0, x_2)] \\ + \delta_1^j(t) + \delta_2^j(t) + \int_0^\tau e^{(\tau-s)A_{\alpha,\rho}} [\nabla^G \tilde{v}(s, X_s^1) - \nabla^G \tilde{v}(s, X_s^2)] dW_s.$$

by regularity properties of v

$$|e^{\tau A_{\alpha,\rho}}(x_1 - x_2)|_H + |e^{\tau A_{\alpha,\rho}}[\tilde{v}^j(0, x_1) - \tilde{v}^j(0, x_2)]|_H \leq C|x_1 - x_2|_H \\ \mathbb{E} \left| \int_0^\tau e^{(\tau-s)A_{\alpha,\rho}} [\nabla^G \tilde{v}^j(s, X_s^1) - \nabla^G \tilde{v}^j(s, X_s^2)] dW_s \right|^2 \leq C \mathbb{E} |X_s^1 - X_s^2|_H^2 ds$$

Gronwall Lemma: for $x_1 = x_2 \rightsquigarrow$ uniqueness.

For $x_1 \neq x_2 \rightsquigarrow$ Lipschitz continuous dependence on the initial data.

By the Yamada-Watanabe Theorem: there exists a strong solution to (2).

A unified BSDE approach

- **wave equation:** Study a FBSDE, set $v(t, x) := Y_t^{t,x}$ that solves

$$\begin{aligned} v(t, x) = & \int_t^T R_{s-t} \left[e^{-(s-t)A} GB(s, \cdot) \right] (x) ds \\ & - \int_t^T R_{s-t} \left[e^{-(s-t)A} \nabla^G v(s, \cdot) B(s, \cdot) \right] (x) ds \end{aligned}$$

- **damped wave equation** Study an approximated FBSDE, set $\tilde{v}^j(t, x) := e^{T_0-\tau}\mathcal{A}_{\alpha,\rho}^j Y_t^{t,x,j}$, \tilde{v}^j satisfies

$$\tilde{v}^j(t, x) := \int_t^{T_0} R_{s-t} \left[e^{(T_0-s)\mathcal{A}_{\alpha,\rho}^j} G \tilde{C}(s, \cdot) \right] (x) ds + \int_t^{T_0} R_{s-t} \left[\nabla^G \tilde{v}^j(s, \cdot) \tilde{C}(s, \cdot) \right] (x) ds$$

- **evolution equations of parabolic type:** start from the integral PDE above, with different approximations of the operator A

$$\begin{cases} dX_\tau^{t,x} = AX_\tau^{t,x}d\tau + GB(\tau, X_\tau^{t,x})d\tau + GdW_\tau, & \tau \in [t, T], 0 \leq t \leq \tau, \\ X_t^{t,x} = x \in H, \end{cases}$$

$X_\tau^{t,x} \rightsquigarrow \Xi_\tau^{0,x}$, if $B = 0$; $\textcolor{red}{R}_t[\Phi](x) := \mathbb{E}[\Phi(\Xi_t^{0,x})]$, $t \geq 0$, $\Phi \in B_b(H, H)$

$$v(t, x) = \int_t^{\mathcal{T}} R_{s-t} \left[e^{(\mathcal{T}-s)A_0} GBs, \cdot \right] (x) ds + \int_t^{\mathcal{T}} R_{s-t} \left[\nabla^G v(s, \cdot) B(s, \cdot) \right] (x) ds,$$

$$u_n^{\mathcal{T}}(t, x) = \int_t^{\mathcal{T}} R_{s-t} \left[e^{(\mathcal{T}-s)A_n} GB(s, \cdot) \right] (x) ds + \int_t^{\mathcal{T}} R_{s-t} \left[\nabla^G u_n^{\mathcal{T}}(s, \cdot) B(s, \cdot) \right] (x) ds.$$

A_0 generator of a C_0 -semigroup; A_n generator of a C_0 - **group** of operators

$$\sup_{t \in [0, T]} \sup_{n \geq 1} \|e^{tA_n}\|_{L(H)} = K_T < \infty, \quad \lim_{n \rightarrow \infty} e^{tA_n} x = e^{tA} x, \quad x \in H, \quad t \geq 0.$$

$$\sup_{x \in H} |u_n^{\mathcal{T}}(0, x+y) - u_n^{\mathcal{T}}(0, x)|_H \leq C_T |y|_H, \quad y \in H,$$

$$\sup_{x \in H} \|\nabla^G u_n^{\mathcal{T}}(t, x+y) - \nabla^G u_n^{\mathcal{T}}(t, x)\|_{L_2(U; H)}^2 \leq h(\mathcal{T} - t) |y|_H^2, \quad t \in (0, \mathcal{T}), \quad y \in H.$$

A unified BSDE approach

$$\begin{cases} dX_\tau^{0,x} = AX_\tau^{0,x}d\tau + GB(\tau, X_\tau^{0,x})d\tau + GdW_\tau & \tau \in [0, T], \\ X_0^{0,x} = x \in H, & X_\tau^x := X_\tau^{0,x} \end{cases}$$

$$X_\tau^x = e^{\tau A}x + \int_0^\tau \left(e^{(\tau-s)A} - e^{(\tau-s)A_n} \right) GB(s, X_s^x)ds + \int_0^\tau e^{(\tau-s)A_n} GB(s, X_s^x)ds \\ + \int_0^\tau e^{(\tau-s)A} GdW_s, \quad \forall \tau \in [0, T].$$

Proposition (Addona-M-Priola) For any $n \in \mathbb{N}$ and any $\tau \in [0, T]$ we have

$$X_\tau^x = e^{\tau A}x + \int_0^\tau \left(e^{(\tau-s)A} - e^{(\tau-s)A_n} \right) GB(s, X_s^x)ds \\ + u_n^\tau(0, x) + \int_0^\tau \nabla^G u_n^\tau(s, X_s^x)dW_s + \int_0^\tau e^{(\tau-s)A} GdW_s, \quad \mathbb{P}\text{-a.s..}$$

Theorem $\exists c = c(T) > 0$ s.t. $\forall x_1, x_2 \in H$

$$\sup_{t \in [0, T]} \mathbb{E}[|X_t^{x_1} - X_t^{x_2}|_H^2] \leq c|x_1 - x_2|_H^2,$$

~~~ uniqueness + Lipschitz continuous dependence

- Formally  $u_n^\mathcal{T}$  solves

$$\begin{cases} \frac{\partial u_n^\mathcal{T}(t, x)}{\partial t} + \mathcal{L}_t[u_n^\mathcal{T}(t, \cdot)](x) = -e^{(\mathcal{T}-t)A_n}GB(t, x), & x \in H, t \in [0, \mathcal{T}], \\ u_n^\mathcal{T}(\mathcal{T}, x) = 0, & x \in H, \end{cases}$$

where  $\mathcal{L}_t f(x) := \frac{1}{2}\text{Tr}[GG^*\nabla^2 f(x)] + \langle Ax, \nabla f(x) \rangle + \langle GB(t, x), \nabla f(x) \rangle$

~~~ we perform regularization by noise for stochastic heat equation in **dimension  $d = 3$**  not reached in Da Pranto-Flandoli JFA 2010

Da Pranto-Flandoli JFA 2010 $-e^{(\mathcal{T}-t)A_n}GB$ is replaced by **GB** .

- $v_n := e^{-(\mathcal{T}-t)A_n}u_n^\mathcal{T}$ formally solves

$$\begin{cases} \frac{\partial v_n^\mathcal{T}(t, x)}{\partial t} + \mathcal{L}_t[v_n^\mathcal{T}(t, \cdot)](x) = A_n v_n^\mathcal{T}(t, x) - GB(t, x), & x \in H, t \in [0, \mathcal{T}], \\ v_n^\mathcal{T}(\mathcal{T}, x) = 0, & x \in H, \end{cases}$$

similar to the PDE used for the wave equation

Further developments

- stochastic (damped) wave equations with multiplicative noise: gradient estimates to prove regularizing properties of the transition semigroup
- application to regularization by noise
- application of dilations theorems (van Neerven, Veraar) that give a group of operators in a larger space.
- BSDE approach for other problems of regularization by noise

Short Biblio

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