Regulation of natural resources exploitation.

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†II. *Scaling limit for stochastic control problems in population dynamics*, arXiv

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How incentivize optimally an agent for higher interests than his/her owns?

A renewable natural resource is managed by a natural resource manager. A regulator incentivizes the natural resource manager to ensure the sustainability of the resource.

The natural ressource

The logistic equation

$$X_t = X_0 + \int_0^t X_s(\nu - \mu - \lambda X_s)ds \ , \quad t \in [0, T] \ ,$$

where

- ν, μ are the birth and death rates;
- λ is the interspecies competition rate.

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The stochastic logistic equation.

$$X_t = X_0 + \int_0^t X_s(\nu - \mu - \lambda X_s)ds + \int_0^t \sigma X_s dW_s \;, \quad t \in [0, T].$$

The natural ressource

The natural resource abundance X_t^λ under the harvesting/renewing strategy is given by

$$X_t^\lambda = x + \int_0^t \left(X_s^\lambda(\nu - \mu - \lambda(X_s^\lambda)) - \alpha_s X_s^\lambda\right) ds + \int_0^t \sigma X_s^\lambda dW_s^\alpha \ , \quad t \in [0, T] \ .$$

- ν, μ are the birth and death rates;
- λ is the interspecies competition rate.
- $\alpha_t X_t^\lambda$ is the speed of the exploitation of the resource at time t.
- Change of Brownian motion (weak formulation).

A bilevel optimization

$$\textcolor{red}{\text{(Regulator's Problem)}} \qquad \sup_{\xi} \mathbb{E}^{\alpha^\star(\xi)}\Big[\xi - f(X_t^\lambda)\Big],$$

where

- ξ is a compensation/tax proposes to the NRM;
- f is a cost function depending on the size of the resource at T;

subjected to

▷ for ξ fixed,

$$\textcolor{blue}{\text{(NRM's Problem)}} \quad V^A(\xi) = \sup_{\alpha\in\mathcal{A}} V^A(\xi;\alpha) = V^A(\xi;\alpha^\star(\xi)),$$

with

$$V^A(\xi;\alpha) := \mathbb{E}^\alpha\Big[-\exp\Big(-\gamma\big(\underbrace{\int_0^T p(X_s^\lambda)X_s^\lambda\alpha_s ds}_{\text{incomes of the NRM}} - \underbrace{\int_0^T \frac{|\alpha_s|^2}{2} ds}_{\text{Exploitation costs}} - \underbrace{\xi}_{\text{tax}}\big)\Big)\Big].$$

▷ $V_0^A(\alpha^\star(\xi)) \geqslant R_0$.

Step 1. NRM optimization.

Theorem

Let $\xi \in \Xi$. There exists a unique pair (Y_0, Z) such that

- *the tax has the following decomposition*

$$\xi = Y_T^{Y_0,Z} = Y_0 - \int_0^T \left(g(X_t^\lambda, Z_t) + \frac{\sigma^2}{2}\gamma|Z_t|^2\right)dt + \int_0^T \sigma Z_t dW_t \ ,$$

  where g is defined for any $(x,z) \in \mathbb{R}_+ \times \mathbb{R}$ by

$$g(x,z) = \frac{|a^\star(x,z)|^2}{2} - p(x)xa^\star(x,z) - a^\star(x,z)z,$$

  with

$$a^\star(x,z) = \left((p(x)x + z) \vee (-\underline{M})\right) \wedge \overline{M} \ ,$$

- $V^A(\xi) = -\exp(\gamma Y_0)$ *, and the process $\alpha^\star(\xi)$ defined by $\alpha_t^\star(\xi) = a^\star(X_t^\lambda, Z_t)$ is the unique optimal effort associated with the tax ξ.*

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The proof uses the existence and uniqueness of the BSDE associated with the NRM's problem.

Step 2. The optimal contract

$$\text{(Regulator's Problem)} \quad \sup_{Z} \mathbb{E}^{\alpha^\star(X^\lambda, Z)}\left[Y_T^{\tilde{R},Z} - f(X_t^\lambda) \right],$$

with $\tilde{R} := \log(-R_0)/\gamma$.

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$$\Longleftrightarrow$$

$$\begin{cases} -\partial_t v - H\Big(x, \partial_x v(t,x), \partial_{xx} v(t,x)\Big) = 0\ , & (t,x) \in [0,T) \times \mathbb{R}_+^* \ , \\ v(T,x) = -f(x)\ , \quad x \in \mathbb{R}_+^*\ , \end{cases}$$

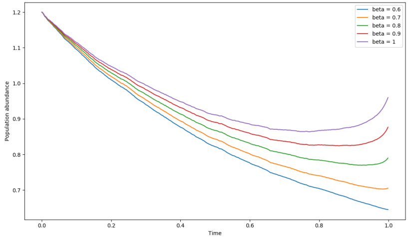
where the Hamiltonian H is given by

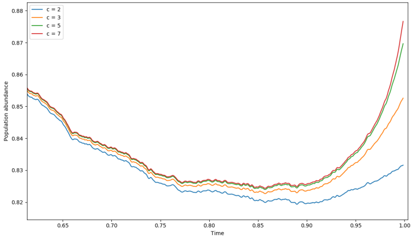
$$\begin{aligned} H(x,\delta_1,\delta_2) = \sup_{z\in\mathbb{R}} \Big\{ & xp(x)\alpha^\star(x,z) - k(\alpha^\star(x,z)) - \frac{\sigma^2}{2}\gamma z^2 + x(\nu - \mu - \lambda(x) - \alpha^\star(x,z))\delta_1 \Big\} \\ & + \frac{\sigma^2}{2} x^2 \delta_2\ , \quad (x,\delta_1,\delta_2) \in \mathbb{R}_+^* \times \mathbb{R} \times \mathbb{R}. \end{aligned}$$

Up to technical conditions, we apply a verification result to get the optimal Z and so the optimal contract proposed to the NMR by the regulator.

Numerical analyzis

$f(x) = c(\beta - x)^+$ for a target $\beta > 0$ and a cost $c > 0$.





Part II: scaling limit in population dynamics

1. What about the relevancy of using a continuous process compared with a (natural) birth and death process to model the dynamic of the natural resource?

2. Convergence of the solutions for associated stochastic control problems with these models?

Scaling limit of birth/death process

- Scaling parameter $K > 0$ of the population size;
- the number of birth by a Poisson process N^b with intensity $\lambda_t^{K,b} = \nu X_t^K K + \frac{\sigma^2}{2} X_t^K K^2$;
- the number of death by a Poisson process N^d with intensity $\lambda_t^{K,d} = \mu X_t^K K + \frac{\sigma^2}{2} X_t^K K^2$;
- the rescaled population process X^K by

$$X^K = x_0 + \frac{N^b - N^d}{K}, \; x_0 \in \mathbb{N}.$$

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Theorem

The sequence of processes $(X_t^K, \; t \in [0,T])_{K>0}$ converges in law (for the Skorohod topology) to the continuous diffusion process $(X_t, \; t \in [0,T])_{K>0}$ solution to the stochastic Feller differential equation

$$X_t = x_0 + \int_0^t (\nu - \mu) X_s ds + \int_0^T \sigma \sqrt{X_s} dW_s,$$

where W is a brownian motion under "a larger" probability space.

Controlled problem: a toy model in the discrete case

We consider a natural resource manager modifying the death rate of the resource with an action α so that

$$\lambda_t^{K,d,\alpha} := KX_t^K(\mu + K\frac{\sigma^2}{2}) + KX_t^K\alpha_t.$$

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The agent is assumed to be penalized

- if he fails at reaching a fixed level $\tilde{x} > 0$ of the resource at time T determined by a regulator.
- by the instantaneous amount $\frac{|\alpha X^K|^2}{2}$ per unit of time for a given effort α fixed.

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The problem of the resource manager is thus to solve

$$(\mathbf{TM})_K : \;\; V_0^K = \sup_{\alpha \in \mathcal{A}^K} \mathbb{E}^{K,\alpha}[-\gamma(X_T^K - \tilde{x})^2 - \int_0^T \frac{(\alpha_s X_s^K)^2}{2} \mathrm{d}s]$$

The Hamilton Jacobi Bellman equation associated to the control problem $(\mathbf{TM})_K$ is given by

$$(\mathbf{HJB})_K \begin{cases} \partial_t U^K(t,x) + H^K(x, D_+^K U^K(t,x), D_-^K U^K(t,x)) = 0, \\ U^K(T,x) = -\gamma(x-\tilde{x})^2, \quad x \in (\mathbb{N}^*/K), \end{cases}$$

for some Hamiltonian H^K with $D_\pm^K U^K(t,x) = U^K(t, x \pm 1/K) - U^K(t,x)$.

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$$\Longleftrightarrow \begin{cases} Y_t^K & = \xi^K + \int_t^T \frac{(KZ_s^{K,d})^2}{2} \mathbf{1}_{X_s^K > 0} ds - \int_t^T Z_s^K \cdot dM_s^K. \\ \xi^K & = -\gamma(X_T^K - \tilde{x})^2. \end{cases}$$

Corresponding problem in the continuous case

In the continuous framework, the problem becomes

$$(\mathbf{TM}): \;\; V_0 = \sup_{\alpha \in \mathcal{A}} \mathbb{E}[-\gamma(X_T - \tilde{x})^2 - \int_0^T \frac{(\alpha_s X_s)^2}{2} \mathrm{d}s],$$

with

$$\mathrm{d}X_t = (\nu - \mu - \alpha_t) X_t \mathrm{d}t + \sigma \sqrt{X_s} \mathrm{d}W_t.$$

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$$(\mathbf{HJB}) \begin{cases} \partial_t U(t,x) + H(x, DU(t,x), \Delta U(t,x)) = 0, & (t,x) \in [0,T) \times \mathbb{R}^+, \\ U(T,x) = -\gamma(x - \tilde{x})^2, \quad x \in \mathbb{R}^+, \end{cases}$$

$$\Longleftrightarrow$$

$$Y_t = \xi + \int_t^T \frac{Z_s^2}{2} \mathbf{1}_{X_s > 0} ds - \int_t^T Z_s \sigma \sqrt{X_s} dW_s$$

$$\xi = -\gamma(X_T - \tilde{x})^2.$$

Illustrative example: convergence of solutions

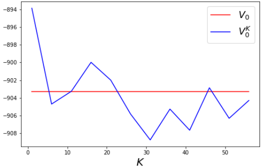

Figure: $\lim_{K \to +\infty} Y_0^K = Y_0$

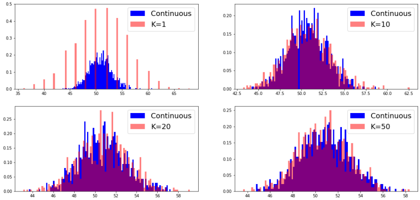

Figure: $\lim_{K \to +\infty} \alpha^{*,K} \sim Z^K \overset{\text{in law}}{=} \alpha^* \sim Z$

Extension to non-Markovian problems

For non-Markovian problems, we need to investigate the convergence of BSDE driven by sequences of martingales.

- Extension of Briand, Delyon, and Mémin (2002). *On the robustness of backward stochastic differential equations.*
- We have the convergence of the corresponding value functions (Y components of the BSDEs considered) and the weak convergence of the Z component.
- See other stability results for general classes of BSDEs in Papapantoleon, Possamaï and Saplaouras. (2022).