Optimal switching problems with an infinite set of modes: an approach by randomization and constrained backward SDEs

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9th BSDE international Colloquium
Université Savoie Mont Blanc (27/06 - 01/07/2022)

Outline of the talk

I- Preliminaries & motivations

- The Optimal Switching problem (OSP): primal vs dual formulation.
- Assumptions for the dual formulation.
- Why choosing the "dual" approach ?

II- Main results & perspectives

- The two main results:
  - (i): equality between the two value functions;
  - (ii): new BSDE characterization.
- Perspectives

I.1 Primal optimal switching problem and value function

On a standard prob. space $(\Omega, \mathbb{F}, \mathbb{P})$, let

- W: standard d-dim. Brownian Motion, W $\mathbb{F}$-adapted.
  usually: $\mathbb{F} = \mathcal{F}^W \vee \mathcal{N}$.
- T fixed finite horizon; A set of modes (possibly **infinite**).
- $\forall\ (x_0, e) \in \mathbb{R}^n \times A$, let X^e proc. s.t.

$$\forall\ t \in [0, T],\quad X^e_t = x_0 + \int_0^t \left(b^e(s, X^e_\cdot) ds + \sigma^e(s, X^e_\cdot) dW_s \right),$$

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Let $(f^e)_e$, $(g^e)_e$ and $(c_{e,e'})_{(e,e')}$: 3 families of (possib. **random**) real-valued data

(i) $f^e(s, X_\cdot)$: instant. profit (when system in mode e)

(ii) $g^e(X_\cdot)$: payoff at time T when syst. in mode e,

(iii) $c_{e,e'}(s, X_\cdot)$: *nonnegative* penalty costs incurred at time s when switching from e to e'.

I.1 Primal optimal switching problem and value function

- Mathematical assumptions:
  • A: Borel set (example: any subspace of $\mathbb{R}^d$);
  • Both $(b^e, \sigma^e)_e$, (f^e, g^e), $(c_{e,e'})_{e,e'}$ may be path-dependent;
  • Let $\mathbb{C}^n$: set of continuous paths $(s \mapsto x(s))_{s\in[0,T]}$
  Topology on $\mathbb{C}^n$: $|x|_* = \sup_{s\in[0,T]} |x(s)|$

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  • Measurability

  $(t, \omega, e) \mapsto b^e(t, x(\omega), \omega)$, $\sigma^e(t, \omega, x(\omega), e)$ are $Prog(\mathbb{C}^n) \otimes \mathcal{B}(A)$ meas.; (similar for $f^e, g^e, c_{e,e'}$)

  $Prog(\mathbb{C}^n)$: σ-algebra of prog. measurable maps on $[0, T] \times \Omega$.

I.1 Primal optimal switching problem and value function

- Math. Assumptions (cont'):

• For every t in $[0, T]$,
$(x, e) \mapsto b_t(x, e)\ \sigma_t(x, e),\ \ f_t(x, e), g(x, e)$ are continuous on $\mathbb{C}^n \times A$ $(x, e, e') \mapsto c_t(x, e, e')$ is continuous on $\mathbb{C}^n \times A \times A$.

• Regularity & growth assumpt (wrt x):
$\exists\ \ K > 0$ s.t. $\forall\ (t, x, x', e, e') \in [0, T] \times \mathbb{C}^n \times \mathbb{C}^n \times A \times A$,

(i) $|b_t(x, e) - b_t(x', e)| + |\sigma_t(x, e) - \sigma_t(x', e)| \leq K|x - x'|_{t*}$
Similar for other data.

(ii) $|b(t, 0, e)| + |\sigma(t, 0, e)| \leq K$;

I.1 Primal optimal switching problem and value function

• Growth assumpt wrt x (cont')

$\exists\ r,\ K > 0$ s.t. $\forall\ (t, x, x', e, e') \in [0, T] \times \mathbb{C}^n \times \mathbb{C}^n \times A \times A$,

(iii) $|f(t, x, e| + |g(x, e)| + |c(t, x, e, e')| \leq K\big(1 + |x|^r_{t*}\big)$

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Comment

(i)-(iii) standard to obtain estim.

(a) Estim. (of Hilbertian norm) of process X^e (see Cosso-Confortola-Fuhrman '18);
(b) Estim. of the value function (well known in Markovian case).

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I.1 Primal optimal switching problem and value function

1. Let $\alpha = (\tau^n, \xi^n)_{n\geq 1}$ with $\tau^1 > 0$. α = *management* strategy
2. To α, we associate the state proc. a as follows

$$a_s = \xi^1 \mathbf{1}_{s<\tau_1} + \sum_{n\geq 1} \xi^{n+1} \mathbf{1}_{\tau^n \leq s < \tau_{n+1}} \mathbf{1}_{\tau^n < T}$$

a: piecewise constant proc. A-valued

By abuse, one may replace α by a.

I.1 Primal value function: Admissible set $\mathcal{A}$

$a = (\tau^n, \xi^n)$ is said *admissible* (a in $\mathcal{A}$ if

H$_1$ $(\tau^n(\cdot), \xi^n(\cdot))_n$-$\mathbb{R}^+ \times A$-valued $\mathbb{F}$-adapt. such that $\tau_n(\omega) \to +\infty$ and $\tau^n < \tau^{n+1}$, $\mathbb{P}$-a.s

simultaneous switchings prohibited: equivalent to

$$\forall (a_1, a_2, a_2) \in A^3, \quad c_{a_1,a_2}(t,x) + c_{a_2,a_3}(t,x) > c_{a_1,a_3}(t,x)$$

Stronger than the no-loop property (in finite case).

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H₂ **H₁** implies: $N_T^a(\omega) = \text{Card}\{\tau^n(\omega), \tau^n < T\} < \infty$, $\mathbb{P}$-a.s

H₃ Impose $\tau^n \neq T$: **no switching at terminal time.**
In finite case, equivalent to:

$$\forall\, (i,j) \in A \times A, \;\; g^i(X_T^i) > g^j(X_T^j) - c_{i,j}(T, X_T).$$

I.1 Primal optimal switching problem and value function

1. For a in $\mathcal{A}$, let X^α (or X^a) the controlled proc. s.t.

$$dX^a = b^a(s, X^a)ds + \sigma^a(s, X^a_s)dW_s$$

with $b^a(s,x) = b^{\xi_0}(s,x)\mathbf{1}_{s<\tau^1} + \sum_{n\geq 1} b^{\xi^n}(s,x)\mathbf{1}_{\tau^n\leq s<\tau^{n+1}}$.

Similar definition for $\sigma^a(s,x)$.

Remark: b and σ path-dependent $\Rightarrow$ X^a no more Markovian (PDE approach not available).

I.1 Primal control problem and (primal) value function

1. Primal value function $\mathcal{V}$

$$\mathcal{V} = \sup_{\alpha \in \mathcal{A}} \left(\mathcal{J}(\alpha)\right), \text{ where}$$

$$J(\alpha) = \mathbb{E}\left(g^{a_T}(X_\cdot) + \int_0^T f^{a_s}(s, X_\cdot^a)ds - \sum_{\substack{n\geq 1,\\ \tau n < T}} c_{\xi_{n-1},\xi_n}(\tau^n, X_{\tau^n}^a)\right)$$

$$= J_1(\alpha) - J_2(\alpha)$$

Objective: choose the best a (or α) to optimize $J(\alpha)$ and minimize $\mathcal{J}_2(\alpha)$.

A (non exhaustive) review of the literature

(1) **OSP with finite set of modes:**

- (i) Using PDE approaches: Ishii-Koike '91, Yong-Zhou '99, Ludkowski '05, Carmona-Ludkovski '07-08, ...
- (ii) Using BSDE and analyt. tools: Hamadène-Jeanblanc '02, Djehiche-Hamadene-Popier '08, Hu-Tang '07 Chassagneux-Elie Kharroubi; Elie-Kharroubi '08 '11, ...
- (iii) Standard OSP with refinements: infinite horizon, partial information, non positive costs: Lundstrom -Olofsson, R. Martyr, B. El Asri, ..

(2) **Connection between "finite" OSP & constrained BSDE:**

(a) Ma-Pham-Kharroubi '08 (Markovian setting)

(b) Elie-Kharroubi ('14) (Non Markovian case)

I.2 Randomized set-up & dual formulation

1. On $(\Omega', \mathbb{F}', \mathbb{P}')$ let $\mu = \sum_{n\geq 0} \delta_{\sigma^m, \zeta^m}$ be a Poisson random meas. s.t.

   (i) Random dates & marks $(\sigma^m, \zeta^m)_m$ $\mathbb{R}^+ \times A$-valued;

   (ii) μ **indep.** of W with $\hat{\mu}(de, ds) = \lambda(de)ds$ s.t

   (a) $\tilde{\mu} = \mu - \hat{\mu}$ is a martingale measure;

   (b) $\lambda(de)$ has **full support** and $\lambda(\mathbf{A}) < +\infty$.

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2. The *randomized* dual set up := $(\overline{\Omega}, \overline{\mathbb{P}}, \overline{\mathcal{F}}, \overline{W}, \overline{\mu})$:

   (2.i) Let $\overline{\Omega} := \Omega \times \Omega'$, $\overline{\mathbb{P}} = \mathbb{P} \otimes \hat{\mathbb{P}}'$ and $\overline{\mathcal{F}} = \mathbb{F}^{W,\mu}$, with

   $$\mathbb{F}^{W,\mu} := (\mathbb{F}^W \vee \mathbb{F}^\mu) \vee \mathcal{N}$$

   (2.ii) $\overline{W}(\omega, \omega') = W(\omega)$ remains a $\mathbb{F}^{W,\mu}$- Brownian motion;
   $\overline{\mu} := (\overline{\sigma}^m, \overline{\zeta}^m)_m$ Poisson r.m. with $\mathbb{F}^{W,\mu}$-prog. meas random marks and **same determ. compensator** $\widehat{\overline{\mu}} = \hat{\mu}$.

I.2. The randomized set-up and dual formulation

1. Let I (resp. $\bar{I}$) the Poisson point proc. assoc. with μ (resp. $\overline{\mu}$) as follows

$$\forall\, t \in [0, T], \quad I_t = \zeta^0 \mathbf{1}_{t<\sigma^1} + \sum_{m\geq 1} \zeta^m \mathbf{1}_{\sigma^m \leq t < \sigma^{m+1}}.$$

Note that $N_T^I := \mathrm{Card}\{m \geq 1,\ \sigma_m(\omega') < T\} < \infty$, $\mathbb{P}'$-a.s.

2. On *randomized* prob. space, $(\bar{I}, X^{\bar{I}})$ is a **forward uncontrolled proc.** with

$$X_t^{\bar{I}} = x_0 + \int_0^t \left(b^{\bar{I}_s}(s, X^{\bar{I}})ds + \sigma^{\bar{I}_s}(s, X^{\bar{I}})dW_s\right)$$

I.2. The randomized set-up and dual formulation

1. To any proc. $\overline{\nu}$ $\mathbb{F}^{W,\mu}$-meas., associate process $\kappa^{\overline{\nu}}$

$$\kappa^{\overline{\nu}}_T = \mathcal{E}_T((\overline{\nu}-1)\star\tilde{\mu}) = e^{-\int_0^T\int_A(\overline{\nu}_s(e)-1)\lambda(de)ds}\prod_{\substack{m\geq 1\\ \zeta_m<T}}\overline{\nu}_{\sigma^m}(\zeta^m)$$

2. Let $\bar{\mathbb{P}}^{\overline{\nu}}$ with density $\kappa^{\overline{\nu}}$, i.e. $\dfrac{d\bar{\mathbb{P}}^{\hat{\nu}}}{d\bar{\mathbb{P}}} = \kappa^{\overline{\nu}}$
   then, under $\bar{\mathbb{P}}^{\hat{\nu}}$,
   (a) $\overline{I}$ remains Poisson point proc.;
   (b) New compensated meas. $\overline{\nu}_s(e)\lambda(de)ds$
3. Set of dual controls

$$\mathcal{A}^R := \{\overline{\nu} : \overline{\Omega}\times[0,T]\times A \mapsto]0;\infty[\ \text{ meas. and essentially bounded}\}$$

I.2 The randomized set-up: dual formulation

1. Let $\mathcal{V}_0^R = \sup_{\overline{\nu} \in \mathcal{A}^R} J^R(\overline{\nu})$ be the dual value function with

$$J^R(\overline{\nu}) = \underbrace{\bar{\mathbb{E}}^{\overline{\nu}}\left(g(X^I, I_T) + \int_t^T f(s, X^I, I_s)ds\right)}_{=J_1^R(\overline{\nu})}$$
$$- \underbrace{\bar{\mathbb{E}}^{\overline{\nu}}\left(\sum_{m \geq 1} c_{\zeta_{m-1},\zeta_m}(\sigma^m, X_{\sigma^m})\right)}_{=J_2^R(\overline{\nu})}$$

$\bar{\mathbb{E}}^{\overline{\nu}}$ stands for expectation under meas. $\mathbb{P}^{\overline{\nu}}$.

I.2 The randomized set-up: Major comments

1. Unique assumption on A: it is a **Borel** space
   No compactness assumption.
   Desirable properties: A both **metric and separable**.
2. Exogeneous proc. X (resp. $\bar{X}$) not necess. Markovian
3. The controlled volatility process may be degenerate (contrary to papers using PDE approaches).
4. If b, σ only depends on (x, a) not on ω, then the pair (I, X^I) is a Markov process.

Main results

II First main result & comments

Under all previous assumptions on the primal & dual version of the OSP, one claims

$$\mathcal{V}_0 = \mathcal{V}_0^{\mathcal{R}} = v_0(x_0, a_0).$$

This **deterministic** common value function only depends on $X_0 = x_0$ and initial mode a_0 and not of the choice of the randomized set up:
(i.e. **neither on the construction of the extended dual set-up nor on the choice of intensity measure λ).**

II. Second main result: BSDE characterization

Let $Y^{\mathcal{R}}$ be the *minimal* solution of following BSDE

$$\begin{cases} Y_t^{\mathcal{R}} = g(X, I_T) + \int_t^T f_s(X, I_s)\, ds + K_T - K_t \\ \qquad\qquad - \int_t^T Z_s dW_s - \int_{(t,T]}\int_A U_s(a)\, \tilde{\mu}(ds\, da), \\ U_t(a) \leq c_t(X, I_{t-}, a), \quad \lambda(da)ds\ \mathbb{P} - \text{a.s.} \end{cases} \tag{1}$$

then it holds

$$Y_0^{\mathcal{R}} = \mathcal{V}_0^{\mathcal{R}}.$$

Remark: (1) is a BSDE with constrained jumps & non decreas. proc K: K **only càdlàg** .
$Y_t^{\mathcal{R}}$ $\mathcal{F}_t^{W,\mu}$-adapted.

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1. when A infinite (even uncountable), the *infinite* system of RBSDEs does not seem well posed (at least to us..)

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3. the randomized set up allows to tackle general cases: path-dependency, possibly degenerate diffusions, case of an infinite set of modes.
4. Another motivation: in the Markovian setting, connection already proved by R.Elie & I.Kharroubi ('09, '10).

Main results: comments

Connection with BSDE associated with the OSP (finite set of modes)

Let $\mathcal{J}$ set of modes and let $(Y^i)_{i\in\mathcal{J}}$ solving

$$\begin{cases} Y_t^i = & g(X_T, i) + \int_t^T f_s(X_s, i)\, ds + K_T^i - K_t^i \\ & - \int_t^T Z_s^i dW_s, \\ Y_s^i \geq \max_{\{j\in\mathcal{J}\setminus\{i\}\}} \left(Y_s^j - c_{i,j}(s, X_s) \right) & \text{and} \\ \int_0^T (Y_s^i - \max_{\{j\in\mathcal{J}\setminus\{i\}\}} \left(Y_s^j - c_{i,j}(s, X_s) \right) dK_s^i = 0 \end{cases} \tag{2}$$

If BSDE system (2) has a solution, the *minimal* solution of dual BSDE (1) is s.t.

$$Y_t^{\mathcal{R}} = Y_t^{I_t} \text{ and } U_t(i) = Y_t^i - Y_t^{I_{t-}}.$$

The minimal BSDE

Let Y the *minimal* solution of following BSDE

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then it holds: $Y_0^{\mathcal{R}} = \mathcal{V}_0^{\mathcal{R}}$. Combined with first main result

$$Y_0^{\mathcal{R}} = \mathcal{V}_0^{\mathcal{R}} = \mathcal{V}_0 = \sup_{\alpha \in \mathcal{A}} \mathcal{J}(\alpha).$$

$Y^{\mathcal{R}}$: obtained as the increasing limit of penalized scheme.

Probabilistic representation

Let (Y^n) solving

$$\begin{cases} Y_t^n = g(X, I_T) + \int_t^T f_s(X, I_s)\, ds + K_T^n - K_t^n \\ \qquad\qquad - \int_t^T Z_s^n dW_s - \int_{(t,T]} \int_A U_s^n(a)\, \tilde{\mu}(ds\, da), \\ \text{with } dK_s^n = n \int_A \big(U_s^n(a) - c_s(X, I_s, a)\big)^+ \lambda(da) ds. \end{cases} \tag{4}$$

Main results: the BSDE characterization

Probabilistic representation

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$$\left\{\begin{array}{l} Y_t^n \;=\; g(X, I_T) + \int_t^T f_s(X, I_s)\, ds + K_T^n - K_t^n \\ \qquad\qquad - \int_t^T Z_s^n dW_s - \int_{(t,T]} \int_A U_s^n(a)\, \tilde{\mu}(ds\, da), \\ \text{with } dK_s^n = n \int_A \big(U_s^n(a) - c_s(X, I_s, a)\big)^+ \lambda(da) ds. \end{array}\right. \tag{4}$$

It holds

$$Y_t^n = \operatorname*{ess\,sup}_{\substack{\nu \in \mathcal{A}^{\mathcal{R}} \\ |\nu|_\infty \leq \mathbf{n}}} \mathbb{E}^\nu \Big(g(X_T, I_T) + \int_t^T f_r(X, I_r) dr - \int_t^T \int_A c_r(X_r, I_{r-}, a) \mu(da, ds) \Big| \mathcal{F}_t^{W,\mu} \Big) \tag{5}$$

Concluding remarks

Some perspectives: theoretical & numerical

1. Stability results: Approximating the general OSP by the OSP with finite number of modes
   Objective: explicit rate of convergence

2. Refinements in Markovian setting ((I, X^I) Markov process)

3. Numerical perspectives
   Numerical solving of the "dual" BSDE.

   Note when Card(A) $< \infty$ but too large, simulating the solution of multidim BSDE system becomes unfeasible.

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Thanks for your attention !