Optimal switching problems with an infinite set of modes: an approach by randomization and constrained backward SDEs

M.-A. Morlais (LMM - IRA, Le Mans Université, France)
j.w.w. M. Fuhrman (Milano, Italy)

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Outline of the talk

I- Preliminaries & motivations
• The Optimal Switching problem (OSP): primal vs dual formulation.
• Assumptions for the dual formulation.
• Why choosing the "dual" approach?

II- Main results & perspectives
• The two main results:
  (i): equality between the two value functions;
  (ii): new BSDE characterization.
• Perspectives
Motivations & preliminaries

I.1 Primal optimal switching problem and value function

On a standard prob. space \((\Omega, \mathcal{F}, \mathbb{P})\), let

- \(W\): standard \(d\)-dim. Brownian Motion, \(W\ \mathcal{F}\)-adapted.

usually: \(\mathcal{F} = \mathcal{F}^W \vee \mathcal{N}\).

- \(T\) fixed finite horizon; \(A\) set of modes (possibly infinite).

- \(\forall (x_0, e) \in \mathbb{R}^n \times A\), let \(X^e\) proc. s.t.

\[
\forall \ t \in [0, T], \ X^e_t = x_0 + \int_0^t (b^e(s, X^e)ds + \sigma^e(s, X^e)dW_s),
\]
Motivations & preliminaries

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- $\forall (x_0, e) \in \mathbb{R}^n \times A$, let $X^e$ proc. s.t.

$$\forall t \in [0, T], \quad X^e_t = x_0 + \int_0^t (b^e(s, X^e)\,ds + \sigma^e(s, X^e)\,dW_s),$$

Let $(f^e)_e$, $(g^e)_e$ and $(c_{e,e'})_{(e,e')}$: 3 families of (possib. random) real-valued data

- (i) $f^e(s, X.)$: instant. profit (when system in mode $e$)
- (ii) $g^e(X.)$: payoff at time $T$ when syst. in mode $e$,
- (iii) $c_{e,e'}(s, X.)$ : nonnegative penalty costs incurred at time $s$ when switching from $e$ to $e'$. 
I.1 Primal optimal switching problem and value function

- Mathematical assumptions:
  - $A$: Borel set (example: any subspace of $\mathbb{R}^d$);
  - Both $(b^e, \sigma^e)_e, (f^e, g^e), (c_{e,e'})_{e,e'}$ may be path-dependent;
  - Let $C^n$: set of continuous paths $s \mapsto x(s)_{s \in [0,T]}$

Topology on $C^n$: $|x|_* = \sup_{s \in [0,T]} |x(s)|$
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- Topology on $\mathcal{C}^n$: $|x|_* = \sup_{s \in [0, T]} |x(s)|$
- Measurability
  $$(t, \omega, e) \mapsto b^e(t, x(\omega), \omega), \sigma^e(t, \omega, x(\omega), e)$$
  are $\text{Prog}(\mathcal{C}^n) \otimes \mathcal{B}(A)$ meas.; (similar for $f^e, g^e, c_{e,e'}$)
- $\text{Prog}(\mathcal{C}^n)$: $\sigma$-algebra of prog. measurable maps on $[0, T] \times \Omega$. 

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I.1 Primal optimal switching problem and value function

Math. Assumptions (cont’):

• For every \( t \in [0, T] \),
  \((x, e) \mapsto b_t(x, e)\) \(\sigma_t(x, e)\), \(f_t(x, e), g(x, e)\) are continuous on \(\mathbb{C}^n \times A\)
  \((x, e, e') \mapsto c_t(x, e, e')\) is continuous on \(\mathbb{C}^n \times A \times A\).

• Regularity & growth assumpt (wrt \( x \)):
  \(\exists \ K > 0 \ \text{s.t.} \ \forall (t, x, x', e, e') \in [0, T] \times \mathbb{C}^n \times \mathbb{C}^n \times A \times A,\)
  \(\ |b_t(x, e) - b_t(x', e)| + |\sigma_t(x, e) - \sigma_t(x', e)| \leq K|x - x'|_{t^*} \)
  Similar for other data.
  \(\ |b(t, 0, e)| + |\sigma(t, 0, e)| \leq K;\)
I.1 Primal optimal switching problem and value function

- Growth assumpt wrt $x$ (cont’)

  $\exists \ r, \ K > 0$ s.t. $\forall (t, x, x', e, e') \in [0, T] \times \mathbb{C}^n \times \mathbb{C}^n \times A \times A,$

  (iii) $|f(t, x, e) + g(x, e)| + |c(t, x, e, e')| \leq K(1 + |x|_{t^*})$
Motivations & preliminaries

I.1 Primal optimal switching problem and value function

- Growth assumpt wrt $x$ (cont’)

\[ \exists \ r, \ K > 0 \ \text{s.t.} \ \forall (t, x, x', e, e') \in [0, T] \times \mathbb{C}^n \times \mathbb{C}^n \times A \times A, \]

\[(iii) \ |f(t, x, e) + g(x, e)| + |c(t, x, e, e')| \leq K(1 + |x|_{t*})\]

Comment

(i)-(iii) standard to obtain estim.

(a) Estim. (of Hilbertian norm) of process $X^e$ (see Cosso-Confortola-Fuhrman ’18);
(b) Estim. of the value function (well known in Markovian case).
I.1 Primal optimal switching problem and value function

1. Let $\alpha = (\tau^n, \xi^n)_{n \geq 1}$ with $\tau^1 > 0$. $\alpha =$ management strategy

2. To $\alpha$, we associate the state proc. $a$ as follows

$$a_s = \xi^1 1_{s < \tau_1} + \sum_{n \geq 1} \xi^{n+1} 1_{\tau^n \leq s < \tau_{n+1}} 1_{\tau^n < T}$$

$a$: piecewise constant proc. $A$-valued

By abuse, one may replace $\alpha$ by $a$. 
I.1 Primal value function: Admissible set $\mathcal{A}$

$a = (\tau^n, \xi^n)$ is said **admissible** ($a$ in $\mathcal{A}$) if

$H_1 (\tau^n(\cdot), \xi^n(\cdot))_{n-\mathbb{R}^+} \times A$-valued $\mathbb{F}$-adapt. such that

$\tau_n(\omega) \rightarrow +\infty$ and $\tau^n < \tau^{n+1}$, $\mathbb{P}$-a.s

**simultaneous switchings prohibited**: equivalent to

$$\forall (a_1, a_2, a_2) \in A^3, \quad c_{a_1,a_2}(t, x) + c_{a_2,a_3}(t, x) > c_{a_1,a_3}(t, x)$$

Stronger than the no-loop property (in finite case).
Motivations & preliminaries

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$H_1$ $(\tau^n(\cdot), \xi^n(\cdot))_n : \mathbb{R}_+^n \times A$-valued $\mathbb{F}$-adapt. such that

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Stronger than the no-loop property (in finite case).

$H_2$ $H_1$ implies: $N^a_T(\omega) = \text{Card}\{\tau^n(\omega), \tau^n < T\} < \infty$, $\mathbb{P}$-a.s

$H_3$ Impose $\tau^n \neq T$: no switching at terminal time.

In finite case, equivalent to:

$$\forall (i, j) \in A \times A, \quad g^i(X^i_T) > g^j(X^j_T) - c_{i,j}(T, X_T).$$
I.1 Primal optimal switching problem and value function

1. For $a$ in $\mathcal{A}$, let $X^\alpha$ (or $X^a$) the controlled proc. s.t.

$$dX^a = b^a(s, X^a)ds + \sigma^a(s, X^a_s)dW_s$$

with $b^a(s, x) = b^{\xi_0}(s, x)1_{s<\tau^1} + \sum_{n \geq 1} b^{\xi_n}(s, x)1_{\tau^n \leq s < \tau^{n+1}}$. 

Similar definition for $\sigma^a(s, x)$.

**Remark:** $b$ and $\sigma$ path-dependent $\Rightarrow$ $X^a$ no more Markovian (PDE approach not available).
I.1 Primal control problem and (primal) value function

1. Primal value function \( \mathcal{V} \)

\[
\mathcal{V} = \sup_{\alpha \in \mathcal{A}} \left( \mathcal{J}(\alpha) \right), \text{ where}
\]

\[
J(\alpha) = \mathbb{E} \left( g_{aT}(X_\cdot) + \int_0^T f_{as}(s, X^a_\cdot) ds - \sum_{n \geq 1, \tau_n < T} c_{\xi_{n-1}, \xi_n}(\tau^n, X^a_{\tau^n}) \right)
\]

\[
= J_1(\alpha) - J_2(\alpha)
\]

**Objective:** choose the best \( a \) (or \( \alpha \)) to optimize \( J(\alpha) \) and minimize \( \mathcal{J}_2(\alpha) \).
Motivations & preliminaries

A (non exhaustive) review of the literature

(1) **OSP with finite set of modes:**
   
   (i) Using PDE approaches: Ishii-Koike ’91, Yong-Zhou ’99, Ludkowski ’05, Carmona-Ludkovski ’07-08, ...
   
   (ii) Using BSDE and analyt. tools: Hamadène-Jeanblanc ’02, Djehiche-Hamadene-Popier ’08, Hu-Tang ’07 Chassagneux-Elie Kharroubi; Elie-Kharroubi ’08 ’11, ...
   
   (iii) **Standard OSP with refinements:** infinite horizon, partial information, non positive costs: Lundstrom -Olofsson, R. Martyr, B. El Asri, ..

(2) **Connection between "finite" OSP & constrained BSDE:**
   
   (a) Ma-Pham-Kharroubi ’08 (Markovian setting)
   
   (b) Elie-Kharroubi (’14) (Non Markovian case)
Motivations & preliminaries

I.2 Randomized set-up & dual formulation

1. On $(\Omega', \mathbb{F}', \mathbb{P}')$ let $\mu = \sum_{n \geq 0} \delta_{\sigma^m, \zeta^m}$ be a Poisson random meas. s.t.

   (i) Random dates & marks $(\sigma^m, \zeta^m) \sim \mathbb{R}^+ \times A$-valued;
   (ii) $\mu$ ind. of $W$ with $\hat{\mu}(de, ds) = \lambda(de)ds$ s.t
   (a) $\tilde{\mu} = \mu - \hat{\mu}$ is a martingale measure;
   (b) $\lambda(de)$ has full support and $\lambda(A) < +\infty$. 
Motivations & preliminaries

1.2 Randomized set-up & dual formulation

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   (a) $\tilde{\mu} = \mu - \hat{\mu}$ is a martingale measure;
   (b) $\lambda(de)$ has full support and $\lambda(A) < +\infty$.

2. The randomized dual set up := $(\bar{\Omega}, \bar{\mathbb{P}}, \bar{\mathbb{F}}, \bar{W}, \bar{\mu})$:

(2.i) Let $\bar{\Omega} := \Omega \times \Omega'$, $\bar{\mathbb{P}} = \mathbb{P} \otimes \hat{\mathbb{P}}'$ and $\bar{\mathbb{F}} = \mathbb{F}^{W, \mu}$, with

$$\mathbb{F}^{W, \mu} := (\mathbb{F}^W \vee \mathbb{F}^\mu) \vee \mathcal{N}$$

(2.ii) $\bar{W}(\omega, \omega') = W(\omega)$ remains a $\mathbb{F}^{W, \mu}$- Brownian motion;
$\bar{\mu} := (\bar{\sigma}^m, \bar{\zeta}^m)_m$ Poisson r.m. with $\mathbb{F}^{W, \mu}$-prog. meas random marks and same determ. compensator $\hat{\mu} = \hat{\mu}$. 
I.2. The randomized set-up and dual formulation

1. Let \( I \) (resp. \( \bar{I} \)) the Poisson point proc. assoc. with \( \mu \) (resp. \( \bar{\mu} \)) as follows

\[
\forall \ t \in [0, T], \quad I_t = \zeta^0 \mathbf{1}_{t<\sigma^1} + \sum_{m \geq 1} \zeta^m \mathbf{1}_{\sigma^m \leq t < \sigma^{m+1}}.
\]

Note that \( N^I_T := \text{Card}\{m \geq 1, \ \sigma^m(\omega') < T\} < \infty, \ \mathbb{P}'\text{-a.s.} \)

2. On randomized prob. space, \((\bar{I}, X^{\bar{I}})\) is a **forward uncontrolled proc.** with

\[
X^\bar{I}_t = x_0 + \int_0^t (b^\bar{I}_s(s, X^\bar{I})ds + \sigma^\bar{I}_s(s, X^\bar{I})dW_s)
\]
I.2. The randomized set-up and dual formulation

1. To any proc. \( \nu \) \( \mathbb{P}^W, \mu \)-meas., associate process \( \kappa_\nu \)

\[
\kappa_\nu_T = \mathcal{E}_T((\nu - 1) \ast \tilde{\mu}) = e^{- \int_0^T \int_A (\nu_s(e) - 1) \lambda(de)ds} \prod_{m \geq 1} \nu_{\sigma m}(\zeta^m)
\]

2. Let \( \mathbb{P}^\nu \) with density \( \kappa_\nu \), i.e. \( \frac{d\mathbb{P}^\nu}{d\mathbb{P}} = \kappa_\nu \)
then, under \( \mathbb{P}^\nu \),
(a) \( \tilde{T} \) remains Poisson point proc.;
(b) New compensated meas. \( \nu_s(e) \lambda(de)ds \)

3. Set of dual controls

\[
\mathcal{A}^R := \{ \nu : \Omega \times [0, T] \times A \mapsto 0; \infty[ \text{ meas. and essentially bounded} \} \]
I.2 The randomized set-up: dual formulation

1. Let $\nu_0^R = \sup_{\nu \in \mathcal{A}^R} J^R(\nu)$ be the dual value function with

$$J^R(\nu) = \bar{E}^\nu \left( g(X^l, I_T) + \int_t^T f(s, X^l, I_s) ds \right)$$

$$= J_1^R(\nu)$$

$$- \bar{E}^\nu \left( \sum_{m \geq 1} c_{\zeta \sigma} m \left( \sigma^m, X_{\sigma^m} \right) \right)$$

$$= J_2^R(\nu)$$

$\bar{E}^\nu$ stands for expectation under meas. $\mathbb{P}^\nu$. 
I.2 The randomized set-up: Major comments

1. Unique assumption on $A$: it is a **Borel** space
   No compactness assumption.
   Desirable properties: $A$ both **metric and separable**.

2. Exogeneous proc. $X$ (resp. $\tilde{X}$) not necess. Markovian

3. The controlled volatility process may be degenerate
   (contrary to papers using PDE approaches).

4. If $b$, $\sigma$ only depends on $(x, a)$ not on $\omega$, then the pair $(I, X^I)$
   is a Markov process.
Il First main result & comments
Under all previous assumptions on the primal & dual version of the OSP, one claims

$$\mathcal{V}_0 = \mathcal{V}_0^R = v_0(x_0, a_0).$$

This **deterministic** common value function only depends on $X_0 = x_0$ and initial mode $a_0$ and not of the choice of the randomized set up:

(i.e. **neither on the construction of the extended dual set-up nor on the choice of intensity measure $\lambda$**).
II. Second main result: BSDE characterization

Let $Y^R$ be the \textit{minimal} solution of following BSDE

\[
\begin{aligned}
Y_t^R &= g(X, I_T) + \int_t^T f_s(X, I_s) \, ds + K_T - K_t \\
&\quad - \int_t^T Z_s dW_s - \int_{(t, T]} \int_A U_s(a) \tilde{\mu}(ds \, da), \\
U_t(a) &\leq c_t(X, I_t, a), \quad \lambda(da) \, ds \, \mathbb{P} - \text{a.s.}
\end{aligned}
\]

then it holds

\[
Y_0^R = V_0^R.
\]

\textbf{Remark:} (1) is a BSDE with constrained jumps & non decreas. proc $K$: $K$ only \textit{càdlàg}.

$Y_t^R$ $\mathcal{F}_t$-$\mu$-adapted.
I.3. Why choosing randomization to study the OSP?

1. when $A$ infinite (even uncountable), the *infinite* system of RBSDEs does not seem well posed (at least to us..)
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2. For the primal OSP, many ingredients *deeply* use the finiteness of $A$. 
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1. When $A$ infinite (even uncountable), the \textit{infinite} system of RBSDEs does not seem well posed (at least to us..)
2. For the primal OSP, many ingredients \textbf{deeply} use the finiteness of $A$.
3. The randomized set up allows to tackle general cases: path-dependency, possibly degenerate diffusions, case of an infinite set of modes.
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Main results: comments

Connection with BSDE associated with the OSP (finite set of modes)

Let $\mathcal{J}$ set of modes and let $(Y^i)_{i \in \mathcal{J}}$ solving

$$
\begin{cases}
Y^i_t = g(X_T, i) + \int_t^T f_s(X_s, i) \, ds + K^i_T - K^i_t \\
- \int_t^T Z^i_s \, dW_s,
\end{cases}
Y^i_s \geq \max_{j \in \mathcal{J} \setminus \{i\}} \left( Y^j_s - c_{i,j}(s, X_s) \right) \quad \text{and}
\int_0^T (Y^i_s - \max_{j \in \mathcal{J} \setminus \{i\}} \left( Y^j_s - c_{i,j}(s, X_s) \right)) \, dK^i_s = 0
$$

If BSDE system (2) has a solution, the minimal solution of dual BSDE (1) is s.t.

$$
Y^R_t = Y^l_t \quad \text{and} \quad U_t(i) = Y^i_t - Y^{l-}_t.
$$
Main results: the BSDE characterization

The minimal BSDE
Let $Y$ the \textit{minimal} solution of following BSDE

$$
\begin{cases}
Y_t^R = g(X, I_T) + \int_t^T f_s(X, I_s) \, ds + K_T - K_t \\
- \int_t^T Z_s dW_s - \int_{(t,T]} \int_A U_s(a) \tilde{\mu}(ds \, da), \\
U_s(a) \leq c_s(X, I_{s-}, a), \lambda(da)ds \mathbb{P} - \text{a.s.}
\end{cases}
$$

then it holds: $Y_0^R = \mathcal{V}_0^R$. Combined with first main result

$$
Y_0^R = \mathcal{V}_0^R = \mathcal{V}_0 = \sup_{\alpha \in \mathcal{A}} \mathcal{J}(\alpha).
$$

$Y^R$: obtained as the increasing limit of penalized scheme.
Main results: the BSDE characterization

Probabilistic representation

Let \((Y^n)\) solving

\[
\begin{aligned}
Y^n_t &= g(X, I_T) + \int_t^T f_s(X, I_s) \, ds + K^n_T - K^n_t \\
&\quad - \int_t^T Z^n_s \, dW_s - \int_{(t,T]} \int_A U^n_s(a) \, \tilde{\mu}(ds \, da),
\end{aligned}
\]

with \(dK^n_s = n \int_A (U^n_s(a) - c_s(X, I_s, a))^+ \lambda(da) ds.\)

(4)
Main results: the BSDE characterization

Probabilistic representation

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\end{cases}
\]

with $dK^n_s = n \int_A (U^n_s(a) - c_s(X, I_s, a))^+ \lambda(da) \, ds.$

It holds

\[
Y^n_t = \text{ess sup}_{\nu \in A^R \atop |\nu|_\infty \leq n} \mathbb{E}^\nu \left( g(X_T, I_T) + \int_t^T f_r(X, I_r) \, dr \\
- \int_t^T \int_A c_r(X_r, I_{r-}, a) \mu(da, ds) \big| \mathcal{F}_{t}^{W, \mu} \right).
\]

(5)
Concluding remarks

Some perspectives: theoretical & numerical

1. Stability results: Approximating the general OSP by the OSP with finite number of modes
   **Objective:** explicit rate of convergence

2. Refinements in Markovian setting \(((I, X^I)\) Markov process)

3. Numerical perspectives
   Numerical solving of the "dual" BSDE.
   **Note** when \(\text{Card}(A) < \infty\) but too large, simulating the solution of multidim BSDE system becomes unfeasible.

Thanks for your attention!
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