Probabilistic solutions to Stefan equations with supercooling

Sergey Nadtochiy
(based on joint works with F. Delarue, M. Shkolnikov, X. Zhang)

Department of Applied Mathematics
Illinois Institute of Technology

June 28, 2022

9th colloquium on BSDEs and Mean Field Systems
Annecy, France
Stefan equation (Visintin 1998)

\begin{equation}
  u_t - \mathcal{L}(u_{xx}, u_x, x, t) = 0, \quad x \in D(t) \text{ or } \mathbb{R}^d \setminus \partial D(t), \quad t > 0, \quad u(x, 0) = \phi(x),
\end{equation}

\begin{equation}
  u|_{\partial D(t)} = aV + bH, \quad H \text{ is the mean curvature of } \partial D(t),
\end{equation}

where \( \mathcal{L} \) is an elliptic operator (e.g., \( \mathcal{L} = \frac{1}{2} u_{xx} \)) and the normal velocity \( V(x, t) := \dot{D}(x, t) \) of \( \partial D(t) \) at \( x \in \partial D(t) \) must, in addition, satisfy

\[ V(x, t) = \frac{1}{2} \left[ \lim_{D^c(t) \ni y \to x} u_x(y, t) \cdot \nu(t, x) - \lim_{D(t) \ni y \to x} u_x(y, t) \cdot \nu(t, x) \right], \]

with \( \nu \) being the outer unit normal to \( \partial D(t) \).

- Elliptic version of this problem is known as Hele-Shaw equation. Various modifications of (1) form Laplacian growth models.
- **Applications**: melting/solidification, condensation, crystal growth, aging of alloys, interaction of fluids with different viscosities, dynamics of membrane potentials in a network of neurons, tumor growth, etc.
Stefan equation

\[ \dot{D}(x, t) = V(x, t) = \frac{1}{2} \left[ \lim_{D^c(t) \ni y \to x} u_x(y, t) \cdot \nu(t, x) - \lim_{D(t) \ni y \to x} u_x(y, t) \cdot \nu(t, x) \right] \]
Stefan equation as a mean-field system

Regular Stefan problem

Probab-c rep-n for regular \((\phi \geq 0, \ a = b = 0)\), single-phase, one-dim. \((d = 1)\) Stefan problem

\[
\begin{align*}
&u_t - \frac{1}{2} u_{xx} = 0, \quad x > \Lambda(t), \quad t > 0, \quad u(x, 0) = \phi(x), \\
&u(\Lambda(t)^+, t) = 0, \quad -\dot{\Lambda}(t) = \frac{1}{2} u_x(\Lambda(t)^+, t), \quad \Lambda(0) = 0
\end{align*}
\]

- Assume \(\phi \geq 0\), \(\int_0^\infty \phi = 1\), and let \(\varphi(\cdot, t)\) be Gaussian kernel with variance \(t\).
- Feynman-Kac formula and time reversal of BM \(W\) yield

\[
\sigma := \inf\{s \geq 0 : x + W_s \leq \Lambda(t - s)\}, \quad u(x, t) = \mathbb{E}[\phi(x + W_t)1_{\sigma > t}]
\]

\[
\begin{align*}
&= \int_0^\infty \phi(y) \varphi(x - y, t) \mathbb{P}\left(\inf_{s \in [0,t]} (x + W_s - \Lambda(t - s)) > \mathbb{E} \left( x + W_s - \Lambda(t - s) \right) \right) dy \\
&= \int_0^\infty \phi(y) \varphi(x - y, t) \mathbb{P}\left(\inf_{s \in [0,t]} (y + W_s - \Lambda(s)) > \mathbb{E} \left( y + W_s - \Lambda(s) \right) \right) dy \\
&= \mathbb{P}(X_t \in dx, \tau > t)/dx, \quad X_t = \xi + W_t, \quad \tau := \inf\{s \geq 0 : \xi + W_s \leq \Lambda(s)\},
\end{align*}
\]

with an independent r.v. \(\xi\) having density \(\phi\).
Probab-c representation of the growth condition

\[ u(x, t) = \mathbb{P}(X_t \in dx, \tau > t)/dx, \quad x > \Lambda(t), \]
\[ X_t = (\xi + W_{t\wedge \tau}), \quad \tau := \inf\{s \geq 0 : \xi + W_s \leq \Lambda(s)\}, \]
\[ -\dot{\Lambda}(t) = \frac{1}{2} u_x(\Lambda(t)^+, t), \quad \Lambda(0) = 0 \tag{2} \]

- We have established that \( u(\cdot, t) \) is the density of the marginal distribution of a BM absorbed at \( \Lambda \), before absorption.
- To derive a probabilistic representation for \( \Lambda \) (to replace (2)), we notice that

\[
\frac{d}{dt} \mathbb{P}(\tau > t) = \frac{d}{dt} \int_{\Lambda(t)} u(x, t) dx = -\dot{\Lambda}(t) u(\Lambda(t)^+, t) + \int_{\Lambda(t)} u_t(x, t) dx \\
= \frac{1}{2} \int_{\Lambda(t)} u_{xx}(x, t) dx = -\frac{1}{2} u_x(\Lambda(t), t) = \dot{\Lambda}(t), \\
\Lambda(t) = -\mathbb{P}(\tau \leq t)
\]
Stefan problem as a McKean-Vlasov equation: single-phase, $d = 1$, regular

$$u(x, t) = \mathbb{P}(X_t \in dx, \tau > t)/dx, \quad x > \Lambda(t), \quad \Lambda(t) = -\mathbb{P}(\tau \leq t),$$

$$X_t = (\xi + W_{t\wedge \tau}), \quad \tau := \inf\{s \geq 0 : \xi + W_s \leq \Lambda(s)\}, \quad \xi \sim u(0, x) \, dx$$

- The above system is a McKean-Vlasov equation, as the dynamics of $X$ depend explicitly on its distribution.

- Levine-Peres 2010 show that the regular Stefan (and Hele-Shaw) equation can be obtained as a scaling limit of Internal Diffusion Limited Aggregation (DLA) model.

- This probabilistic connection offers a new perspective (and a new numerical method), but the well-posedness of regular Stefan equation (for any $d$ and with any number of phases) was established a long time ago (e.g., in Kamenomostskaja 1961).
Supercooled Stefan problem

\[ u_t - \mathcal{L}(u_{xx}, u_x, x, t) = 0, \quad x \in D(t) \text{ or } \mathbb{R}^d \setminus \partial D(t), \quad t > 0, \quad u(x, 0) = \phi(x), \]

\[ u|_{\partial D(t)} = aV + bH, \quad H \text{ is the mean curvature of } \partial D(t), \]

\[ \dot{D}(x, t) = V(x, t) = \frac{1}{2} \left[ \lim_{D^c(t) \ni y \to x} u_x(y, t) \cdot \nu(t, x) - \lim_{D(t) \ni y \to x} u_x(y, t) \cdot \nu(t, x) \right] \]

- If \( \phi \leq 0, \ a = b = 0 \) (and w.l.o.g. \( \int_0^\infty \phi = -1 \)), we call this Stefan problem supercooled, as the liquid temperature is below the freezing point.

- Such problems appear in the modeling of: solidification process (e.g., to produce glassy metals), crystal growth, macroscopic dynamics of neurons' membrane potentials, Hele-Shaw cell, etc.

- Main challenges (as compared to regular case):
  1. No global \textbf{comparison principle}.
  2. \textbf{Time-singularity} of \( D(\cdot) \).
Regular Stefan: comparison principle

\[ u^1(\cdot, 0) \geq u^2(\cdot, 0) \implies u^1(\cdot, t) \geq u^2(\cdot, t) \]
Supercooled Stefan: no comparison principle

\[ u^1(\cdot, 0) \geq u^2(\cdot, 0) \implies u^1(\cdot, t) \geq u^2(\cdot, t) \]
Time-singularity of the boundary

The faster the boundary moves up, the steeper is the graph of $u(\cdot, t)$ at the boundary.
Jump size of the boundary

\[ \Lambda(t) - \Lambda(t^-) = \inf\{z > 0 : -\int_{\Lambda(t^-)}^{\Lambda(t^-)+z} u(y, t^-) \, dy < z \} \]
Probabilistic representation for single-phase supercooled Stefan, $d = 1$

\[
\begin{align*}
  u_t - \frac{1}{2}u_{xx} &= 0, \quad x > \Lambda(t), \quad t > 0, \quad u(x, 0) = \phi(x) \leq 0, \\
  u(\Lambda(t)^+, t) &= 0, \quad -\dot{\Lambda}(t) = \frac{1}{2}u_x(\Lambda(t)^+, t), \quad \Lambda(0) = 0
\end{align*}
\]

- Similar to the regular case, one can derive the following probabilistic representation of a single-phase supercooled Stefan problem, with $d = 1$:

  \[
  X_t = \xi + W_{t \wedge \tau}, \quad \xi \sim -\phi(x)dx \geq 0, \quad x > 0, \\
  \tau := \inf\{s \geq 0 : \xi + W_s \leq \Lambda(s)\}, \quad \Lambda(t) = \mathbb{P}(\tau \leq t), \\
  \Lambda(t) - \Lambda(t-) = \inf\{z > 0 : \mathbb{P}(\tau \geq t, X_{t-} \in (\Lambda(t-), \Lambda(t-)+z)) < z\}
  \]

- **Theorem.** For **small initial data**, $-1 < \phi \leq 0$, $\int \phi = -1$, there exists a unique (smooth) solution to the above system.

- This result was first established in 1980s: see, e.g., *Fasano-Primicerio 1981* and *Chayes-Swindle 1996* for PDE methods, and *Ledger-Sojmark 2019* for probabilistic approach.
Sketch of the proof: small initial data

\[ X_t = \xi + W_{t \wedge \tau}, \quad \xi \sim -\phi(x)dx, \quad x > 0, \]
\[ \tau := \inf\{s \geq 0 : \xi + W_s \leq \Lambda(s)\}, \quad \Lambda(t) = \mathbb{P}(\tau \leq t) \]

The mapping \( \Gamma \mapsto \Lambda \),
\[ \Lambda(t) := \mathbb{P}(\inf_{s \in [0,t]} (\xi + W_s - \Gamma(s)) \leq 0) = -\int_0^\infty \phi(x) \mathbb{P}(\inf_{s \in [0,t]} (W_s - \Gamma(s)) \leq -x) \, dx, \]
is a contraction:
\[ \Lambda(t) - \tilde{\Lambda}(t) \leq \int_0^\infty \left| \phi(x) \right| \mathbb{P} \left( \inf_{s \in [0,t]} (W_s - \Gamma(s)) \in [-x - \sup_{[0,T]}|\Gamma - \tilde{\Gamma}|, -x] \right) \, dx \]
\[ \leq \sup_{\mathbb{R}_+} |\phi| \mathbb{E} \int_0^\infty 1_{\left\{-x - \sup_{[0,T]} |\Gamma - \tilde{\Gamma}| \leq \inf_{s \in [0,t]} (W_s - \Gamma(s)) \leq -x \right\}} dx \leq \sup_{\mathbb{R}_+} |\phi| \sup_{[0,T]} |\Gamma - \tilde{\Gamma}| \]
Single-phase supercooled Stefan, \( d = 1 \): well-posedness w/o smallness

\[
X_t = \xi + W_{t^\wedge \tau}, \quad \xi \sim -\phi(x)dx \geq 0, \quad x > 0,
\]
\[
\tau := \inf\{s \geq 0 : \xi + W_s \leq \Lambda(s)\}, \quad \Lambda(t) = \mathbb{P}(\tau \leq t),
\]
\[
\Lambda(t) - \Lambda(t^-) = \inf\{z > 0 : \mathbb{P}(\tau \geq t, X_{t^-} \in (\Lambda(t^-), \Lambda(t^-) + z]) < z\}
\]

- If we do **not** assume that \( |\phi| < 1 \), the previous argument fails. Nevertheless, existence (by approximation) and uniqueness (by a different argument) of the solution have been established.

- **Theorem.** *(Delarue et al 2015, N.-Shkolnikov 2019, Ledger-Sojmark 2019, Cuchiero et al 2021)* There **exists** a probabilistic solution \((X, \Lambda)\) for any absolutely integrable \( -\phi \geq 0 \), \( \int \phi = -1 \).

- **Theorem.** *(Delarue-N.-Shkolnikov 2019)* Assume that \( -\phi \geq 0 \) is bounded and changes its monotonicity finitely many times on any compact. Then, the probabilistic solution \((X, \Lambda)\) is **unique**.
Stefan equation as a mean-field system

Supercooled Stefan: results for $d = 1$

**Single-phase supercooled Stefan, $d = 1$: existence via particle representation**

- Particles $\{X^{i,N}_i\}_{i=1}^N$ perform random walks, started from i.i.d. $\{X^{0,N}_i\}_{i=1}^N$ generated from $-\phi(x) \, dx$.
- Each particle is absorbed when hitting the aggregate $[0, \Lambda(\cdot)]$, increasing the aggregate by $1/N$.

**Theorem** (Delarue et al 2015, N.-Shkolnikov 2019, Ledger-Sojmark 2019, Cuchiero et al 2020) As $N \to \infty$, there exists at least one (weak) limit point $(X, \Lambda)$ of $\{(X^{1,N}_i, \Lambda^N)\}_N$, and any such limit point is a (probabilistic) solution to the single-phase supercooled Stefan problem with $d = 1$. 
Single-phase supercooled Stefan, $d = 1$: proof of uniqueness

$$X_t = \xi + W_{t \wedge \tau}, \quad \xi \sim -\phi(x)dx \geq 0, \quad x > 0,$$

$$\tau := \inf\{s \geq 0 : \xi + W_s \leq \Lambda(s)\}, \quad \Lambda(t) = \mathbb{P}(\tau \leq t),$$

$$\Lambda(t) - \Lambda(t^-) = \inf\{z > 0 : \mathbb{P}(\tau \geq t, X_{t^-} \in (\Lambda(t^-), \Lambda(t^-) + z] < z\}$$

**Proposition.** Assume that $u(0^-, \cdot) = -\phi(\cdot)$ is bounded and changes its monotonocity finitely many times on any compact. Then, for any two probabilistic solutions $(X_1^t, \Lambda^1(t)), (X_2^t, \Lambda^2(t))$, with $\Lambda^1(0^-) = \Lambda^2(0^-)$, there exist $\delta, \varepsilon > 0$ s.t. $u^1(0, x) = u^2(0, x) > -1$ for $x \in (0, \delta]$, and hence the two solutions coincide for $t \in [0, \varepsilon]$.

Thus, it remains to prove that, for any $t$, $u(t^-, \cdot)$ changes its monotonocity finitely many times on any compact.
Points of monotonicity changes cannot multiply

Extending the results of *Angenent-Fiedler 1988*, we show that the curves that separate the domains of positive and negative values of

$$\nu(t, \cdot) := -u_x(t, \cdot)$$

look as follows:

![Graph showing points of monotonicity changes](image)
Prob-c rep-n: single-phase supercooled Stefan, $d \geq 2$ (N.-Shkolnikov-Zhang 2022)

\[ u_t - \frac{1}{2}u_{xx} = 0, \quad x \in D(t), \quad t > 0, \]
\[ u|_{\partial D(t)} = 0, \quad u(x, 0) = \phi(x) \leq 0, \]
\[ \dot{D}(x, t) = V(x, t) = \frac{1}{2} \left[ \lim_{D^c(t) \ni y \to x} u_x(y, t) \cdot \nu(t, x) - \lim_{D(t) \ni y \to x} u_x(y, t) \cdot \nu(t, x) \right] \]

The first two lines of the above are expressed probabilistically as:

\[ X_t = \xi + W_{t \wedge \tau}, \quad \xi \sim -\phi(x)dx \geq 0, \]
\[ \tau := \inf\{s \geq 0 : \xi + W_s \in \Gamma(s)\}, \quad \Gamma(t) := \mathbb{R}^d \setminus \overline{D(t)} \]

**Growth condition** (3) is equivalent to: $\mathbb{P}(X_t \in dx, \tau \leq t) = 1_{\Gamma(t)}(x) \, dx$. (See also Kim-Kim 2021 for connections to optimal transport.)
$d = 2$: Diffusion Limited Aggregation (DLA)

- Particles $\{X^i, N\}_{i=1}^N$ perform random walks.
- Each particle is absorbed when hitting the aggregate $\Gamma(\cdot)$, adding the square in which the particle is located to the aggregate.
- This is a Multiparticle External Diffusion Limited Aggregation.
Negative result for $d = 2$

N.-Shkolnikov-Zhang 2021. Every limit point of $\{(X_{1,N}, \Lambda^N)\}_N$ (at least one limit point does exist) corresponds to a BM absorbed at hitting the aggregate, but the growth condition is only satisfied with inequality: $\mathbb{P}(X_t \in dx, \tau \leq t) < \mathbf{1}_{\Gamma(t)}(x)dx$. 
Two-phase Stefan with surface tension

\[ u_t - \mathcal{L}(u_{xx}, u_x, x, t) = 0, \quad x \in \mathbb{R}^d \setminus \partial D(t), \quad t > 0, \quad u(x, 0) = \phi(x), \]

\[ u|_{\partial D(t)} = bH, \quad H \text{ is the mean curvature of } \partial D(t), \]

\[ \dot{D}(x, t) = V(x, t) = \frac{1}{2} \left[ \lim_{D^c(t) \ni y \to x} u_x(y, t) \cdot \nu(t, x) - \lim_{D(t) \ni y \to x} u_x(y, t) \cdot \nu(t, x) \right] \]

- Why surface tension? It regularizes \( \partial \Gamma(t)(= \partial D(t)) \) and eliminates the degeneracy observed in limiting DLA aggregate.

- New challenges:
  - Single-phase simplification is impossible (only two-phase).
  - It is typically impossible to avoid supercooling, even if \( \phi \geq 0 \).
  - Partial maximum principle (available in 1-phase 1-dim. supercooled Stefan, with \( b = 0 \)) no longer holds.

- Merimanov 1994 shows that classical solution fails to exist globally even in a radially symmetric case.

- Luckhaus 1990 constructs weak solutions, but shows that the proposed notion of a solution is too weak to yield uniqueness.
Two-phase Stefan w. surface tension: probab-c representation?

One can try to construct an analogous McKean-Vlasov (or particle) system.

Main problem: if $u(\cdot, t)$ was the density of a limiting process $X_t$, absorbed at $\Gamma$, we would have $u|_{\partial \Gamma(t)} = 0$!

What is the probabilistic interpretation of $u|_{\partial \Gamma(t)} = H$?
Main idea: particles are not only absorbed at hitting the aggregate, but new particles are expelled from the aggregate.

Consider Brownian particles generated at Poisson times with rate $1/(2\delta)$, stated from the initial location $\delta > 0$, and absorbed at zero.

The (non-probability) measure given by the expected sum of distributions of all survived particles at time $t$ is

$$u^\delta(x, t) = t \frac{\mathbb{P}(B^\delta_{t-\tau} \in dx, \inf_{s \in [0, t-\tau]} B^\delta_s > 0)}{2\delta \, dx}, \quad x > 0,$$

where $B^\delta$ is a BM started from $\delta > 0$, independent of $\tau \sim \text{Unif}(0, t)$.

**Claim.** For any $x > 0$, we have

$$\lim_{\delta \downarrow 0} u^\delta(x, t) = 2(1 - \Phi(x/\sqrt{t})) =: u^0(x, t), \quad u^0_t = \frac{1}{2} u^0_{xx}, \quad u^0(0, t) = 1,$$

where $\Phi$ is the standard normal c.d.f.
Stefan w. surface tension, under radial symmetry

\[ u_t - \frac{1}{2} u_{xx} = 0, \quad x \in \mathbb{R}^d, \quad |x| \neq \Lambda(t), \quad t > 0, \quad u(x, 0) = \phi(x), \]

\[ u|_{|x| = \Lambda(t)} = H(\Lambda(t)) = -1/\Lambda(t), \]

\[ \dot{\Lambda}(t) = \frac{1}{2} \left[ \lim_{y \to x, \; |y| < |x|} u_x(y, t) \cdot \frac{x}{|x|} - \lim_{y \to x, \; |y| > |x|} u_x(y, t) \cdot \frac{x}{|x|} \right], \quad |x| = \Lambda(t) \]

Due to radial symmetry, we expect that \( u(x, t) \) depends on \( x \) only via \( |x| \). Thus, we can reduce the above Stefan problem to an equation in spatial dimension one:

\[ u_t - \frac{1}{2} u_{xx} - \frac{d - 1}{2x} u_x = 0, \quad x \in \mathbb{R} \setminus \{\Lambda(t)\}, \quad t > 0, \quad u(x, 0) = \phi(x), \]

\[ u(\Lambda(t)^\pm, t) = -1/\Lambda(t), \]

\[ \dot{\Lambda}(t) = \frac{1}{2} \left[ u_x(\Lambda(t)^-, t) - u_x(\Lambda(t)^+, t) \right] \]
Stefan w. surface tension, under radial symmetry: probabilistic representation

\[ u_t - \frac{1}{2} u_{xx} - \frac{d - 1}{2x} u_x = 0, \quad x \in \mathbb{R} \setminus \{\Lambda(t)\}, \quad t > 0, \quad u(x, 0) = \phi(x), \]

\[ u(\Lambda(t)^\pm, t) = H(\Lambda(t)) := -1/\Lambda(t), \]

\[ \dot{\Lambda}(t) = \frac{1}{2} \left[ u_x(\Lambda^-, t) - u_x(\Lambda^+, t) \right] \]

The following Feynman-Kac formula gives a probabilistic representation for \( v := -u \):

\[ v(t, x) := \mathbb{E}^x[\mathbf{1}_{\{\tau \leq t\}} \cdot H(R_\tau)] + \mathbb{E}^x[\mathbf{1}_{\{\tau > t\}} \cdot (-\phi(R_t))], \]

\[ \frac{1}{d} (\Lambda^d(t) - \Lambda^d(0^-)) = \int_{\mathbb{R}^+} (-\phi(x)) \nu(dx) - \int_{\mathbb{R}^+} v(t, x) \nu(dx), \]

where \( R \) is a Bessel process, \( \nu(dx) := x^{d-1} dx \), and \( \tau \) is the first crossing time of \( \Lambda(t - \cdot) \) by \( R \).
Probabilistic solution to Stefan problem w. surface tension, under radial symmetry: definition

\[
v(t, x) := \mathbb{E}^x[1_{\{\tau \leq t\}} \cdot H(R_\tau)] + \mathbb{E}^x[1_{\{\tau > t\}} \cdot (-\phi(R_t))],
\]

\[
\frac{1}{d} (\Lambda^d(t) - \Lambda^d(0^-)) = \int_{\mathbb{R}^+} (-\phi(x)) \nu(dx) - \int_{\mathbb{R}^+} v(t, x) \nu(dx),
\]

To have a chance for uniqueness, we add the jump condition:

\[
\Lambda(t^-) - \Lambda(t) = \inf \left\{ z \in (0, \Lambda(t^-)] : \int_{\Lambda(t^-) - z}^{\Lambda(t^-)} v(t^-, x) \nu(dx) > \int_{\Lambda(t^-) - z}^{\Lambda(t^-)} (H(x) - 1) \nu(dx) \right\},
\]

\[
\Lambda(t) - \Lambda(t^-) = \inf \left\{ z > 0 : \int_{\Lambda(t^-)}^{\Lambda(t^-) + z} v(t^-, x) \nu(dx) < \int_{\Lambda(t^-)}^{\Lambda(t^-) + z} (H(x) + 1) \nu(dx) \right\}.
\]
Theorem (N.-Shkolnikov 2022). Assume that $\phi \geq -1$ and that $|\phi(x)| \leq C \exp(-Cx)$. Then, for $d \geq 3$, there exists a solution to the above system.
Numerical experiment

Figure: Graph of $\Lambda(\cdot)$, for $d=3$ and the initial data $\Lambda(0^-) = 0.9$, $\phi = -1_{[0,0.81]}$. 
About the proof

For a given piece-wise constant cadlag $\Lambda^\Delta$, consider:

- a $d$-dimensional Bessel process $X$ started according to the density $-\phi(x) x^{d-1}$

- and its crossing time $\tau^\Delta := \inf\{t > 0 : (X_t - \Lambda^\Delta_t)(X_0 - \Lambda^\Delta_0) < 0\}$ of $\Lambda^\Delta$;

- for $m \geq 1$, the $d$-dim. Bessel processes $\{X^m,i,\Delta\}_{i \geq 1}$ started at time $m\Delta$ from the atoms of an independent Poisson random measure of intensity $H$ in the interval between $\Lambda^\Delta_{(m-1)\Delta}$ and $\Lambda^\Delta_{m\Delta}$,

- and their crossing times $\tau^m,i,\Delta$ of $\Lambda^\Delta$;

- the jumps times $\{T^\delta,i,\Delta\}_{i \geq 1}$ of a Poisson process with rate $2\delta^{-1}(\Lambda^\Delta)^{d-2}$, for $\delta > 0$;

- $[-1,1]$-valued independent uniform random variables $\{\gamma_i\}_{i \geq 1}$;

- independent $d$-dimensional Bessel processes $\{Y^\delta,i,\Delta\}_{i \geq 1}$ started at the times $\{T^\delta,i,\Delta\}_{i \geq 1}$ from $\{(\Lambda^\Delta_{T^\delta,i,\Delta} + \delta \gamma_i) \vee 0\}_{i \geq 1}$, respectively,

- and their crossing times $\tau^\delta,i,\Delta$ of $\Lambda^\Delta$. 
Implicit Euler scheme

Fix $\Delta > 0$ and define $\Lambda^\Delta$ by solving the following recursive equations, for all $m = 0, 1, \ldots$:

$$
\frac{1}{d} \left( (\Lambda^\Delta_{m\Delta})^d - (\Lambda^\Delta_{0\Delta})^d \right) = \mathbb{P}(\tau^\Delta \leq m\Delta) - \sum_{n=1}^{m} \sum_{i \geq 1} \mathbb{P}(\tau^n_{i,\Delta} > m\Delta)
$$

$$
- \lim_{\delta \downarrow 0} \sum_{i \geq 1} \mathbb{P}(T_{i,\delta,\Delta} \leq m\Delta < \tau_{i,\delta,\Delta})
$$

**Theorem** *(N.-Shkolnikov 2022)* $\{\Lambda^\Delta\}_{\Delta \downarrow 0}$ is pre-compact in $\text{M}1$, and its every limit point yields a probabilistic solution to the supercooled Stefan problem with surface tension, under radial symmetry.