Probabilistic solutions to Stefan equations with supercooling

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Stefan equation (Visintin 1998)

$$u_t - \mathcal{L}(u_{xx}, u_x, x, t) = 0, \quad x \in D(t) \text{ or } \mathbb{R}^d \setminus \partial D(t), \quad t > 0, \quad u(x,0) = \phi(x), \tag{1}$$

$$u|_{\partial D(t)} = aV + bH, \quad H \text{ is the mean curvature of } \partial D(t),$$

where $\mathcal{L}$ is an **elliptic** operator (e.g., $\mathcal{L} = \frac{1}{2}u_{xx}$) and the **normal velocity** $V(x,t) := \dot{D}(x,t)$ of $\partial D(t)$ at $x \in \partial D(t)$ must, in addition, satisfy

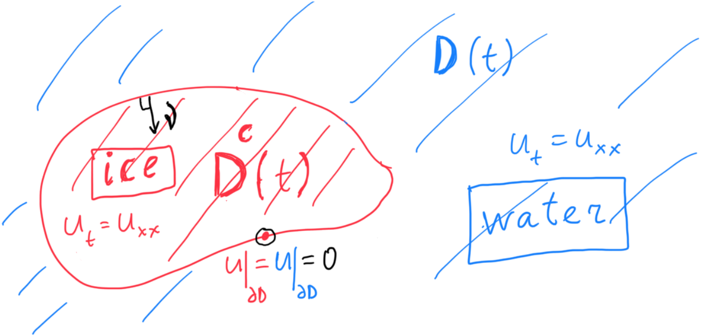
$$V(x,t) = \frac{1}{2}\left[\lim_{D^c(t)\ni y\to x} u_x(y,t)\cdot\nu(t,x) - \lim_{D(t)\ni y\to x} u_x(y,t)\cdot\nu(t,x)\right],$$

with ν being the **outer unit normal** to $\partial D(t)$.

- Elliptic version of this problem is known as **Hele-Shaw** equation. Various modifications of (1) form **Laplacian growth** models.
- **Applications**: melting/solidification, condensation, crystal growth, aging of alloys, interaction of fluids with different viscosities, dynamics of membrane potentials in a network of neurons, tumor growth, etc.

Stefan equation

$$\dot{D}(x,t) = V(x,t) = \frac{1}{2}\left[\lim_{D^c(t)\ni y\to x} u_x(y,t)\cdot\nu(t,x) - \lim_{D(t)\ni y\to x} u_x(y,t)\cdot\nu(t,x)\right]$$



Probab-c rep-n for regular ($\phi \geq 0$, $a = b = 0$), single-phase, one-dim. ($d = 1$) Stefan problem

$$u_t - \frac{1}{2}u_{xx} = 0, \quad x > \Lambda(t), \quad t > 0, \quad u(x,0) = \phi(x),$$
$$u(\Lambda(t)^+, t) = 0, \quad -\dot{\Lambda}(t) = \frac{1}{2}u_x(\Lambda(t)^+, t), \quad \Lambda(0) = 0$$

- Assume $\phi \geq 0$, $\int_0^\infty \phi = 1$, and let $\varphi(\cdot, t)$ be Gaussian kernel with variance t.
- Feynman-Kac formula and time reversal of BM W yield

$$\sigma := \inf\{s \geq 0 :\ x + W_s \leq \Lambda(t-s)\}, \quad u(x,t) = \mathbb{E}\left[\phi(x + W_t)\mathbf{1}_{\sigma > t}\right]$$
$$= \int_0^\infty \phi(y)\varphi(x-y,t)\mathbb{P}\left(\inf_{s\in[0,t]}(x + W_s - \Lambda(t-s)) > 0 \,|\, x + W_t = y\right) dy$$
$$= \int_0^\infty \phi(y)\varphi(x-y,t)\mathbb{P}\left(\inf_{s\in[0,t]}(y + W_s - \Lambda(s)) > 0 \,|\, y + W_t = x\right) dy$$
$$= \mathbb{P}(X_t \in dx, \tau > t)/dx, \quad X_t = \xi + W_t, \quad \tau := \inf\{s \geq 0 :\ \xi + W_s \leq \Lambda(s)\},$$

with an independent r.v. ξ having density ϕ.

Probab-c representation of the growth condition

$$
\begin{aligned}
&u(x,t) = \mathbb{P}(X_t \in dx, \tau > t)/dx, \quad x > \Lambda(t),\\
&X_t = (\xi + W_{t\wedge\tau}), \quad \tau := \inf\{s \geq 0 : \xi + W_s \leq \Lambda(s)\},\\
&-\dot{\Lambda}(t) = \frac{1}{2} u_x(\Lambda(t)^+, t), \quad \Lambda(0) = 0
\end{aligned} \tag{2}
$$

- We have established that $u(\cdot, t)$ is the density of the marginal distribution of a BM absorbed at Λ, before absorption.
- To derive a probabilistic representation for Λ (to replace (2)), we notice that

$$
\begin{aligned}
&\frac{d}{dt}\mathbb{P}(\tau > t) = \frac{d}{dt}\int_{\Lambda(t)} u(x,t)dx = -\dot{\Lambda}(t)u(\Lambda(t)+,t) + \int_{\Lambda(t)} u_t(x,t)dx\\
&= \frac{1}{2}\int_{\Lambda(t)} u_{xx}(x,t)dx = -\frac{1}{2}u_x(\Lambda(t),t) = \dot{\Lambda}(t),\\
&\Lambda(t) = -\mathbb{P}(\tau \leq t)
\end{aligned}
$$

Stefan problem as a McKean-Vlasov equation: single-phase, $d = 1$, regular

$$u(x,t) = \mathbb{P}(X_t \in dx, \tau > t)/dx, \quad x > \Lambda(t), \quad \Lambda(t) = -\mathbb{P}(\tau \leq t),$$
$$X_t = (\xi + W_{t\wedge\tau}), \quad \tau := \inf\{s \geq 0 : \xi + W_s \leq \Lambda(s)\}, \quad \xi \sim u(0,x)\,dx$$

- The above system is a McKean-Vlasov equation, as the dynamics of X depend explicitly on its distribution.
- *Levine-Peres 2010* show that the regular Stefan (and Hele-Shaw) equation can be obtained as a scaling limit of Internal **Diffusion Limited Aggregation** (**DLA**) model.
- This probabilistic connection offers a new perspective (and a new numerical method), but the well-posedness of **regular** Stefan equation (for any d and with any number of phases) was established a long time ago (e.g., in *Kamenomostskaja 1961*).

Supercooled Stefan problem

$$u_t - \mathcal{L}(u_{xx}, u_x, x, t) = 0, \quad x \in D(t) \text{ or } \mathbb{R}^d \setminus \partial D(t), \quad t > 0, \quad u(x,0) = \phi(x),$$

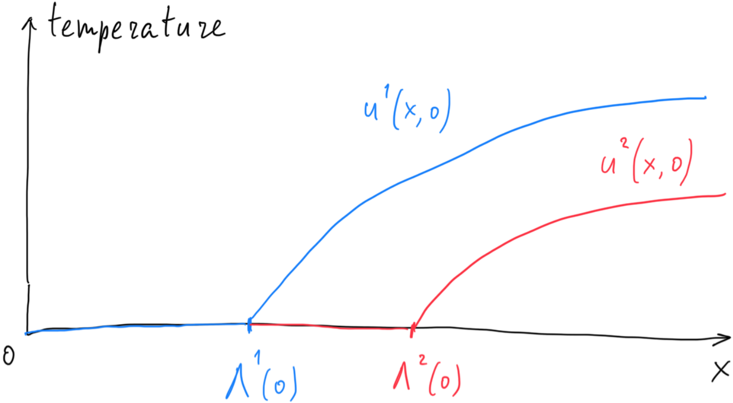
$$u|_{\partial D(t)} = aV + bH, \quad H \text{ is the mean curvature of } \partial D(t),$$

$$\dot{D}(x,t) = V(x,t) = \frac{1}{2}\left[\lim_{D^c(t)\ni y\to x} u_x(y,t)\cdot\nu(t,x) - \lim_{D(t)\ni y\to x} u_x(y,t)\cdot\nu(t,x)\right]$$

- If $\phi \leq 0$, $a = b = 0$ (and w.l.o.g. $\int_0^\infty \phi = -1$), we call this Stefan problem **supercooled**, as the liquid temperature is below the freezing point.
- Such problems appear in the modeling of: solidification process (e.g., to produce glassy metals), crystal growth, macroscopic dynamics of neurons' membrane potentials, Hele-Shaw cell, etc.
- Main challenges (as compared to regular case):
  1. No global **comparison principle**.
  2. **Time-singularity** of $D(\cdot)$.

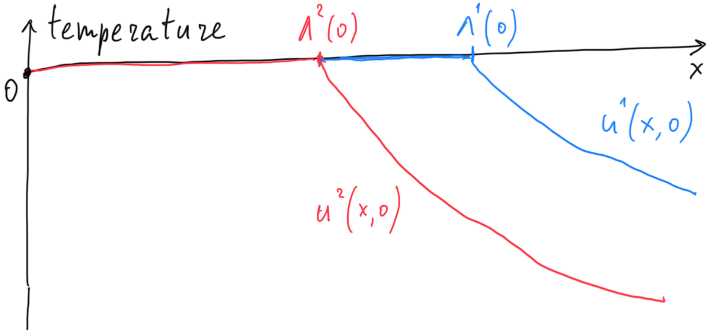
Regular Stefan: comparison principle

$$u^1(\cdot, 0) \geq u^2(\cdot, 0) \;\Rightarrow\; u^1(\cdot, t) \geq u^2(\cdot, t)$$

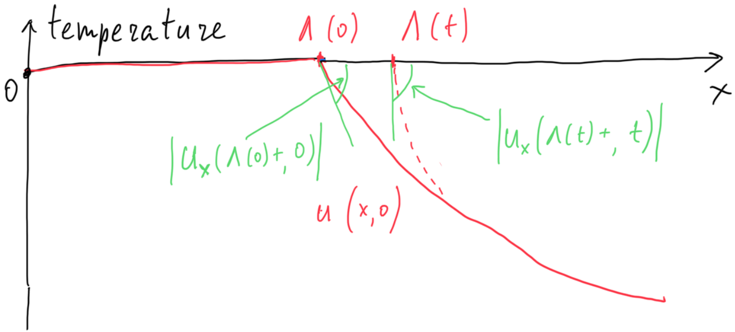


Supercooled Stefan: no comparison principle

$$u^1(\cdot,0) \geq u^2(\cdot,0) \;\;\not\Rightarrow\;\; u^1(\cdot,t) \geq u^2(\cdot,t)$$

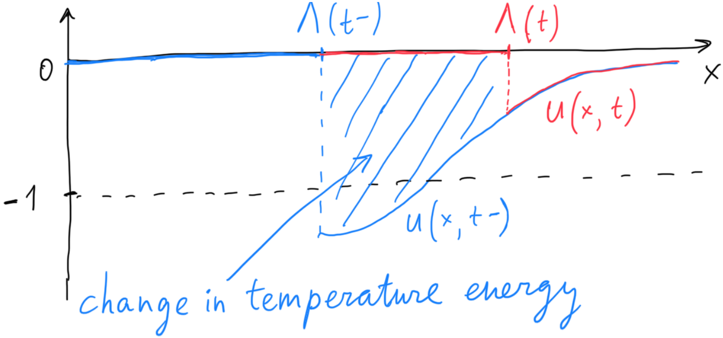


Time-singularity of the boundary



The faster the boundary moves up, the steeper is the graph of $u(\cdot,t)$ at the boundary.

Jump size of the boundary



$$\Lambda(t) - \Lambda(t^-) = \inf\{z > 0 : -\int_{\Lambda(t^-)}^{\Lambda(t^-)+z} u(y, t^-)\, dy < z\}$$

Probabilistic representation for single-phase supercooled Stefan, $d = 1$

$$u_t - \frac{1}{2}u_{xx} = 0, \quad x > \Lambda(t), \quad t > 0, \quad u(x,0) = \phi(x) \leq 0,$$
$$u(\Lambda(t)^+, t) = 0, \quad -\dot{\Lambda}(t) = \frac{1}{2}u_x(\Lambda(t)^+, t), \quad \Lambda(0) = 0$$

- Similar to the regular case, one can derive the following probabilistic representation of a single-phase supercooled Stefan problem, with $d = 1$:

$$X_t = \xi + W_{t\wedge\tau}, \quad \xi \sim -\phi(x)dx \geq 0, \quad x > 0,$$
$$\tau := \inf\{s \geq 0 : \xi + W_s \leq \Lambda(s)\}, \quad \Lambda(t) = \mathbb{P}(\tau \leq t),$$
$$\Lambda(t) - \Lambda(t-) = \inf\{z > 0 : \mathbb{P}(\tau \geq t,\, X_{t-} \in (\Lambda(t-), \Lambda(t-) + z]) < z\}$$

- **Theorem**. For **small initial data**, $-1 < \phi \leq 0$, $\int \phi = -1$, there exists a unique (smooth) solution to the above system.
- This result was first established in 1980s: see, e.g., *Fasano-Primicerio 1981* and *Chayes-Swindle 1996* for PDE methods, and *Ledger-Sojmark 2019* for probabilistic approach.

Sketch of the proof: small initial data

$$X_t = \xi + W_{t\wedge\tau}, \quad \xi \sim -\phi(x)dx, \quad x > 0,$$
$$\tau := \inf\{s \geq 0 : \xi + W_s \leq \Lambda(s)\}, \quad \Lambda(t) = \mathbb{P}(\tau \leq t)$$

The mapping $\Gamma \mapsto \Lambda$,

$$\Lambda(t) := \mathbb{P}(\inf_{s\in[0,t]}(\xi + W_s - \Gamma(s)) \leq 0) = -\int_0^\infty \phi(x)\,\mathbb{P}(\inf_{s\in[0,t]}(W_s - \Gamma(s)) \leq -x)\,dx,$$

is a **contraction**:

$$\Lambda(t) - \tilde{\Lambda}(t) \leq \int_0^\infty |\phi(x)|\,\mathbb{P}\left(\inf_{s\in[0,t]}(W_s - \Gamma(s)) \in [-x - \sup_{[0,T]}|\Gamma - \tilde{\Gamma}|, -x]\right)dx$$

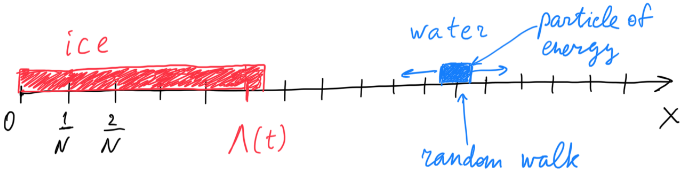
$$\leq \sup_{\mathbb{R}_+}|\phi|\,\mathbb{E}\int_0^\infty \mathbf{1}_{\left\{-x-\sup_{[0,T]}|\Gamma-\tilde{\Gamma}| \leq \inf_{s\in[0,t]}(W_s-\Gamma(s)) \leq -x\right\}}\,dx \leq \sup_{\mathbb{R}_+}|\phi| \sup_{[0,T]}|\Gamma - \tilde{\Gamma}|$$

Single-phase supercooled Stefan, $d = 1$: well-posedness w/o smallness

$$X_t = \xi + W_{t\wedge\tau}, \quad \xi \sim -\phi(x)dx \geq 0, \quad x > 0,$$
$$\tau := \inf\{s \geq 0 : \xi + W_s \leq \Lambda(s)\}, \quad \Lambda(t) = \mathbb{P}(\tau \leq t),$$
$$\Lambda(t) - \Lambda(t^-) = \inf\{z > 0 : \mathbb{P}\left(\tau \geq t,\, X_{t^-} \in (\Lambda(t^-), \Lambda(t^-) + z] < z\right)$$

- If we do **not** assume that $|\phi| < 1$, the previous argument fails. Nevertheless, existence (by approximation) and uniqueness (by a different argument) of the solution have been established.
- **Theorem**. (*Delarue et al 2015, N.-Shkolnikov 2019, Ledger-Sojmark 2019, Cuchiero et al 2021*) There **exists** a probabilistic solution (X, Λ) for any absolutely integrable $-\phi \geq 0$, $\int \phi = -1$.
- **Theorem**. (*Delarue-N.-Shkolnikov 2019*) Assume that $-\phi \geq 0$ is bounded and changes its monotonicity finitely many times on any compact. Then, the probabilistic solution (X, Λ) is **unique**.

Single-phase supercooled Stefan, $d = 1$: existence via particle representation



- Particles $\{X^{i,N}\}_{i=1}^N$ perform random walks, started from i.i.d. $\{X_0^{i,N}\}_{i=1}^N$ generated from $-\phi(x)\,dx$.
- Each particle is absorbed when hitting the **aggregate** $[0, \Lambda(\cdot)]$, increasing the aggregate by $1/N$.
- **Theorem** (*Delarue et al 2015, N.-Shkolnikov 2019, Ledger-Sojmark 2019, Cuchiero et al 2020*) As $N \to \infty$, there **exists** at least one (weak) limit point (X, Λ) of $\{(X^{1,N}, \Lambda^N)\}_N$, and any such limit point is a (probabilistic) **solution** to the single-phase supercooled Stefan problem with $d = 1$.

Single-phase supercooled Stefan, $d = 1$: proof of uniqueness

$$X_t = \xi + W_{t\wedge\tau}, \quad \xi \sim -\phi(x)dx \geq 0, \quad x > 0,$$
$$\tau := \inf\{s \geq 0 : \xi + W_s \leq \Lambda(s)\}, \quad \Lambda(t) = \mathbb{P}(\tau \leq t),$$
$$\Lambda(t) - \Lambda(t^-) = \inf\{z > 0 : \mathbb{P}\left(\tau \geq t,\, X_{t^-} \in (\Lambda(t^-), \Lambda(t^-) + z] < z\right)$$

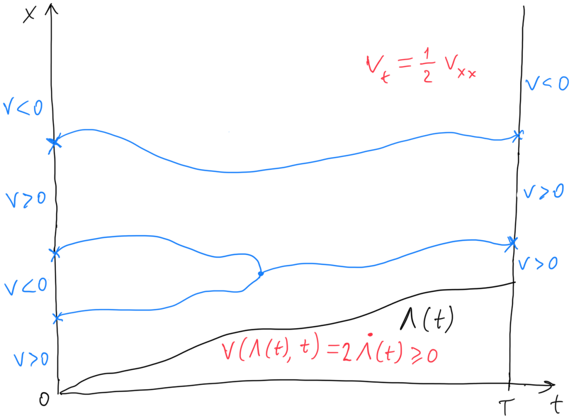
- **Proposition**. Assume that $u(0^-, \cdot) = -\phi(\cdot)$ is bounded and changes its monotonicity finitely many times on any compact. Then, for any two probabilistic solutions $(X^1_t, \Lambda^1(t))$, $(X^2_t, \Lambda^2(t))$, with $\Lambda^1(0^-) = \Lambda^2(0^-)$, there exist $\delta, \varepsilon > 0$ s.t. $u^1(0, x) = u^2(0, x) > -1$ for $x \in (0, \delta]$, and hence the two solutions coincide for $t \in [0, \varepsilon]$.
- Thus, it remains to prove that, for any t, $u(t^-, \cdot)$ changes its monotonicity finitely many times on any compact.

Points of monotonicity changes cannot multiply

Extending the results of *Angenent-Fiedler 1988*, we show that the curves that separate the domains of positive and negative values of

$$v(t,\cdot) := -u_x(t,\cdot)$$

look as follows:



Prob-c rep-n: single-phase supercooled Stefan, $d \geq 2$ (N.-Shkolnikov-Zhang 2022)

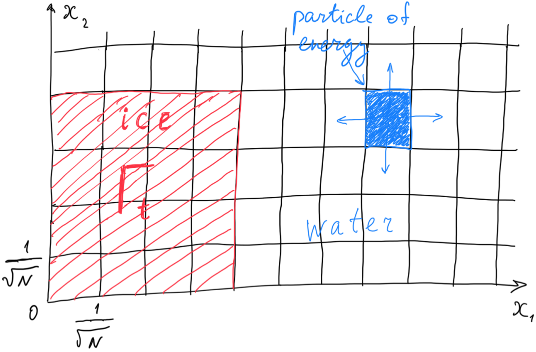
$$
\begin{aligned}
& u_t - \frac{1}{2} u_{xx} = 0, \quad x \in D(t), \quad t > 0, \\
& u|_{\partial D(t)} = 0, \quad u(x,0) = \phi(x) \leq 0, \\
& \dot{D}(x,t) = V(x,t) = \frac{1}{2}\left[\lim_{D^c(t) \ni y \to x} u_x(y,t) \cdot \nu(t,x) - \lim_{D(t) \ni y \to x} u_x(y,t) \cdot \nu(t,x)\right]
\end{aligned}
\tag{3}
$$

- The first two lines of the above are expressed probabilistically as:

$$
\begin{aligned}
& X_t = \xi + W_{t \wedge \tau}, \quad \xi \sim -\phi(x)dx \geq 0, \\
& \tau := \inf\{s \geq 0 : \xi + W_s \in \Gamma(s)\}, \quad \Gamma(t) := \mathbb{R}^d \setminus \overline{D}(t)
\end{aligned}
$$

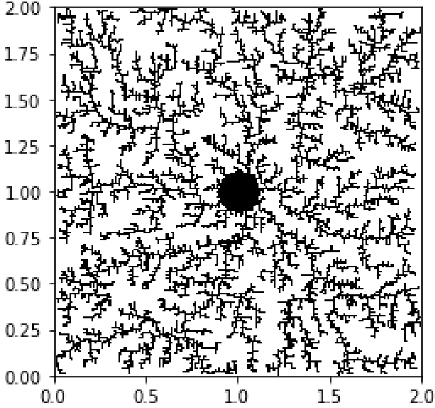
- **Growth condition** (3) is equivalent to: $\mathbb{P}(X_t \in dx, \tau \leq t) = \mathbf{1}_{\Gamma(t)}(x)\, dx$. (See also *Kim-Kim 2021* for connections to optimal transport.)

$d = 2$: Diffusion Limited Aggregation (DLA)



- Particles $\{X^{i,N}\}_{i=1}^{N}$ perform random walks.
- Each particle is absorbed when hitting the **aggregate** $\Gamma(\cdot)$, adding the square in which the particle is located to the aggregate.
- This is a **Multiparticle External Diffusion Limited Aggregation**.

Negative result for $d = 2$



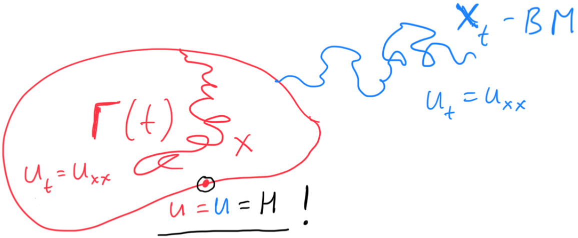
N.-Shkolnikov-Zhang 2021. Every limit point of $\{(X^{1,N}, \Lambda^N)\}_N$ (at least one limit point does exist) corresponds to a BM absorbed at hitting the aggregate, but the growth condition is only satisfied with inequality: $\mathbb{P}(X_t \in dx, \tau \leq t) < \mathbf{1}_{\Gamma(t)}(x)dx$.

Two-phase Stefan with surface tension

$$u_t - \mathcal{L}(u_{xx}, u_x, x, t) = 0, \quad x \in \mathbb{R}^d \setminus \partial D(t), \quad t > 0, \quad u(x,0) = \phi(x),$$
$$u|_{\partial D(t)} = bH, \quad H \text{ is the mean curvature of } \partial D(t),$$
$$\dot{D}(x,t) = V(x,t) = \frac{1}{2}\left[\lim_{D^c(t)\ni y\to x} u_x(y,t)\cdot\nu(t,x) - \lim_{D(t)\ni y\to x} u_x(y,t)\cdot\nu(t,x)\right]$$

- Why surface tension? It **regularizes** $\partial\Gamma(t)(=\partial D(t))$ and eliminates the degeneracy observed in limiting DLA aggregate.
- New challenges:
  - Single-phase simplification is impossible (only **two-phase**).
  - It is typically impossible to avoid **supercooling**, even if $\phi \geq 0$.
  - Partial **maximum principle** (available in 1-phase 1-dim. supercooled Stefan, with $b = 0$) no longer holds.
- *Merimanov 1994* shows that classical solution fails to exist globally even in a radially symmetric case.
- *Luckhaus 1990* constructs weak solutions, but shows that the proposed notion of a solution is too weak to yield uniqueness.

Two-phase Stefan w. surface tension: probab-c representation?



- One can try to construct an analogous McKean-Vlasov (or particle) system.
- **Main problem: if $u(\cdot, t)$ was the density of a limiting process X_t, absorbed at Γ, we would have $u|_{\partial\Gamma(t)} = 0$!**
- **What is the probabilistic interpretation of $u|_{\partial\Gamma(t)} = H$?**

Probab-c rep-n of the surface tension condition $u|_{\partial\Gamma(t)} = H$

- Main idea: particles are not only absorbed at hitting the aggregate, but **new particles are expelled from the aggregate**.
- Consider Brownian particles generated at Poisson times with rate $1/(2\delta)$, stated from the initial location $\delta > 0$, and absorbed at zero.
- The (non-probability) measure given by the expected sum of distributions of all survived particles at time t is

$$u^\delta(x,t) = t\,\frac{\mathbb{P}(B^\delta_{t-\tau} \in dx,\ \inf_{s\in[0,t-\tau]} B^\delta_s > 0)}{2\delta\, dx}, \quad x > 0,$$

  where B^δ is a BM started from $\delta > 0$, independent of $\tau \sim Unif(0,t)$.
- **Claim**. For any $x > 0$, we have

$$\lim_{\delta\downarrow 0} u^\delta(x,t) = 2(1-\Phi(x/\sqrt{t})) =: u^0(x,t), \quad u^0_t = \frac{1}{2}u^0_{xx}, \quad u^0(0,t) = 1,$$

  where Φ is the standard normal c.d.f.

Stefan w. surface tension, under radial symmetry

$$u_t - \frac{1}{2} u_{xx} = 0, \quad x \in \mathbb{R}^d, \ |x| \neq \Lambda(t), \quad t > 0, \quad u(x,0) = \phi(x),$$
$$u|_{|x|=\Lambda(t)} = H(\Lambda(t)) = -1/\Lambda(t),$$
$$\dot{\Lambda}(t) = \frac{1}{2} \left[\lim_{y \to x, \, |y|<|x|} u_x(y,t) \cdot \frac{x}{|x|} - \lim_{y \to x, \, |y|>|x|} u_x(y,t) \cdot \frac{x}{|x|} \right], \quad |x| = \Lambda(t)$$

Due to radial symmetry, we expect that $u(x,t)$ depends on x only via $|x|$. Thus, we can reduce the above Stefan problem to an equation in spatial dimension one:

$$u_t - \frac{1}{2} u_{xx} - \frac{d-1}{2x} u_x = 0, \quad x \in \mathbb{R} \setminus \{\Lambda(t)\}, \quad t > 0, \quad u(x,0) = \phi(x),$$
$$u(\Lambda(t)^{\pm}, t) = -1/\Lambda(t),$$
$$\dot{\Lambda}(t) = \frac{1}{2} \left[u_x(\Lambda(t)^-, t) - u_x(\Lambda(t)^+, t) \right]$$

Stefan w. surface tension, under radial symmetry: probabilistic representation

$$u_t - \frac{1}{2}u_{xx} - \frac{d-1}{2x}u_x = 0, \quad x \in \mathbb{R} \setminus \{\Lambda(t)\}, \quad t > 0, \quad u(x,0) = \phi(x),$$
$$u(\Lambda(t)^{\pm}, t) = H(\Lambda(t)) := -1/\Lambda(t),$$
$$\dot{\Lambda}(t) = \frac{1}{2}\left[u_x(\Lambda^-, t) - u_x(\Lambda^+, t)\right]$$

The following Feynman-Kac formula gives a probabilistic representation for $v := -u$:

$$v(t,x) := \mathbb{E}^x\left[\mathbf{1}_{\{\tau \le t\}} \cdot H(R_\tau)\right] + \mathbb{E}^x\left[\mathbf{1}_{\{\tau > t\}} \cdot (-\phi(R_t))\right],$$
$$\frac{1}{d}\left(\Lambda^d(t) - \Lambda^d(0^-)\right) = \int_{\mathbb{R}_+} (-\phi(x))\,\nu(\mathrm{d}x) - \int_{\mathbb{R}_+} v(t,x)\,\nu(\mathrm{d}x),$$

where R is a Bessel process, $\nu(\mathrm{d}x) := x^{d-1}\,dx$, and τ is the first crossing time of $\Lambda(t-\cdot)$ by R.

Probabilistic solution to Stefan problem w. surface tension, under radial symmetry: definition

$$v(t,x) := \mathbb{E}^x\big[\mathbf{1}_{\{\tau\le t\}}\cdot H(R_\tau)\big] + \mathbb{E}^x\big[\mathbf{1}_{\{\tau>t\}}\cdot(-\phi(R_t))\big],$$
$$\frac{1}{d}\big(\Lambda^d(t)-\Lambda^d(0^-)\big) = \int_{\mathbb{R}_+}(-\phi(x))\,\nu(\mathrm{d}x) - \int_{\mathbb{R}_+} v(t,x)\,\nu(\mathrm{d}x),$$

To have a chance for uniqueness, we add the jump condition:

$$\Lambda(t^-)-\Lambda(t) = \inf\Big\{z\in(0,\Lambda(t^-)] : \int_{\Lambda(t^-)-z}^{\Lambda(t^-)} v(t-,x)\,\nu(\mathrm{d}x) > \int_{\Lambda(t^-)-z}^{\Lambda(t^-)} \big(H(x)-1\big)\,\nu(\mathrm{d}x)\Big\},$$

$$\Lambda(t)-\Lambda(t^-) = \inf\Big\{z>0 : \int_{\Lambda(t^-)}^{\Lambda(t^-)+z} v(t-,x)\,\nu(\mathrm{d}x) < \int_{\Lambda(t^-)}^{\Lambda(t^-)+z} \big(H(x)+1\big)\,\nu(\mathrm{d}x)\Big\}.$$

Probab-c Stefan w. surface tension, under radial symmetry: existence via approximation

$$v(t,x) := \mathbb{E}^x\big[\mathbf{1}_{\{\tau\le t\}}\cdot H(R_\tau)\big] + \mathbb{E}^x\big[\mathbf{1}_{\{\tau>t\}}\cdot(-\phi(R_t))\big],$$

$$\frac{1}{d}\big(\Lambda^d(t)-\Lambda^d(0^-)\big) = \int_{\mathbb{R}_+}(-\phi(x))\,\nu(\mathrm{d}x) - \int_{\mathbb{R}_+} v(t,x)\,\nu(\mathrm{d}x),$$

$$\Lambda(t^-)-\Lambda(t) = \inf\Big\{z\in(0,\Lambda(t-)] :\ \int_{\Lambda(t-)-z}^{\Lambda(t-)} v(t-,x)\,\nu(\mathrm{d}x) > \int_{\Lambda(t-)-z}^{\Lambda(t-)}\big(H(x)-1\big)\,\nu(\mathrm{d}x)\Big\},$$

$$\Lambda(t)-\Lambda(t^-) = \ldots$$

- **Theorem** (*N.-Shkolnikov 2022*). Assume that $\phi\ge -1$ and that $|\phi(x)|\le C\exp(-Cx)$. Then, for $d\ge 3$, there exists a solution to the above system.

Numerical experiment

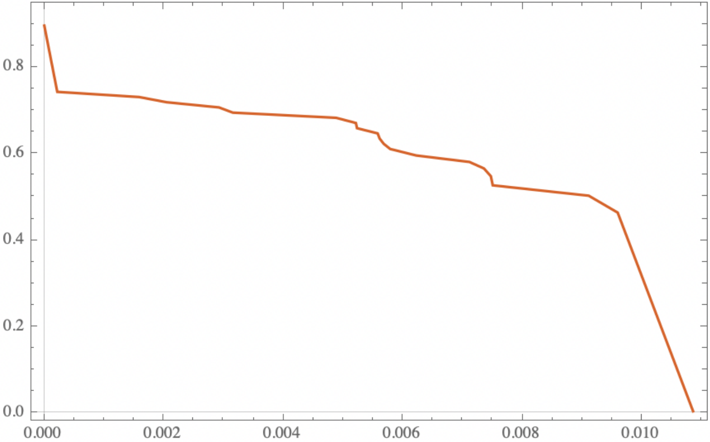


Figure: Graph of $\Lambda(\cdot)$, for $d=3$ and the initial data $\Lambda(0-) = 0.9$, $\phi = -\mathbf{1}_{[0,0.81]}$.

About the proof

For a given piece-wise constant cadlag Λ^Δ, consider:

- a d-dimensional Bessel process X started according to the density $-\phi(x)\,x^{d-1}$
- and its crossing time $\tau^\Delta := \inf\{t > 0\colon (X_t - \Lambda^\Delta_t)(X_0 - \Lambda^\Delta_0) < 0\}$ of $\Lambda^\Delta_\cdot$;
- for $m \geq 1$, the d-dim. Bessel processes $\{X^{m,i,\Delta}\}_{i\geq 1}$ started at time $m\Delta$ from the atoms of an independent Poisson random measure of intensity H in the interval between $\Lambda^\Delta_{(m-1)\Delta}$ and $\Lambda^\Delta_{m\Delta}$,
- and their crossing times $\tau_i^{m,\Delta}$ of $\Lambda^\Delta_\cdot$;
- the jumps times $\{T_i^{\delta,\Delta}\}_{i\geq 1}$ of a Poisson process with rate $2\delta^{-1}(\Lambda^\Delta)^{d-2}$, for $\delta > 0$;
- $[-1,1]$-valued independent uniform random variables $\{\gamma_i\}_{i\geq 1}$;
- independent d-dimensional Bessel processes $\{Y^{\delta,i,\Delta}\}_{i\geq 1}$ started at the times $\{T_i^{\delta,\Delta}\}_{i\geq 1}$ from $\{(\Lambda^\Delta_{T_i^{\delta,\Delta}} + \delta\gamma_i) \vee 0\}_{i\geq 1}$, respectively,
- and their crossing times $\tau_i^{\delta,\Delta}$ of $\Lambda^\Delta_\cdot$.

Implicit Euler scheme

- Fix $\Delta > 0$ and define Λ^Δ by solving the following recursive equations, for all $m = 0, 1, \ldots$:

$$\frac{1}{d}\big((\Lambda^\Delta_{m\Delta})^d - (\Lambda^\Delta_{0-})^d\big) = \mathbb{P}(\tau^\Delta \le m\Delta) - \sum_{n=1}^{m}\sum_{i\ge 1}\mathbb{P}(\tau_i^{n,\Delta} > m\Delta)$$
$$- \lim_{\delta\downarrow 0}\sum_{i\ge 1}\mathbb{P}(T_i^{\delta,\Delta} \le m\Delta < \tau_i^{\delta,\Delta})$$

- **Theorem** (*N.-Shkolnikov 2022*) $\{\Lambda^\Delta\}_{\Delta\downarrow 0}$ is pre-compact in M1, and its every **limit point** yields a **probabilistic solution** to the supercooled Stefan problem with surface tension, under radial symmetry.