Uniqueness of the solution of the filtering equations

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joint work with Dan Crisan

BSDE 2022, Annecy

The classical filtering problem

- Assume that $\{(X_t, Y_t),\ t \geq 0\}$ is given as

$$X_t = X_0 + \int_0^t f(s, X_s)ds + \int_0^t g(s, X_s)dV_s + \int_0^t \bar{g}(s, X_s)dW_s,$$
$$Y_t = \int_0^t h(s, X_s)ds + W_t,$$

where X_t takes its values in $\mathbb{R}^d$, Y_t in $\mathbb{R}^m$, V_t and W_t are mutually independent Brownian motions, resp. k and m dimensional. This is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, equipped with a filtration $\mathcal{F}_t$.

- Let $\mathcal{Y}_t := \sigma\{Y_s,\ 0 \leq s \leq t\}$. We wish to "compute" the conditional law of X_t, given $\mathcal{Y}_t$, for all $t \geq 0$.
- Let

$$Z_t = \exp\left(\int_0^t (h(s, X_s), dW_s) - \frac{1}{2}\int_0^t |h(s, x_s)|^2 ds\right),$$

$\tilde{\mathbb{P}}$ defined by $\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}|_{\mathcal{F}_t} = Z_t,\ \tilde{Z}_t := Z_t^{-1} = \frac{d\mathbb{P}}{d\tilde{\mathbb{P}}}|_{\mathcal{F}_t}.$

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The Zakai equation 1

- The so-called Kallianpur–Striebel formula is easy to verify :

$$\mathbb{E}\left[\varphi(X_t)|\mathcal{Y}_t\right] = \frac{\tilde{\mathbb{E}}\left[\tilde{Z}_t\varphi(X_t)|\mathcal{Y}_t\right]}{\tilde{\mathbb{E}}\left[\tilde{Z}_t|\mathcal{Y}_t\right]}.$$

- Under $\tilde{\mathbb{P}}$, $\{Y_t,\ t \geq 0\}$ is a Brownian motion independent of V_t and the measure–valued process π_t defined by $\pi_t(\varphi) := \tilde{\mathbb{E}}\left[\tilde{Z}_t\varphi(X_t)|\mathcal{Y}_t\right]$ solves (using summation over repeated indices)

$$\pi_t(\varphi) = \pi_0(\varphi) + \int_0^t \pi_s(A_s\varphi)ds + \pi_s(B_s^j\varphi)dY_s^j,$$

- where with $a = gg^T + \bar{g}\bar{g}^T$,

$$(A_s\varphi)(x) = \frac{1}{2}a_{ij}(s,x)\partial^2_{x_i,x_j}(x) + f_i(s,x)\partial_{x_i}\varphi(x),$$
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The Zakai equation is easily obtained as follows (with φ smooth enough) :

- First develop $\tilde{Z}_t\varphi(X_t)$ using Itô's formula.
- Note that
  1. For ξ_t $\mathcal{F}_t$ measurable, $\tilde{\mathbb{E}}[\xi_t|\mathcal{Y}_{t+s}] = \tilde{\mathbb{E}}[\xi_t|\mathcal{Y}_t]$, since $\mathcal{Y}_t$ is the filtration of a Brownian motion under $\tilde{\mathbb{P}}$.
  2. $\tilde{\mathbb{E}}[\int_0^t \tilde{Z}_s\psi(X_s)ds|\mathcal{Y}_t] = \int_0^t \pi_s(\psi)ds$.
  3. $\tilde{\mathbb{E}}[\int_0^t \tilde{Z}_s\psi(X_s)dY_s^j|\mathcal{Y}_t] = \int_0^t \pi_s(\psi)dY_s^j$.
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  4. $\tilde{\mathbb{E}}[\int_0^t \tilde{Z}_s\psi(X_s)dV_s|\mathcal{Y}_t] = 0$.

The Zakai equation 3

- π_t is called the "unnormalized conditional law of X_t, given $\mathcal{Y}_t$". Indeed

$$\mathbb{E}[\varphi(X_t)|\mathcal{Y}_t] = \frac{\pi_t(\varphi)}{\pi_t(1)} \text{ (see the K–S formula)}$$

- With a smooth enough test function $u(t,x)$, the Zakai equation becomes

$$\pi_t(u_t) = \pi_0(u_0) + \int_0^t \pi_s(\partial_s u_s + A_s u_s)ds + \int_0^t \pi_s(B_s^j u_s)dY_s^j,$$

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Uniqueness of the Zakai equation 0

- Given $r \in L^\infty(0, T; \mathbb{R}^m)$, we consider the complex valued process

$$\theta_t = \exp\left(i \int_0^t (r_s, dY_s) + \frac{1}{2} \int_0^t |r_s|^2 ds \right),$$

so that $\theta_t = 1 + i \int_0^t \theta_s(r_s, dY_s)$.

- Consider the set of r.v.'s $S_T = \{\theta_T, \; r \in L^\infty(0, T; \mathbb{R}^m)\}$. If $X \in L^1(\Omega, \mathcal{Y}_T, \tilde{\mathbb{P}})$ is such that $\tilde{\mathbb{E}}[\theta_T X] = 0$ for all $\theta_T \in S_T$, then $X = 0$ a.s.

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Uniqueness of the Zakai equation 1

- From Itô's formula,

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- If

$$\partial_t u_t + A_t u_t + ir_t^j B_t^j u_t = 0,\ 0 \le t \le T,$$
$$u_T = \varphi \text{ and}$$

$$\tilde{\mathbb{E}}\left(\sqrt{\int_0^T \theta_t^2[\pi_t(B_t^j u_t) + r_t^j\pi_t(u_t)]^2 dt}\right) < \infty,\ 1 \le j \le m,\ (*)$$

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Uniqueness of the Zakai equation 2

- Suppose that for any $T > 0$, $r \in L^\infty(0, T; \mathbb{R}^m)$ and φ in a dense subset of $C_b(\mathbb{R}^d)$, the above backward parabolic PDE has a smooth enough solution which satisfies $(*)$. Then we have uniqueness of the solution of the Zakai equation in the space of measure valued processes satisfying some condition to insure $(*)$.
- If all coefficients are bounded, as well as the solution of the backward PDE and its first order derivatives, then we have uniqueness in the set of measure valued processes satisfying $\mathbb{E}[\sup_{0\le t\le T} \pi_t(1)] < \infty$.
- Such a result has been obtained by A. Bensoussan in his book *Stochastic Control of Partially Observable Systems* with no ellipticity assumption, allowing the coefficients f and h to have linear growth, provided a, f and h have bounded derivatives of order 1 and 2 w.r.t. the spatial variables.
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A more general filtering problem

- One can generalize the above filtering problem as follows :

$$X_t = X_0 + \int_0^t f(s, X_s, Y_s)ds + \int_0^t g(s, X_s, Y_s)dV_s + \int_0^t \bar{g}(s, X_s, Y_s)dW_s$$

$$Y_t = \int_0^t h_1(s, Y_s)ds + \int_0^t k(s, Y_s)[h_2(s, X_s, Y_s)ds + dW_s],$$

where the matrix k need not be invertible. All coefficients bounded, appropriate Lipschitz properties.

- In this case, the Zakai equation takes the form

$$\pi_t(\varphi)=\pi_0(\varphi)+\int_0^t \pi_s(A_s\varphi)ds+\sum_{j=1}^m \int_0^t \pi_s(B_s^j\varphi)k^+(s, Y_s)(dY_s^j-h_1(s, Y_s)ds),$$

where here B_s^j is as above, but with h replaced by h_2, and k^+ is the Moore–Penrose pseudo–inverse, which satisfies : $kk^+k = k$, $(k^+k)^* = k^+k$, k^+k is the orthogonal projection on $\mathrm{Im}(A^*)$.

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- The new argument. Itô's formula yields

$$\varphi(X_t)\,\tilde{Z}_t = \varphi(X_0) + \int_0^t \tilde{Z}_s A_s \varphi(X_s)\mathrm{d}s + \int_0^t \tilde{Z}_s(\nabla\varphi\sigma)(s, X_s, Y_s)\,\mathrm{d}V_s$$
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- We show that $\tilde{\mathbb{E}}(\cdot|\mathcal{Y}_t)$ of the second integral on the right vanishes. Hence the Zakai euation :

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Backward SPDE 1

- Again, the Zakai equation takes the form

$$\pi_t(\varphi)=\pi_0(\varphi)+\int_0^t \pi_s(A_s\varphi)ds+\sum_{j=1}^m\int_0^t \pi_s(B_s^j\varphi)d\tilde{W}_t,$$

where $B_t\varphi$ is the previous one, multiplied on the right by $[k^+k](t, Y_t)$.

- We consider the complex valued BSPDE

$$du_t+(A_tu_t+[B_t^jv_t^j+ir_t^jB_t^ju_t+ir_t^jv_t^j])dt=v_t^jd\tilde{W}_t^j,\ u_T=\varphi\,.$$

which is equivalent to the system of real–valued BSPDEs

$$du_t^1+(A_tu_t^1+[B_t^jv_t^{1,j}-r_t^jB_t^ju_t^2-r_t^jv_t^{2,j}])dt=v_t^{1,j}d\tilde{W}_t^j,\ u_T^1=\varphi;$$
$$du_t^2+(A_tu_t^2+[B_t^jv_t^{2,j}+r_t^jB_t^ju_t^1+r_t^jv_t^{1,j}])dt=v_t^{2,j}d\tilde{W}_t^j,\ u_T^2=0\,.$$

Backward SPDE 1

- Again, the Zakai equation takes the form

$$\pi_t(\varphi)=\pi_0(\varphi)+\int_0^t \pi_s(A_s\varphi)ds+\sum_{j=1}^m\int_0^t \pi_s(B_s^j\varphi)d\tilde{W}_t,$$

  where $B_t\varphi$ is the previous one, multiplied on the right by $[k^+k](t,Y_t)$.
- We consider the complex valued BSPDE

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Backward SPDE 2

Adapting to this system known results for BSPDEs, see Du, Meng '10 and Du, Tang, Zhang '13, we can show that if all our coefficients are bounded, together with their derivatives up to order n in x, and φ is smooth, the above system of BSPDEs has a solution such that for $i = 1, 2$, wih $\|\cdot\|_n$ denoting the norm in the Sobolev space H^n,

$$\mathbb{E}\left[\sup_{0\le t\le T}\|u_t^i\|_n^2 + \int_0^T \|v^i\|_n^2 dt\right] < \infty\,.$$

An ad hoc Itô formula

- From the Zakai equation written in weak form, which gives the semimartingale decomposition of $\pi_t(\varphi)$, we have deduced the semimartingale decomposition of $\pi_t(u_t)$ in case $u \in C^{1,2}$.
- Now we need to develop $\pi_t(u_t)$ in case

$$u(t,x) = u(0,x) + \int_0^t \Sigma(s,x)ds + \int_0^t \Lambda^j(s,x)d\tilde{W}_s^j, \ 0 \le t \le T$$

such that the processes $A_t u_t + \Sigma_t + B_t^j \Lambda_t^j$ and $B_t^j u_t + \Lambda_t^j$ are $C_b(\mathbb{R}^d)$ valued.
- We have the formula

$$\pi_t(u_t) = \pi_0(u_0) + \int_0^t \pi_s(A_s u_s + \Sigma_s + B_s^j \Lambda_s^j)ds$$
$$+ \int_0^t \pi_s(B_s^j u_s + \Lambda_s^j)d\tilde{W}_s^j, \ 0 \le t \le T .$$

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Uniqueness of the Zakai equation using a duality argument with BSPDEs

- We assume that the above assumptions hold for some $n > 2 + d/2$. Then we can show that if u is a solution of the above BSPDE, then

$$d\theta_t\pi_t(u_t) = \theta_t\pi_t(B_t^j u_t + v_t^j + ir_t^j u_t)d\tilde{W}_t^j$$

  and provided that $\mathbb{E}\left[\sup_{0\le t\le T}\pi_t(1)^2\right] < \infty$, $\{\theta_t\pi_t(u_t),\ 0\le t\le T\}$ is a martingale

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Theorem

If the coefficients a, f and h are of class C_b^n as functions of x for some $n > 2 + d/2$, then the Zakai equation has a unique solution in the class of $\mathcal{Y}_t$–adapted measure valued processes satisfying for any $T > 0$

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Bibliography

- A. Bain, D. Crisan, *Fundamentals of Stochastic Filtering*, Stochastic Modelling and Applied Probability, Vol 60, Springer Verlag, 2008.
- A. Bensoussan, *Stochastic control of partially observable systems*. Cambridge University Press, Cambridge, 1992.
- K. Du, Q. Meng, A revisit of W_2^n-theory of super-parabolic backward stochastic partial differential equations in $\mathbb{R}^d$, *Stoch. Proc. and Applic.* **120**, 1996–2015, 2010.
- K. Du, S. Tang, Q. Zhang, $W^{m,p}$–solutions ($p \geq 2$) of linear degenerate backward stochastic partial differential equations in the whole space, *J. of Differential Equ.* **254**, 2877 –2904, 2013.
- D.J. Fotsa–Mbogne, É. Pardoux, Nonlinear filtering with degenerate noise, *Elec. Comm. Probab.* **22**, 2017.

THANK YOU FOR YOUR ATTENTION !