Time inconsistent stochastic control

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Outline

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What we should take home from this: human beings have time-depending and slightly inconsistent preferences.

The basic problem

- On space $(\Omega, \mathcal{F})$, let $\mathbb{P}^{\nu}$ be a weak solution to the controlled SDE

$$X_t = x + \int_0^t b_r(X_{r\wedge\cdot}, \nu_r)\mathrm{d}r + \int_0^t \sigma_r(X_{r\wedge\cdot})\mathrm{d}W_r,\ t \in [0, T].$$

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- The reward functional

$$v(t,\nu) := J(t,t,\nu) = \mathbb{E}^{\mathbb{P}^\nu}\bigg[\int_t^T f_r(t, X_{r\wedge\cdot}, \nu_r)\mathrm{d}r + F(t, X_{T\wedge\cdot})\bigg|\mathcal{F}_t\bigg].$$

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- Control problem: because of the dependence in t, classical dynamic programming arguments fail (unless for exponential discounting of course). What to do?

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- Take time-inconsistency seriously: consider a non-cooperative game, where the agent plays against future versions of himself, and look for sub-game perfect Nash equilibria (Barro, Czichowsky, Ekeland, Laibson, Lazrak, Pollak, Privu, Strotz, Zhou,...)

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Equilibria in discrete-time

Perfectly understandable situation. If $(t_i)_{0 \leq i \leq n}$ is a partition of $[0, T]$

- Given actions played by Player 0,..., and Player t_{n-2}, Player t_{n-1} faces a standard optimisation problem.

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- Repeat backwardly.
- Unfortunately, optimisation problems often lose concavity after a few iterations $\Longrightarrow$ equilibria do not exist in general.

Equilibria in continuous-time

Definition [Ekeland, Lazrak (2008)]

$\nu^\star$ is an equilibrium if for any $(t, \nu) \in [0, T) \times \mathcal{V}$

$$\liminf_{\ell \to 0} \frac{J(t, t, \nu^\star) - J(t, t, \nu^\ell)}{\ell} \geq 0,$$

where for $\ell \in [0, T - t]$, ν^ℓ is given by

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- Not completely satisfying, when $\liminf$ is 0.
- Not a 'local' property.

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Definition [Hernández, P. (2019)]

$\nu^\star$ *is an* *equilibrium* *if for any* $\varepsilon > 0$ *there exists* $\ell_\varepsilon > 0$ *such that for any* $(t, \nu, \tau) \in [0, T) \times \mathcal{V} \times \mathcal{T}_{t,t+\ell_\varepsilon}$

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- Definition is roughly speaking the same as before: $\varepsilon-$equilibrium. .

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- Extended by Björk, Khapko, and Murgoci to general Markovian diffusion models.
- However, extended HJB equation only justified formally by passing to the limit in discrete-time models.
- Different from classical control problems, where HJB equation and optimal controls are intimately linked.

Outline

1. The big picture
   - Motivating example
   - Time-inconsistency
   - Three approaches
2. What is an equilibrium?
   - In discrete-time, (almost) all is well
   - Not so much in continuous-time...
3. Main results
   - An extended DPP
   - The characterising BSDE system
   - Verification theorem
   - Extensions

An extended DPP

Theorem [Hernández, P. (2019)]

If $\nu^\star$ is an equilibrium then

$$v(0,\nu^\star)=\sup_{\nu\in\mathcal{V}}\mathbb{E}^{\mathbb{P}^\nu}\bigg[v(\tau,\nu^\tau)+\int_0^\tau\bigg(f(r,r,X_{r\wedge\cdot},\nu_r)-\frac{\partial F}{\partial s}(r)-\int_r^T\frac{\partial f}{\partial s}(r,u,X_{r\wedge\cdot},\nu^\star_u)\mathrm{d}u\bigg)\mathrm{d}r\bigg].$$

HJB BSDE system

Define the Hamiltonian

$$H_t(x,z) := \sup_{a\in A} h_t(t,x,z),\ \ h_t(s,x,z,a) := f_t(s,x,a) + b_t(x,a)\cdot z,$$
$$\nu_t^\star(x,z) := \mathrm{argmax}_{a\in A} h_t(t,x,z,a).$$

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Extended DPP relates value of the agent at equilibrium to

$$\begin{cases} Y_t = F(T, X_{\cdot\wedge T}) + \displaystyle\int_t^T \big(H_r(X,Z_r) - \partial Y_r^r\big)\mathrm{d}r - \int_t^T Z_r\cdot \mathrm{d}X_r, \\ Y_t^s = F(s, X_{\cdot\wedge T}) + \displaystyle\int_t^T h_r(s,X,\nu^\star_r(X_{r\wedge\cdot},Z_r),Z_r^s)\mathrm{d}r - \int_t^T Z_r^s\cdot \mathrm{d}X_r, \end{cases}$$

where

$$\partial Y_t^s := \frac{\partial}{\partial_s} Y_t^s.$$

HJB BSDE system (2)

- For exponential discounting $f_s(t,x,a) = \mathrm{e}^{-\delta(s-t)}\tilde{f}(s,x,a)$, we have $\partial Y_t^s = -\delta Y_s \Longrightarrow$ no coupling and classical HJB BSDE.

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- This is the non-Markovian version of the non-local PDE system derived first by Ekeland and Lazrak (2008).
- Earlier contributions argued by passing informally to the limit from discrete-time. Thanks to our extended DPP

Theorem [Hernández, P. (2019)]

If $\nu^\star$ is an equilibrium, then there exists a solution to the BSDE system, and necessarily $\nu^\star = \nu^\star(X,Z)$.

Verification theorem

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If there exists a sufficiently integrable solution to the BSDE *system, then* $\nu^\star(X,Z)$ *is an equilibrium,*

$$Y_t^s = J(s,t,\nu^\star), \text{ and } Y_t = v(t,\nu^\star).$$

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- Non-Markovian extension of earlier results (Ekeland and Lazrak, Björk et al., Wei et al.).
- Under mild conditions (Lipschitz + integrability) can prove wellposedness of the BSDE system

Extensions

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- More general time-inconsistency

$$J(t,t,\mathbb{M}) := \mathbb{E}^{\mathbb{P}^\nu}\left[\int_t^T f(t,r,X_{r\wedge\cdot},\nu_r)\mathrm{d}r + G(t,X_{T\wedge\cdot})\Big|\mathcal{F}_t\right] + F\Big(t,\mathbb{E}^{\mathbb{P}^\nu}[h(X_{T\wedge\cdot})|\mathcal{F}_t]\Big),$$

including mean–variance: system of 3 coupled (2)BSDEs (need to add $\mathbb{E}^{\mathbb{P}^\nu}[h(X_{T\wedge\cdot})|\mathcal{F}_t]$ as a state).

Extensions (2)

- Link between the BSDE system and BSVIEs: family $(Y^s, Z^s)_{s\in[0,T]}$ solves a type-I extended BSVIE.

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- using standard arguments to prove existence to the system leads to new wellposedness results for BSVIEs themselves.

Thank you for your attention!