Reflected BSDEs in non-convex domains

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joint work with
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why defining a notion of conditional expectation constrainted in a non-convex domain?

Figure: https://www.touteleurope.eu

A known solution

We can define a distance function d in the domain $\mathcal{D}$ and then consider a Fréchet mean for a r.v. X with values in $\mathcal{D}$:

$$\tilde{\mathbb{E}}[X] \in \arg\min_{y\in\mathcal{D}} \mathbb{E}[d^2(y, X)].$$

Rem: If $\mathcal{D}$ is convex, nothing new here...


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We can define a conditional version of the Frechet-mean.

$$\tilde{\mathbb{E}}[X|\mathcal{F}] \in \arg\min_{y \in \mathcal{D}} \mathbb{E}[d^2(y, X)|\mathcal{F}].$$

Problem: the tower property does not hold!

$$\tilde{\mathbb{E}}[X] \neq \tilde{\mathbb{E}}[\tilde{\mathbb{E}}[X|\mathcal{F}]].$$

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Solution: A reflected BSDE (with null generator) satisfies the tower property by definition.

Reflected BSDEs

We consider W a d-dimensional Brownian motion, $(\mathcal{F}_t)_{t\geqslant 0}$ its natural filtration (augmented), $T>0$, $\xi \in L^2(\mathcal{F}_T)$.
Let $\mathcal{D}$ a domain (open connected subset of $\mathbb{R}^d$). We are looking for a triplet $(Y,Z,K) \in \mathscr{S}^2 \times \mathscr{H}^2 \times \mathscr{K}^1$ i.e.

$$\mathbb{E}\left[\sup_{t\in[0,T]} |Y_t|^2 + \int_0^T |Z_t|^2 dt + \int_0^T d\mathrm{Var}_t(K)\right] < +\infty,$$

that satisfies

$$\begin{cases} (i)\ Y_t = \xi + \int_t^T f(s,Y_s,Z_s)\mathrm{d}s - \int_t^T \mathrm{d}K_s - \int_t^T Z_s \mathrm{d}W_s, \quad 0 \leqslant t \leqslant T, \\ (ii)\ Y_t \in \bar{\mathcal{D}} \text{ a.s.}, \quad K_t = \int_0^t \mathfrak{n}(Y_s)\mathrm{d}\mathrm{Var}_s(K), \quad 0 \leqslant t \leqslant T, \end{cases} \tag{1}$$

where $\mathfrak{n}$ is the unit outward normal to $\partial\mathcal{D}$, extended as zero into $\mathcal{D}$.

Some references

- Convex domain: [Gégout-Petit Pardoux '96]
- Extensions to random domains [Klimsiak Rozkosz Słomiński '15], jumps [Fakhouri Ouknine Ren '18], oblique reflection [Chassagneux Richou '20]
- Dimension 1: [El Karoui Kapoudjian Pardoux Peng Quenez '97] [Cvitanic Karatzas '96] [Dumitrescu Quenez Sulem '16] [Grigorova Imkeller Offen Ouknine Quenez '17]...

Applications:

- EDP [Klimsiak Rozkosz Słomiński '18]
- American options
- Optimal switching problems [Hamadène Zhang '10] [Hu Tang '10]

All these results concern convex domains. Non-convex domains:

- [Gayduk Nadtochiy '20] Market microstructure model with a Control-stopping game
- [Briand Hibon '21] Particule system approximation for mean reflected BSDEs.

Results of Gégout-Petit and Pardoux

Assumptions:

- $\mathcal{D}$ convex, $\xi \in \mathcal{D}$,
- f Lipschitz with respect to y and z,
- $\mathbb{E}[|\xi|^2 + \int_0^T |f(.,0,0)|^2 dt] < +\infty$

Theorem

There exists a unique solution to the reflected BSDE.

Proof of the uniqueness

We take two solutions (Y, Z, K) and (Y', Z', K'):

$$
\begin{aligned}
&|\delta Y_t|^2 + \int_t^T |\delta Z_s|^2 ds \\
&= \int_t^T 2(Y_s - Y'_s).(f(s, Y_s, Z_s) - f(s, Y'_s, Z'_s))ds + \int_t^T dM_s \\
&\quad + 2\int_t^T (Y'_s - Y_s)dK_s + 2\int_t^T (Y_s - Y'_s)dK'_s
\end{aligned}
$$

Proof of the existence

We consider a penalization approximation of the BSDE:

$$Y_t^n = \xi + \int_t^T f(t, Y_t^n, Z_t^n)dt - \int_t^T n\phi(Y_t^n)\nabla\phi(Y_t^n)dt - \int_t^T Z_s^n dW_s$$

with $\phi(y) := d(y, \mathcal{D})$. We have following estimates:

- $\mathbb{E}[\sup_{t\in[0,T]} |Y_t^n|^2] + \mathbb{E}[\int_0^T |Z_t^n|^2 dt] \leqslant C$
- $n\mathbb{E}[\sup_{t\in[0,T]} \phi^2(Y_t^n)] + \mathbb{E}[\int_0^T n^2\phi^2(Y_t^n)dt] \leqslant C$.

The non-convex case

Assumptions:

- $\mathcal{D} = \{x \in \mathbb{R}^d | \phi(x) < 0\}$ with $\phi : \mathbb{R}^d \to \mathbb{R}$ C^2,
- $|\nabla\phi(y)| > 0$ for all $y \in \partial\mathcal{D}$,
- $\mathcal{D}$ bounded,
- $\xi \in \mathcal{D}$, $f(., 0, 0)$ bounded.

To simplify the presentation: $f = 0$.

Uniqueness

Exterior sphere property

There exists R_0 such that

$$(y - y') \cdot \mathfrak{n}(y) + \frac{1}{2R_0}|y - y'|^2 \geqslant 0\ , \quad \forall\, y \in \partial\mathcal{D},\ y' \in \bar{\mathcal{D}}. \tag{2}$$

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We take two solutions (Y,Z,K) and (Y',Z',K'):

$$\begin{aligned}&\Gamma_t|\delta Y_t|^2+\int_t^T\Gamma_s|\delta Z_s|^2ds\\ =&\int_t^T dM_s-2\int_t^T\Gamma_s\left(2(Y_s-Y'_s)dK_s+\frac{1}{R_0}|\delta Y_s|^2d\mathrm{Var}_s(K)\right)\\ &-2\int_t^T\Gamma_s\left(2(Y'_s-Y_s)dK'_s+\frac{1}{R_0}|\delta Y_s|^2d\mathrm{Var}_s(K')\right)\end{aligned}$$

with $\Gamma_t=e^{\frac{1}{R_0}(\mathrm{Var}_t(K)+\mathrm{Var}_t(K'))}$.

Uniqueness

Proposition

There exists at most one solution to the reflected BSDE in the class of solutions such that there exists $p > 1$ for which

$$\mathbb{E}\left[e^{\frac{2p}{R_0}\mathrm{Var}_T(K)}\right] < +\infty.$$

Existence

We consider a penalized approximation of the BSDE:

$$Y_t^n = \xi - \int_t^T n\phi^+(Y_t^n)\nabla\phi(Y_t^n)dt - \int_t^T Z_s^n dW_s$$

and we try to get estimates on (Y^n, Z^n) that do not depend on n.

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If we apply Itô formula to $|Y_t^n|^2$ then we get a term

$$-\int_t^T n\phi^+(Y_t^n) Y_t^n \cdot \nabla\phi(Y_t^n) dt$$

which has no clear sign without extra assumption.

Weak star-shape property

We assume that there exists a convex domain $\mathcal{C} \subset \mathcal{D}$ such that

$$\gamma := \inf_{y \in \partial \mathcal{D}} \nabla \phi_{\mathcal{C}}(y) \cdot \frac{\nabla \phi(y)}{|\nabla \phi(y)|} > 0. \tag{3}$$

with $\phi_{\mathcal{C}} = d(., \mathcal{C})$.

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with $\phi_{\mathcal{C}} = d(.,\mathcal{C})$.
Enough to get

$$\mathbb{E}[\sup_{t \in [0,T]} |Y_t^n|^2] + \mathbb{E}[\int_0^T |Z_t^n|^2 dt] \leqslant C$$

but we are not able to obtain

$$n\mathbb{E}[\sup_{t \in [0,T]} |\phi^+(Y_t^n)|^2] + \mathbb{E}[\int_0^T n^2 |\phi^+(Y_t^n)|^2 dt] \leqslant C!$$

A new penalized BSDE

We consider a new penalized approximation of the BSDE:

$$Y_t^n = \xi - \int_t^T n\phi^+(Y_t^n)(1+|Z_t^n|^2)\nabla\phi(Y_t^n)dt - \int_t^T Z_s^n dW_s.$$

Existence and uniqueness of a solution for this penalized BSDE?

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Existence and uniqueness of a solution for this penalized BSDE?

[Xing Žitković '18]

We assume that $\xi = g(W_T)$ with g a α-Hölder function, then there exists a Markovian solution $(Y^n, Z^n) \in \mathscr{S}^2 \times \mathscr{H}^2$.

And we get

$$n\mathbb{E}[\sup_{t\in[0,T]} |\phi^+(Y_t^n)|^2] + \mathbb{E}[\int_0^T n^2|\phi^+(Y_t^n)|^2 dt] \leqslant C.$$

How to pass to the limit?

[Xing Žitković '18]

there exists $u_n : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $0 < \alpha'_n \leqslant \alpha$ such that $Y_t^n = u^n(t, W_t)$ with

$$\sup_{(t,x)\neq(t',x')} \frac{|u^n(t,x) - u^n(t',x')|}{|t-t'|^{\alpha'_n/2} + |x-x'|^{\alpha'_n}} \leqslant C_n.$$

Constants C_n and α'_n depends on n only through $\text{ess-sup}_{\omega\in\Omega, t\in[0,T]} |n\phi^+(Y_t^n)|$.

Proposition

There exists C independent with n such that

$$\text{ess-sup}_{\omega\in\Omega, t\in[0,T]} |n\phi^+(Y_t^n)| < C.$$

Then we can use compacity argument to get $u_n \to u$.

An existence result in the Markovian framework

Theorem

Assumptions: weak star-shape property + Markovian framework.
There exists a unique solution (Y, Z, K) to the reflected BSDE such that, for all $\beta > 0$,

$$\mathbb{E}[e^{\beta \mathrm{Var}_T(K)}] < +\infty.$$

Back to the non Markovian framework

Theorem

Let us assume that the weak star-shape property and

- $|\phi^+_{\mathcal{C}}(\xi)|_{\mathscr{L}^\infty} < \frac{\gamma R_0}{2}$ and
  $\nabla\phi_{\mathcal{C}}(y) \cdot f(s,y,z) \leqslant 0, \quad \forall s,y,z \in [0,T] \times \bar{\mathcal{D}}\backslash\mathcal{C} \times \mathbb{R}^{d\times d'}$,
- or $\sup_{x\in\mathcal{D}} \phi^+_{\mathcal{C}}(x) < \frac{\gamma R_0}{2}$.

Then there exists a unique solution (Y,Z,K) such that

$$\mathbb{E}\left[e^{\frac{2p}{R_0}\mathrm{Var}_T(K)}\right] < +\infty.$$

Proof

- A priori estimate on $\mathrm{Var}(K)$ to get the exponential integrability.
- Uniqueness follows.
- Existence result for the smooth discrete path dependent framework, i.e. $\xi = g(W_{t_0}, \ldots, W_{t_n})$ by iterating the Markovian result.
- For a general ξ, we approximate it by a sequence of discrete path dependent terminal conditions. Need a stability result.

Link with martingales with prescribed terminal condition on manifolds

$f = 0$

- If ξ is on a "concave" part of the boundary, then Y stays on the boundary: we get a martingale on the boundary manifold. References [Picard '91] [Kendall '90] [Picard '94] [Darling '95]
- Uniqueness can fall down when $d > 2$ even with the weak star shape property.
- Good candidate for a notion of martingale in a flat manifold with a boundary. Work in progess with M. Arnaudon, J.-F. Chassagneux and S. Nadtochiy.

Conjecture (Work in progess with M. Arnaudon, J.-F. Chassagneux and S. Nadtochiy)

$d = 2$, $\mathcal{D}$ smooth domain simply connected. We should have an existence and uniqueness result for the solution of a reflected BSDE in $\mathcal{D}$.